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NON-EXISTENCE OF SYMMETRIC WARPED PRODUCTS WITH COMPACT BASE

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Abstract: We investigate obstructions to the representation of symmetric spaces as non-trivial warped product. It is shown that a locally symmetric space cannot be written as (non-trivial) warped product with compact base.

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1. Introduction

Warped product is a construction in the class of Riemannian manifolds that generalizes direct product. This construction was introduced in [4] where it was used to construct a variety of complete Riemannian manifolds with negative sectional curvature. Warped products have significant applications, in general relativity, in the studies related to solutions of Einstein's equations [1, 2]. Besides general relativity, warped product structures have also generated interest in many areas of geometry, especially due to their role in construction of new examples with interesting curvature and symmetry properties cf. [3, 5, 6, 7].

Definition 1.1. Let (B, g_B) and (F, g_F) be Riemannian manifolds with $f : B \to (0, \infty)$ a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ equipped with the metric

$$g = \pi^*(g_B) \oplus (f \circ \pi)^2 \sigma^*(g_F),$$

where $\pi : B \times F \to B$, $\sigma : B \times F \to F$ are usual projections and $*$ denotes

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pullback. (B, g_B) is called the base, (F, g_F) is called the fiber and f the warping function of the warped product.

If the warping function 'f' is constant then the warped product $B \times_f F$ (up to a change of scale) is a (global) Riemannian product, which we call as trivial warped product.

Because of their construction, warped products exhibit nice curvature and symmetry properties and hence provide a rich class of examples of practical interest in Riemannian as well as semi-Riemannian geometry. Yet there are no known examples of representation of compact symmetric spaces as non-trivial warped product. On the other hand, a warped product being a topological product would naturally have restrictions imposed on their existence. The purpose of this paper is to investigate obstructions to the representation of symmetric spaces as (non-trivial) warped product.

The reader is referred to [3, 7] for the fundamental results and properties of warped products.

2. Restrictions on Symmetric Warped Products

The restrictions on the existence of warped product metrics on symmetric spaces are obtained using the fundamental curvature relations of warped products [7].

Proposition 2.1. *Let $M^m = B \times_f F$ be a warped product of an (m-n)-dimensional Riemannian manifold B and an n-dimensional Riemannian manifold F. Then*

$$-\frac{n}{f}\Delta^B f = \sum_{i=n+1}^{m} \sum_{a=1}^{n} g_M(\mathrm{R}^M(e_a, e_i)e_a, e_i) \tag{2.1}$$

where Δ^B is the Laplacian on B and $(e_a)_{a=1}^n$, $(e_i)_{i=n+1}^m$ are local orthonormal bases for TF, TB respectively.

Proof. From the curvature identity ([7], p. 211), relating the Ricci curvature of M and B, we have

$$-\frac{n}{f}\Delta^B f = \sum_{i=n+1}^{m} \mathrm{Ric}^M(e_i, e_i) - \sum_{i=n+1}^{m} \mathrm{Ric}^B(e_i, e_i).$$

Since, for any $q \in F$, the leaves $B \times q$ are totally geodesic we can write

$$\sum_{i=n+1}^{m} \mathrm{Ric}^M(e_i, e_i) = \sum_{i=n+1}^{m} \sum_{a=1}^{n} g_M(\mathrm{R}^M(e_a, e_i)e_a, e_i) + \sum_{i=n+1}^{m} \mathrm{Ric}^B(e_i, e_i)$$

which completes the proof. □

As a consequence we have the following theorem.

Theorem 2.2. *Let $M^m = B \times_f F$ be a warped product either with non-negative sectional curvature or non-positive sectional curvature. If B is compact then the warping function f is constant.*

Proof. From the hypothesis and Proposition 2.1, f is a subharmonic or superharmonic function on B. Since B is compact, f must be constant. □

The above theorem can be applied to obtain restrictions on the representation of compact symmetric spaces as warped products.

Corollary 2.3. *(i) A locally symmetric space of compact type cannot be written as non-trivial warped product.*

(ii) A compact locally symmetric space of non-compact type cannot be written as non-trivial warped product.

Proof. The result follows from the fact that Riemannian symmetric spaces of compact type have non-negative sectional curvature whereas compact locally symmetric spaces of non-compact type have non-positive sectional curvature. □

For non-compact symmetric spaces M, the compactness of the factor B obstructs the construction of warped product $M^m = B \times_f F$.

Corollary 2.4. *An irreducible Riemannian symmetric space of non-compact type can not be written as a warped product $B \times_f F$, if B is compact.*

Proof. Irreducible Riemannian symmetric space of non-compact type have non-positive sectional curvature. The proof then follows directly from Theorem 2.2. □

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References

[1] J.K. Beem, P. Ehrlich, T.G. Powell, *Warped Product Manifolds in Relativity*, Selected Studies: Physics-Astrophysics, Mathematics, History of Science, North Holland, Amsterdam-New York (1982), 41-66.

[2] J.K. Beem, P. Ehrlich, K. Easley, *Global Lorentzian Geometry*, Marcel Dekker Inc., Second Edition, New York (1996).

[3] A.L. Besse, *Einstein Manifolds*, Springer-Verlag (1987).

[4] R.L. Bishop, B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969) 1-49.

[5] F. Dobarro, Dozo E. Lami, Scalar curvatures and warped products of Riemannian geometry, *Trans. Amer. Math. Soc.*, **303** (1987), 161-168.

[6] G. Ganchev, V. Mihoa, Riemannian submanifolds of quasi-constant sectional curvatures, *J. Reine Angew. Math.*, **522** (2000), 119-141.

[7] B. O'Neill, *Semi-Riemannian Geometry*, Academic Publishers (1983).