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EMBEDDINGS OF CURVES COVERING A CURVE AND CONES CONTAINING THEM

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Abstract: Let $f : X \to C$ be a degree k covering between smooth and connected projective curves. Here we consider the projective normality and the cones containing certain embeddings of X.

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1. Cones and Coverings of Curves

Fix an integer $k \geq 2$. We work over an algebraically closed field $\mathbb{K}$ such that either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > k$. Let $f : X \to C$ be a degree k covering between smooth ad connected projective curves. Since either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > k$ shows that $\mathcal{O}_C$ is a direct factor of the rank k vector bundle $f_*(\mathcal{O}_X)$ on C, say $f_*(\mathcal{O}_X) = \mathcal{O}_C \oplus E_f$ with E_f a rank $k-1$ vector bundle on C. We will often say that E_f is the vector bundle associated to f (see [1]). Since X is connected, we have $h^0(C, E_f) = 0$. Set $g := p_a(X)$ and $q := p_a(C)$. We have $\chi(f_*(\mathcal{O}_X)) = \chi(\mathcal{O}_X)$ and hence $\deg(E_f) = k(q-1) + 1 - g$ (Riemann-Roch).

Remark 1. Let $f : X \to C$ be a degree k covering between smooth ad connected projective curves. Fix $M \in \mathrm{Pic}^t(C)$ and set $L := f^*(M)$. Thus $L \in \mathrm{Pic}^{tk}(X)$. If M is spanned, then L is spanned. We have $h^i(X, L) =$

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$h^i(C,M)\otimes h^i(C,E_f\otimes M)$, $i=0,1$ (projection formula). Hence the rational map induced by $|L|$ does not factor through f (i.e. it is constant on a general fiber of f) if and only if $h^0(C,E_f\otimes M)>0$. Assume M spanned, that the morphism $h_M : C\to \mathbf{P}^n$, $n:=h^0(C,M)-1$, is birational onto its image, $h^0(C,M_f)>0$ and that there are no smooth curve Y and $f_1 : X\to Y$, $f_2 : Y\to C$ such that $2\le \deg(f_1)\le k-1$ and $f_2\circ f_1 = f$. Then the morphism $h_L : X\to \mathbf{P}^m$, $m:=n+h^0(C,E_f\otimes M)$ is birational onto its image. The latter condition is satisfied for all C,X,f if k is a prime integer.

Definition 1. Let $f : X\to C$ be a degree k covering between smooth and projective curves and $L\in \mathrm{Pic}(X)\backslash\mathcal{O}_X$ spanned by its global sections. Let $h_L := X\to \mathbf{P}^m$, $m:=h^0(X,L)-1$, be the morphism induced by the complete linear system $|L|$. We will say that L is conical for f if there is $O\in\mathbf{P}^m$ such for a general $P\in C$, $h_L(f^{-1}(P))$ spans a line of $\mathbf{P}^m$ and O contains all these lines. We will say that L is collinear for f if for a general $P\in C$, $h_L(f^{-1}(P))$ spans a line of $\mathbf{P}^m$. Fix integers r,t such that $0\le t<r\le k-1$. We will say that L is r-linear for f, if $\dim(\langle f^{-1}(P)\rangle)=r$ for a general $P\in C$. Assume that L is r-linear for f. We will say that L is t-conical if there is a t-dimensional linear subspace $A\subset\mathbf{P}^m$ such that $A\subset\langle f^{-1}(P)\rangle$ for all general $P\in C$. We will say that L is strictly t-conical if it is t-conical and either $t=r-1$ or $t\le r-2$ and L is not $(t+1)$-conical for f. We will say that L is maximally conical for f if it is r-linear and $(r-1)$-conical for some r.

Proposition 1. *Let $f : X\to C$ be a degree k covering between smooth and projective curves and $M\in\mathrm{Pic}(C)$ such that $h^0(C,M)\ge 2$. Fix integer r,t such that $0\le t<r\le k-1$. Assume $h^0(C,M)\ge r+1$. The line bundle $L:=f^*(M)$ is r-linear if and only if $H^0(C,E_f\otimes M)$ spans a rank r subsheaf of $E_f\otimes M$. The line bundle $L:=f^*(M)$ is r-linear and strictly t-conical if and only if $H^0(C,E_f\otimes M)$ spans a rank r subsheaf of $E_f\otimes M$ and $H^0(C,E_f\otimes M(-P))$ spans a rank $r-t$ subsheaf of $E_f\otimes M(-P)$ for a general $P\in C$.*

Proof. Notice that $h^0(X,L(-f^{-1}(P))=h^0(C,M(-P))+h^0(C,E_f\otimes M(-P))$ for any $P\in C$. Since $h^0(C,M)>0$, we have $h^0(C,M)-h^0(C,M(-P))=1$ for a general $P\in C$. Hence $h^0(X,L(-f^{-1}(P))=h^0(X,L)-(r+1)$ for a general $P\in C$ if and only if $h^0(C,E_f\otimes M(-P))=h^0(C,E_f\otimes M)-r$ for a general $P\in C$, i.e. if and only if $H^0(C,E_f\otimes M)$ spans a rank r subsheaf of $E_f\otimes M$. Hence the first assertion. Assume L r-linear for f. Again, since $h^0(C,M(-P-Q))=h^0(C,M)-2$ for general $(P,Q)\in C\times M$, L is strictly t-conical if and only if $h^0(C,E_f\otimes M(-P-Q))=h^0(C,E_f\otimes M(-P))-(r-t)$ for a general $(P,Q)\in C\times M$, proving the second part. □

We single out the following particular case of Proposition 1.

Proposition 2. *Let $f : X \to C$ be a degree k covering between smooth and projective curves and $M \in \mathrm{Pic}(C)$ such that M is spanned and $h^0(C, M) \geq 2$. The line bundle $L := f^*(M)$ is conical for f if and only if $h^0(C, E_f \otimes M) = 1$.*

Proof. Since $h^0(C, E_f \otimes M) = 1$, we have $h^0(C, E_f \otimes (-P)) = 0$ for a general $P \in C$. Thus $h_L : X \to \mathbf{P}^{n+1}$, $n := h^0(C, M) - 1$, maps the fiber $f^{-1}(P)$ into a line D_P of $\mathbf{P}^{n+1}$. Since $h^0(C, M(-P-Q)) = h^0(C, M) - 2$ for general $P, Q \in C$, we have $h^0(X, L(-f^{-1}(P) - f^{-1}(Q))) = h^0(X, L) - 3$ for general P, Q, we see that $D_P \cap D_Q \neq \emptyset$ and $D_P \neq D_Q$ for general $P, Q \in C$. Thus the family of all lines D_P consists of pairwise intersecting lines spanning $\mathbf{P}^{n+1}$. Since $n + 1 \geq 3$, this implies that all these lines contain a common point $O \in \mathbf{P}^{n+1}$. □

In the same way we prove the corresponding result when $r \geq 2$ and get the following result.

Proposition 3. *Let $f : X \to C$ be a degree k covering between smooth and projective curves and $M \in \mathrm{Pic}(C)$ such that M is spanned and $h^0(C, M) \geq 2$. Fix an integer r such that $1 \leq r \leq k - 1$. The line bundle $L := f^*(M)$ is r-linear and maximally conical for f if and only if $h^0(C, E_f \otimes M) = r$ and $H^0(C, E_f \otimes M)$ spans a rank r subsheaf of E_f.*

In the maximally conical case the vertex of the cone does not intesect the image curve $h_L(X)$. More precisely, we have the following result.

Proposition 4. *Let $f : X \to C$ be a degree k covering between smooth and projective curves and $M \in \mathrm{Pic}(C)$ such that M is spanned and induces a morphism $h_M : C \to \mathbf{P}^n$. Fix an integer r such that $1 \leq r \leq k - 1$. Assume that line bundle $L := f^*(M)$ is r-linear and maximally conical. Let $V \subset \mathbf{P}^m$, $m := h^0(X, L) - 1 = n + r$ the associated $(r-1)$-dimensional linear subspace, i.e. the vertex of the associated cone. Then $V \cap h_L(X) = \emptyset$.*

Proof. Since M is spanned, L is spanned. Furthermore, h_L maps X birationally onto its image. Use that $\deg(h_L(X)) = \deg(L) = k \cdot \deg(M) = k \cdot \deg(h_M(C))$ and that the linear projection from V is a degree k morphism from $h_L(X)$ onto $h_M(C)$. □

To be r-linear for some $r \leq k - 2$ is a strong restriction on the quadruple (C, X, f, M), as shown by the following corollary of Proposition 1.

Corollary 1. *Fix a degree k covering $f : X \to C$ between smooth and connected projective curves. There is an integer d_f such that for all $d \ge d_f$ and $M \in \mathrm{Pic}^d(C)$ the line bundle $f^*(M)$ is $(r-1)$-linear and not 0-conical.*

Proof. There is an integer d_f such that for all $x \ge d-2$ and $R \in \mathrm{Pic}^x(C)$ the vector bundle $E_f \otimes R$ is spanned. Apply this observation and Proposition 1 taking $R := M(-P-Q)$ for general $P, Q \in C$. □

Proposition 5. *Fix a degree 2 covering $f : X \to C$ between smooth and connected projective curves and a very ample $M \in \mathrm{Pic}(C)$. The line bundle $L := f^*(M)$ is very ample if and only if $E_f \otimes M$ is a spanned line bundle.*

Proof. For the "if" part it is sufficient to show that $h_L|Z$ is an embedding for all length two subschemes of X. If Z is mapped isomorphically by f into C, then this is true, because $h_M|f(Z)$ is an embedding. If Z is not mapped isomorphically by f into C, then Z is a fiber of f, say $Z = f^{-1}(P)$. We have $h^0(X,L) - h^0(X,L(-Z)) = h^0(C,M) - h^0(C,M(-P)) + h^0(C,E_f \otimes M) - h^0(C,E_f \otimes M(-P))$. We easily conclude. □

Remark 2. Fix a degree k covering $f : X \to C$ between smooth and connected projective curves and a very ample $M \in \mathrm{Pic}(C)$. It is easy to check that the line bundle $L := f^*(M)$ is $(k-1)$-linear if and only if $E_f \otimes M$ is generically spanned. The vector bundle $E_f \otimes M$ is spanned if and only if every fiber of f spans a $(k-1)$-dimensional linear space. If $E_f \otimes M$ is spanned, then L is very ample.

Definition 2. Fix an integer $t > 0$, a smooth and connected projective curve A and $R \in \mathrm{Pic}(A)$. R is said to be t-normal if the multiplication map $H^0(A,R)^{\otimes t} \to H^0(A,R^{\otimes t})$ is surjective. R is said to be strongly t-normal if it is z-normal for all $1 \le z \le t$.

Theorem 1. *Fix a degree k covering $f : X \to C$ between smooth and connected projective curves and an ample, spanned and strongly t-normal $M \in \mathrm{Pic}(C)$. Assume that for all integers z such that z, w such that $z > 0$, $w > 0$ and $z + z \le t$ the multiplication map $H^0(C, E_f \otimes M^{\otimes z}) \otimes H^0(C, M^{\otimes w}) \to H^0(C, E_f \otimes M^{\otimes (z+w)})$ is surjective. Then the spanned line bundle $L := f^*(M)$ is strongly t-normal.*

Proof. Just use that $H^0(X, L^{\otimes a}) \cong H^0(C, M^{\otimes a}) \oplus H^0(C, E_f \otimes M^{\otimes a})$ (projection's formula) and that this decomposition is compatible with the multiplication map

$$H^0(C, E_f \otimes M^{\otimes z}) \otimes H^0(C, M^{\otimes w}) \to H^0(C, E_f \otimes M^{\otimes (z+w)}).$$ □

Now we apply Theorem 1 to one of the examples constructed in [1].

Example 1. Take $f : X \to C$ as in the statement of [1], Theorem 1.7. Hence $f_*(\mathcal{O}_X) \cong \mathcal{O}_C \oplus A^{\oplus(k-1)}$ with $A \in \mathrm{Pic}(C)$ such that $\deg(A) < 0$. Take M strictly t-normal such that the multiplication map $H^0(C, M^{\otimes a} \otimes A) \otimes H^0(C, M) \to H^0(C, M^{\otimes(a+1)} \otimes A)$ is surjective for all $0 \le a \le t-1$.

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