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COHERENT SYSTEMS ON REAL LOW GENUS CURVES

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Abstract: Here we study the stability of coherent systems defined over $\mathbb{R}$ on genus 0 and genus 1 smooth real projective curves.

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1. The Main Result

Let K be a non-algebraically closed field and $\mathbb{K}$ an algebraic closure of K. For any algebraic K-scheme Y and any field L such that $K \subseteq L \subseteq \mathbb{K}$ set $Y_L := Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. Let $Y(L)$ or $Y_L(L)$ denote the set of all equivalence classes of L-morphism $\mathrm{Spec}(L) \to Y$ with L as their residue field. For any vector bundle E on Y, i.e. for any locally free coherent algebraic sheaf E on Y, let E_L (or just E) the induced locally free sheaf on Y. If Y is complete, then $\dim_K(H^0(Y,E)) = \dim_L(H^0(Y_L,E_L))$, because the morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is flat ([7], Proposition III.9.3). Let X be a smooth and geometrically connected projective curve defined over K. A coherent pair on X over K is a pair (E,V) such that E is a vector bundle on X defined over K and $V \subseteq H^0(X,E)$ is a linear subspace. The pair (E,V) is of type (n,d,k) if $\mathrm{rank}(E) = n$, $\deg(E) = d$ and $\dim(V) = k$. Hence (E,V) is uniquely determined by E and by a map of $\mathcal{O}_X$-sheaves ${\mathcal{O}_X}^{\oplus k} \to E$ which induces an injection of

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global sections. Fix $\alpha \in \mathbb{R}$. Let $\mu(E) := d/n$ denote the slope of E. Set $\mu_\alpha(E,V) := \mu(E) + \alpha k/n$. The real number μ_α is called the α-slope of the pair (E,V). A subcoherent system $(F,W) \subseteq (E,V)$ is a coherent system such that $F \subseteq E$ and $W \subseteq V \cap H^0(X,F)$. The pair (E,V) is said to be α-stable (resp. α-semistable) if $\mu_\alpha(F,W) < \mu_\alpha(E,V)$ (resp. $\mu_\alpha(F,W) \le \mu_\alpha(E,V)$) for all subcoherent systems (F,W) of (E,V). Let $G(X;\alpha : n,d,k)_K$ denote the moduli scheme of all α-stable (resp. α-semistable) coherent systems of type (n,d,k) on X. For the general theory of coherent systems and several results on the moduli schemes $G(X;\alpha : n,d,k)_{\mathbb{K}}$, see [12], [8], [4], [9], [10] and [5] (at least in characteristic zero).

Remark 1. Fix $\alpha \in \mathbb{R}$. Let (E,V) a coherent system on X defined over K. As in the vector bundle case done in [12], Proposition 3 and Proposition 4, it is easy to check that (E,V) is α-stable over K if and only if the induced coherent system $(E_{\mathbb{K}}, V_{\mathbb{K}})$ over $\mathbb{K}$ is α-stable over $\mathbb{K}$. Obviously, if $(E_{\mathbb{K}}, V_{\mathbb{K}})$ is α-stable over $\mathbb{K}$, then (E,V) is α-stable over K. If (E,V) is α-stable over K, then $(E_{\mathbb{K}}, V_{\mathbb{K}})$ is α-polystable over $\mathbb{K}$ and the indecomposable factors of $(E_{\mathbb{K}}, V_{\mathbb{K}})$ are permuted transitively by the absolute Galois group of K. Hence all these indecomposable factors are defined and α-stable over a finite extension of K.

Remark 2. Let (E,V) be a coherent system on X defined over K. As in the case $K = \mathbb{K}$ and $\mathrm{char}(\mathbb{K}) = 0$ done in [3], Lemma 3.14, we easily see that the set of all $\alpha \in \mathbb{R}$ such that (E,V) is α-semistable (resp. α-stable) over K is connected (resp. connected and open).

In this note we will study the case $K = \mathbb{R}$ and X of genus 0 or 1.

Remark 3. Let X be a real projective curve and E a real vector bundle on X. Fix $\alpha \in \mathbb{R}$. Since $H^0(X(\mathbb{C}), E_{\mathbb{C}}) = H^0(X,E) \otimes_{\mathbb{R}} \mathbb{C}$, for any integer k such that $0 \le k \le h^0(X,E)$ the set of all complexifications of all k-dimensional real linear subspaces of $H^0(X,E)$ is Zariski dense in the Grassmannian of all k-dimensional complex linear subspaces of $H^0(X,E) \otimes_{\mathbb{R}} \mathbb{C}$. By the opennes of α-stability we get that there is a k-dimensional complex linear subspace W of $H^0(X,E) \otimes_{\mathbb{R}} \mathbb{C}$ such that $(E_{\mathbb{C}}, W)$ is α-stabile if and only if there is a k-dimensional real linear subspace V of $H^0(X,E)$ such that (E,V) is stable as a complex vector bundles and the latter condition implies that (E,V) is stable over $\mathbb{R}$. When $X = \mathbf{P}^1_{\mathbb{R}}$, then all vector bundles on $X_{\mathbb{C}}$ are defined over $\mathbb{R}$ (see e.g. [1], Proposition 3.1). Hence in this case we get the following meta-statement: Every existence or non-existence result for coherent systems on $\mathbf{P}^1_{\mathbb{C}}$ is equivalent to the corresponding statement for $\mathbf{P}^1_{\mathbb{R}}$.

Here we consider the case $K = \mathbb{R}$. We recall that ω_X is always a real line bundle and that its class is the double of an element of $\mathrm{Pic}(X)(\mathbb{R})$ ([6], Corollary 4.3), but that if $X(\mathbb{R}) = \emptyset$, it is not assured the existence of a real line bundle L such that $L^{\oplus 2} \cong \omega_X$. $\mathbb{P}^1$ has two real structures: the usual one with $\mathbb{R}\mathbb{P}^1 \cong S^1$ as its real locus and the one, N, with no real point ([6], p. 170). Let F be a real vector bundle on N which is indecomposable as a real vector bundle. For the classification of all real vector bundles on N, see [1]. We recall that a line bundle on N is real if and only if its degree is odd. Hence we immediately get that either F is a line bundle of even degree or it is a rank two vector bundle which over $\mathbb{C}$ is isomorphic to the direct sum of two line bundles with the same odd degree. Hence any such F is stable over $\mathbb{R}$. For any odd integer a let $A(a)$ denote the rank two indecomposable real vector bundle on N with degree $2a$. By [2], Theorem 2, and Remark 3 we immediately get the following result.

Theorem 1. *Fix positive integers n, a, k such that n is even and $n < k \leq \lfloor 3n/2 \rfloor$. Set $E := A(a)^{\oplus n/2}$. Let $V \subset H^0(N, E)$ be a general linear suspace such that* $\dim(V) = k$. *Then (E, V) is α-stable over $\mathbb{R}$ and over $\mathbb{C}$ for all real $\alpha > 0$.*

Now we study the case $g = 1$. As in the case $g = 0$ we need to distinguish between the case $X(\mathbb{R}) \neq \emptyset$ (the easy case) and the case $X(\mathbb{R}) = \emptyset$ (the "strange" case).

Remark 4. Let Y be a smooth and geometrically connected projective curve defined over $\mathbb{R}$ and A, B vector bundles on X defined over $\mathbb{R}$. Every real extension of A by B is a vector bundle defined over $\mathbb{R}$. Since $H^1(Y(\mathbb{C}), \mathrm{Hom}(B_{\mathbb{C}}, A_{\mathbb{C}})) = H^1(Y, \mathrm{Hom}(B, A)) \otimes_{\mathbb{R}} \mathbb{C}$, $H^1(Y, \mathrm{Hom}(B, A))$ is Zariski dense in the affine variety $H^1(Y(\mathbb{C}), \mathrm{Hom}(B_{\mathbb{C}}, A_{\mathbb{C}}))$. Hence if a general extension of $A_{\mathbb{C}}$ by $B_{\mathbb{C}}$ is indecomposable (resp. stable, resp. polystable, resp. semistable), then a general real extension of A by B is a vector bundle defined over $\mathbb{R}$ which is geometrically indecomposable (resp. stable over $\mathbb{C}$ and hence over $\mathbb{R}$, resp. polystable over $\mathbb{C}$ and hence over $\mathbb{R}$, resp. semistable over $\mathbb{C}$ and over $\mathbb{R}$).

Remark 5. Let X be a smooth and geometrically connected elliptic curve X defined over $\mathbb{R}$ such that $X(\mathbb{R}) \neq \emptyset$. Hence $X(\mathbb{R})$ has one or two connected components, each of them diffeomorphic to S^1. Hence $X(\mathbb{R})$ is infinite. By Remark 4 we see that for all integers $n > 0$ and d there are infinitely many pairwise non-isomorphic (not even over $\mathbb{C}$) polystable vector bundles with rank n and degree d and defined over $\mathbb{R}$.

By Remarks 2, 4 and 5 we may go through the proofs in [9], §5, and get the following result.

Theorem 2. *Let X be a smooth and geometrically connected elliptic curve X defined over $\mathbb{R}$ such that $X(\mathbb{R}) \neq \emptyset$. Take $\alpha \in \mathbb{R}$ and integers $n > 0$, $d \geq k > 0$ such that $d > n$ and either $d \neq k$ or $(k, d) = 1$. There are a rank n vector bundle E on X defined over $\mathbb{R}$ with $\mathrm{rank}(E) = n$ and $\deg(E) = d$ and a k-dimensional linear subspace V of $H^0(X, E)$ such that the coherent system (E, V) is stable over $\mathbb{R}$ if and only if the same is true with (E, V) stable over $\mathbb{C}$ if and only if one of the following conditions is satisfied:*

(i) $0 < k < n$ *and* $0 < \alpha < d/(n-k)$;

(ii) $k \geq n$ *and* $\alpha > 0$.

Furthermore, in all these cases we may take E polystable over $\mathbb{C}$ and with pairwise non-isomorphic factors (over $\mathbb{C}$).

Remark 6. Let X be a smooth and geometrically connected elliptic curve X defined over $\mathbb{R}$ such that $X(\mathbb{R}) = \emptyset$. Such curves do exists: take the normalization of the projective closure of the real affine curve $y^2 = -(x^2 + a)(x^2 + b)$, a, b real and $b > a > 0$. By Riemann-Roch every line bundle of positive degree has a non-identically zero global section. Hence there is no odd degree line bundle (and hence no odd degree vector bundle) on X defined over $\mathbb{R}$. Let $\sigma : X(\mathbb{C}) \to X(\mathbb{C})$ be the complex conjugation. Fix $P \in X(\mathbb{C})$. Since σ has no fixed point, $P + \sigma(P)$ is a reduced degree two effective divisor on X defined over $\mathbb{R}$. Taking instead of P infinitely many distinct points we see that for every even integer d there are infinitely many isomorphism classes of degree d line bundles on X defined over $\mathbb{R}$. $X(\mathbb{C})$ is infinite. By Remark 4 we see that for all integers $n > 0$ and d such that d is even there are infinitely many pairwise non-isomorphic (not even over $\mathbb{C}$) polystable vector bundles with rank n and degree d and defined over $\mathbb{R}$.

Using Remark 6 instead of Remark 5 we also get the following result: By the complex case considered in [9] and the non-existence of odd degree real line bundles done in Remark 6 we also get that the result is sharp.

Theorem 3. *Let X be a smooth and geometrically connected elliptic curve X defined over $\mathbb{R}$ such that $X(\mathbb{R}) = \emptyset$. Take $\alpha \in \mathbb{R}$ and integers $n > 0$, $d \geq k > 0$ such that $d > n$, d is even, and either $d \neq k$ or $(k, d) = 1$. There are a rank n vector bundle on X defined over $\mathbb{R}$ with $\mathrm{rank}(E) = n$ and $\deg(E) = d$ and a k-dimensional linear subspace V of $H^0(X, E)$ such that the coherent system (E, V) is stable over $\mathbb{R}$ if and only if the same is true with (E, V) stable over $\mathbb{C}$ if and only if one of the following conditions is satisfied:*

(i) $0 < k < n$ *and* $0 < \alpha < d/(n-k)$;

(ii) $k \geq n$ *and* $\alpha > 0$.

Furthermore, in all these cases we may take E *polystable over* $\mathbb{C}$ *and with pairwise non-isomorphic factors (over* $\mathbb{C}$*).*

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