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DEGREES OF STABILITY OF COHERENT SYSTEMS AND HOLOMORPHIC TRIPLES ON CURVES

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Abstract: Here we introduce the degrees of α-stability of coherent systems and holomorphic triples. For a rank n vector bundle E on $\mathbf{P}^1$ we find all coherent systems (E, V), $\dim(V) = k < n$ which are "maximally α-unstable" in rank one.

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1. Degrees of Stability

Let X be a smooth and connected projective curve of genus g. A holomorphic triple on X is a triple $T = (E_1, E_2, \phi)$ such that E_1 and E_2 are vector bundles on X and $\phi : E_2 \to E_1$. A subtriple of T is a triple (E_1', E_2', ϕ') on X such that there are inclusions (as sheaves) $i : E_2' \to E_2$, $j : E_1' \to E_1$ and $\phi \circ i = j \circ \phi'$. If we drop the assumption that i and j are injective, we get the definition of morphism between two triples. Set $\mathrm{rank}(T) := \mathrm{rank}(E_1) + \mathrm{rank}(E_2)$. Fix any $\sigma \in \mathbb{R}$. Define the σ-degree of T by the formula $\deg_\sigma(T) := \deg(E) + \deg(F) + \sigma\mathrm{rank}(E)$ and the σ-slope of T by the formula $\mu_\sigma(T) := \deg_\sigma(T)/\mathrm{rank}(T)$. The σ-slope allows us to define the notions of σ-semistable and σ-stable triples on

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X ([3], Definition 3.3). For the theory of holomorphic triples, see [2], [3] and [6]). In this way the usual properties of stable and semistable vector bundles are true for stable and semistable triples (e.g. a stable triples are simple, i.e. their only endomorphisms are induced by the multiplication by a scalar ([6], Lemma 2.1 and Corollary 2.3). Furthermore, in the category of holomorphic triples on X with respect to the σ-slope has Harder-Narasimhan filtrations and Hölder-Schreier filtrations of σ-semistables objects. For the construction of moduli spaces of σ-stable holomorphic triples on X, see [3], [1], [14] or [15]. A coherent system on X is a pair (E, V) such that E is a vector bundle on X and $V \subseteq H^0(X, E)$ is a linear subspace. The pair (E, V) is of type (n, d, k) if $\mathrm{rank}(E) = n$, $\deg(E) = d$ and $\dim(V) = k$. Hence (E, V) is uniquely determined by E and by a map of $\mathcal{O}_X$-sheaves $\mathcal{O}_X{}^{\oplus k} \to E$ which induces an injection of global sections. Fix $\alpha \in \mathbb{R}$. Let $\mu(E) := d/n$ denote the slope of E. Set $\mu_\alpha(E, V) := \mu(E) + \alpha k/n$. The real number μ_α is called the α-slope of the pair (E, V). A coherent subsystem $(F, W) \subseteq (E, V)$ is a coherent system such that $F \subseteq E$ and $W \subseteq V \cap H^0(X, F)$. The pair (E, V) is said to be α-stable (resp. α-semistable) if $\mu_\alpha(F, W) < \mu_\alpha(E, V)$ (resp. $\mu_\alpha(F, W) \le \mu_\alpha(E, V)$) for all proper coherent subsystems (F, W) of (E, V). For the general theory of coherent systems and several results on the moduli schemes of α-stable coherent systems, see [13], [7], [4], [11], [12] and [5]. In this paper we extend to coherent systems and holomorphic triples the notion of "degree of stability" introduced and studied for vector bundles in [8] (see also [9] for a summary later results).

Definition 1. Fix $\alpha \in \mathbb{R}$, $\alpha \ge 0$. Let (E, V) be a coherent system of type (n, d, k) on X. For all integer t such that $0 < t < n$ let $\mathfrak{G}(E, V; t)$ denote the set of all coherent subsystem (F, W) of (E, V) such that $\mathrm{rank}(F) = t$. Set

$$\delta_\alpha(E, V; t) = \sup_{(F,W) \in \mathfrak{G}(E,V;t)} t(d + k\alpha) - n(\deg(F) + \dim(W) \cdot \alpha).$$

We will say that $\delta_\alpha(E, V; t)$ is the t-degree of α-stability of (E, V).

Definition 2. Fix $\alpha \in \mathbb{R}$, $\alpha \ge 0$. Let (E_1, E_2, ϕ) be a holomorphic triples on X. Set $r_i := \mathrm{rank}(E_i)$ and $d_i := \deg(E_i)$, $i = 1, 2$. For all integers s_1, s_2 such that $0 \le s_1 \le r_1$, $0 \le s_2 \le r_2$, $(s_1, s_2) \ne (0, 0)$ and $(s_1, s_2) \ne (r_1, r_2)$ let $\mathfrak{G}(E_1, E_2, \phi; s_1, s_2)$ denote the set of all holomorphic subtriples $(F_1, F_2, \phi|F_2)$ of (E_1, E_2, ϕ) such that $\mathrm{rank}(F_i) = s_i$, $i := 1, 2$. Set

$$\begin{aligned} &\delta_\alpha(E_1, E_2, \phi; s_1, s_2) \\ := &\sup_{F_1,F_2,\phi|F_2) \in \mathfrak{G}(E_1,E_2,\phi;s_1,s_2)} s_2(d_1 + d_2 + r_2\alpha) - r_2(\deg(F_1) + \deg(F_2) + s_2\alpha. \end{aligned}$$

We will say that $\delta_\alpha(E_1, E_2, \phi; s_1, s_2)$ is the (s_1, s_2)-degree of α-stability of (E_1, E_2, ϕ).

Let E, F be rank r vector bundles on X such that there is an inclusion $j : E \to F$. Set $\delta(F, E) := \mathrm{length}(F/j(E))$. Thus $\delta(F, E) = \deg(F) - \deg(E)$. We will say that F is obtained from E making $\delta(F, E)$ positive elementary transformations. Fix an integer $\delta \geq 0$. It is well-known that the set of all isomorphism classes of vector bundles on X obtained from E making δ positive elementary transformations is parametrized (perhaps not one-to-one or not even generically finite-to-one) by an irreducible variety of dimension $r\delta$. For any coherent system (E, V) of type (r, d, k) the inclusion $j : E \to F$ allows us to see V as a linear subspace of $H^0(X, F)$. We will say that the corresponding coherent system (F, V) of type $(r, d + \delta(F, E), k)$ is induced from (E, V) by the inclusion j. For any holomorphic triple (E, E_1, ϕ) the holomorphic triple $(F, E_2, j \circ \phi)$ will be said to be induced from (E, E_1, ϕ) by the inclusion j.

Proposition 1. *Fix $\alpha \geq 0$. Let E, F be rank r vector bundles on X such that there is an inclusion $j : E \to F$. Let (E, V) (resp. (E, E_2, ϕ)) be a coherent system (resp. holomorphic triple) and (F, V) (resp. $(F, E_2, j \circ \phi)$) the coherent system (resp. holomorphic triple) induced by j. Set $d := \deg(E)$, $r_2 := \mathrm{rank}(E_2)$ and $k := \dim(V)$. Fix integers t, s_1, s_2 such that $0 < t < r$, $0 \leq s_1 \leq r$, $0 \leq s_2 \leq r_2$, $(s_1, s_2) \neq (0, 0)$ and $(s_1, s_2) \neq (r, s_2)$. Then $\delta_\alpha(F, V, t) \leq \delta_\alpha(E, V, t) + \delta(F, E)$ and $\delta_\alpha(F, E_2, j \circ \phi; s_1, s_2) \leq \delta_\alpha(E_1, E_2, \phi; s_1, s_2) + \delta(F, E)$. We have $\delta_\alpha(F, E_2, j \circ \phi; 0, s_2) = \delta_\alpha(E, E_2, \phi; 0, s_2)$. For all (fixed) (E, V), t and $\delta(F, E)$ there is an inclusion $j : E \to F$ such that $\mathrm{length}(F/j(E))$ is the prescribed integer $\delta(F, E)$ and $\delta_\alpha(F, V, t) = \delta_\alpha(E, V, t) + \delta(F, E)$. For all holomorphic triples (E, E_2, ϕ) and integers $s_1, s_2, \delta(F, E)$ as above with the additional restriction $s_1 \neq 0$ there is an inclusion $j : E \to F$ such that $\mathrm{length}(F/j(E))$ is the prescribed integer $\delta(F, E)$ and $\delta_\alpha(F, E_2, j \circ \phi; s_1, s_2) = \delta_\alpha(E_1, E_2, \phi; s_1, s_2) + \delta(F, E)$.*

Proof. Take (A, W) (resp. (A, A_2)) computing $\delta_\alpha(F, V, t)$ (resp. $\delta_\alpha(F, E_2, j \circ \phi; s_1, s_2)$). If $s_1 = 0$, then $A = 0$. In all other cases the sheaf $A' := A \cap E$ has the same rank as A and $0 \leq \delta(A, A') \leq \delta(F, A)$. Hence the first part follows. Now we check the last assertion. By the maximality of the α-slope of $(A, V \cap H^0(X, A)$ or of $(A_1, A_2, \phi|A_2)$ we see the A is saturated in F. Hence A' is saturated in E. For the last assertion use that for fixed E, A, A' with A' saturated in E and $A' \neq 0$ there is an inclusion $j : E \to F$ such that $\mathrm{rank}(F) = \mathrm{rank}(E)$, $A' := j^{-1}(j(E) \cap A)$ and $\delta(F, E) = \delta(A, A')$. □

It is easy to prove the following result which, however, may be applied in a huge number of cases; notice that the assumption $\deg(G) \leq (t - w)\alpha$ for all $w \in \{1, \ldots, t\}$ and all $G \in \mathfrak{G}(F, 0; w)$ is always satisfied when $\alpha \gg 0$.

Theorem 1. *Fix integers $n > k \geq t > 0$, $\alpha \in \mathbb{R}$ such that $\alpha > 0$ and a vector bundle E on X such that $\deg(F) = d$ and $\mathrm{rank}(F) = n - k$. Assume $\deg(G) \leq (t - w)\alpha$ for all $w \in \{1, \dots, t\}$ and all $G \in \mathfrak{G}(F, 0; w)$. Let E be any extension of F by $\mathcal{O}_X^{\oplus k}$. This extension induces a coherent system (E, V) of type (n, d, k). Then $\delta_\alpha(E, V; t) = t(d + (k - n)\alpha)$ and any $(G, W) \in \mathfrak{G}(E, V; t)$ such that $t(d + k\alpha) - n(\deg(G) + \dim(W) \cdot \alpha) = t(d + (k - n)\alpha)$ is a rank t trivial factor of the subbundle $\mathcal{O}_X^{\oplus k}$ of E given by the extension defining E.*

Now we may give a classification of all "maximally unstable in rank one" coherent systems (E, V) on $\mathbf{P}^1$ for any fixed vector bundle E on $\mathbf{P}^1$. Its proof is a dimensional count whose details are left to the interested reader.

Theorem 2. *Fix $\alpha \in \mathbb{R}$, $\alpha > 0$ and integers $n > k > 0$ and $a_1 \geq \cdots \geq a_n$ such that $a_1 \geq k - 1$. Let e be the maximal integer $i \in \{1, \dots, n\}$ such that $a_i = a_1$. Fix an integers j such that $1 \leq j \leq e$. Set $E := \oplus_{i=1}^n \mathcal{O}_{\mathbf{P}^1}(a_i)$. For any k-dimensional linear subspace $V \subseteq H^0(\mathbf{P}^1, E)$ we have $\delta_\alpha(E, V; j) \leq j(a_1 + \cdots + a_n) - nja_1 - (n - j)\alpha$, i.e. $\mu_\alpha(F, W) \leq a_1/j + (k/j)\alpha$ for all coherent subsystem (F, W) of (E, V) such that $\mathrm{rank}(F) = j$. Let $B(E; j)$ denote the set all k-dimensional linear subspaces V of $H^0(\mathbf{P}^1, E)$ such that $\delta_\alpha(E, V; j) = j(a_1 + \cdots + a_n) - nja_1 - (n - j)\alpha$. Then $B(E; j) \neq \emptyset$, $B(E; j)$ is an irreducible closed algebraic subset of the Grassmannian $G(k; H^0(\mathbf{P}^1, E))$ of all k-dimensional linear subspaces of $H^0(\mathbf{P}^1, E)$, and $\dim(B(E; j)) = j(e - j) + k(ea_1 + e - k)$. Every $V \in B(E; j)$ arises in the following way (and the converse is true and easier). Take any inclusion $j : \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus j} \to E$, any k-dimensional linear subspace M of $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus j})$ and set $V := j_*(M)$. Take any $V \subseteq H^0(\mathbf{P}^1, E)$ such that $\dim(V) = k$ and $(F, V) \notin B(E; j)$.*

(i) Assume $\alpha > 1$ and $\delta_\alpha(F, V) \geq j(a_1 + \cdots + a_n) - nja_1 - (n - j)\alpha - n$. Then $\delta_\alpha(F, V) = j(a_1 + \cdots + a_n) - nja_1 - (n - j)\alpha - n$ and V is constructed in the following way; take a rank j saturated subbundle of E with degree $ea_1 - 1$ and take as V any k-dimensional linear subspace of $H^0(\mathbf{P}^1, F)$. If either $e = n$ or $a_{e+1} \leq a_1 - 2$, then the set of all such V's is an irreducible closed algebraic subset of the Grassmannian $G(k; H^0(\mathbf{P}^1, E))$ of all k-dimensional linear subspaces of $H^0(\mathbf{P}^1, E)$ with dimension $e + k(ja_1 + j - k - 1)$.

(ii) Assume $\alpha < 1$ and $\delta_\alpha(F, V) \geq j(a_1 + \cdots + a_n) - nja_1 - (n - j - 1)\alpha$. Then $\delta_\alpha(F, V) = j(a_1 + \cdots + a_n) - nja_1 - (n - j - 1)\alpha$ and V is constructed in the following way (and the converse is true). Take any inclusion $j : \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus j} \to E$, any $(k-1)$-dimensional linear subspace M of $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus j})$ and set $W := j_*(M)$. Take as V any k-dimensional linear subspace of $H^0(\mathbf{P}^1, E)$ containing W, but not contained in $j_*(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus j}))$. The set of all such V's is an irreducible algebraic subset of $G(k; H^0(\mathbf{P}^1, E))$ of dimension $j(e - j) + (k -$

$1)(ea_1 + e - k) - k + \sum_{i=1}^{n} \max\{0, a_i + 1\}$.

(iii) Assume $\alpha = 1$ *and* $\delta_\alpha(F, V) \geq j(a_1 + \cdots + a_n) - nja_1 - (n - j - 1)$. *Then* $\delta_\alpha(F, V) = j(a_1 + \cdots + a_n) - nja_1 - (n - j - 1)$ *and* V *arieses in one of the two ways described in (i) and (ii).*

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