International Journal of Pure and Applied Mathematics
Volume 28 No. 1 2006, 35-54

ASYMPTOTIC PROPERTY OF THE SOLUTION OF A REPAIRABLE, STANDBY, HUMAN AND MACHINE SYSTEM

Geni Gupur
College of Mathematics and System Science
Xinjiang University
Urumqi, 830046, P.R. CHINA
e-mails: geni@xju.edu.cn, genigupur@263.net

Abstract: In this paper, we study asymptotic property of the solution of a repairable, standby, human and machine system. First we consider the expression of the solution of a simple system derived from the original system, thus give the expression of the C_0-semigroup corresponding to the simple system. Second we prove this C_0-semigroup is a quasi-compact operator. Third by using perturbation theory we obtain that the C_0-semigroup corresponding to the original system is a quasi-compact operator. Fourth we prove that 0 is an eigenvalue of the operator corresponding to the original system with algebraic multiplicity one. Last by using all the above results we deduce that the time-dependent solution of the original system converges strongly to the static solution as time tends to infinite.

AMS Subject Classification: 47A10, 47D99
Key Words: C_0-semigroup, quasi-compact operator, eigenvalue

1. Introduction

According to [5], a repairable, standby, human and machine system can be described by the following system of equations:

Received: February 24, 2005

$$\frac{dp_0(t)}{dt} = -(\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta)p_0(t) + \sum_{i=1}^{2} \mu_i p_i(t) + \sum_{i=3}^{5} \int_0^\infty \mu_i(x) p_i(x,t)dx, \quad (1)$$

$$\frac{dp_1(t)}{dt} = \lambda p_0(t) - (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)p_1(t), \quad (2)$$

$$\frac{dp_2(t)}{dt} = \eta p_0(t) - (\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)p_2(t), \quad (3)$$

$$\frac{\partial p_i(x,t)}{\partial t} + \frac{\partial p_i(x,t)}{\partial x} = -\mu_i(x)p_i(x,t), \quad i = 3,4,5, \quad (4)$$

$$p_3(0,t) = \lambda[p_1(t) + p_2(t)], \quad t > 0, \quad (5)$$

$$p_4(0,t) = \sum_{i=0}^{2} \lambda_{c_i} p_i(t), \quad t > 0, \quad (6)$$

$$p_5(0,t) = \sum_{i=0}^{2} \lambda_{h_i} p_i(t), \quad t > 0, \quad (7)$$

$$p_0(0) = 1,\ p_i(0) = 0,\ i = 1,2;\ p_j(x,0) = 0,\ j = 3,4,5. \quad (8)$$

Here $(x,t) \in [0,\infty) \times [0,\infty)$, $p_i(t)$ represents the probability that the system is in state i at time t $(i = 0,1,2)$; $p_j(x,t)$ represents the probability that at time t, the failed system is in state j $(j = 3,4,5)$ and has an elapsed repair time of x; λ_{c_i} represents common-cause failure rates from state i to state 4, $i = 0,1,2$; λ_{h_i} represent human-error rates from state i $(i = 0,1,2)$ to state 5; η represents hardware failure rate for standby unit; λ represents hardware failure rate for operating unit; μ_i represents repair rate of failed unit in state i ($i = 1,2$); $\mu_j(x)$ represents time-dependent system repair rate when system is in state j and has an elapsed repair time of x which satisfies

$$\mu_j(x) \geq 0, \quad \int_0^\infty \mu_j(x)dx = \infty, \quad j = 3,4,5.$$

λ_{c_i} $(i = 0,1,2)$, λ_{h_i} $(i = 0,1,2)$, λ, μ_i $(i = 1,2)$ and η are positive constants.

In [5], the author established this model by using supplementary variable technique and studied the time-dependent availability of the system by using Laplace transform, and discovered the time-dependent availability decreases as time increases for exponential repair time distribution. The availability of the system depends on the solution of the model. The author used the steady-state solution and the time-dependent solution during calculating the system availability. But the author did not discuss the time-dependent solution of the system and its asymptotic behavior. In [1], the author firstly converted the model into an abstract Cauchy problem in a Banach space, then by using the Hille-Yosida Theorem, the Phillips Theorem and the Fattorini Theorem obtained the existence and uniqueness of positive time-dependent solution of the model for general repair time distribution. In [6], the author obtained asymptotic behavior of the solution of the model by using the relation between resolvent set of the operator corresponding to the model and resolvent set of its adjoint operator for exponential repair time distribution, that is, $\mu_j(x) = \mu, (j = 3, 4, 5)$. In this paper, we consider asymptotic behavior of the time-dependent solution of the model for general repair time distribution. Firstly, we consider expression of the solution of a simple system derived from the system (1)-(8), that is, we give expression of C_0-semigroup generated by the operator corresponding to the simple system. Secondly we will prove the C_0-semigroup is a quasi-compact operator. Thirdly by using perturbation theory we will show that C_0-semigroup generated by the operator corresponding to (1)-(8) is a quasi-compact operator. Fourthly we will prove that 0 is an eigenvalue of the operator corresponding to the system (1)-(8) with algebraic multiplicity one. Lastly, by summing up all the above results we will prove that the time-dependent solution of the system (1)-(8) converges strongly to the steady-state solution of the system (1)-(8).

For simplicity, we introduce a notation as follows.

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda & \lambda & 0 & 0 & 0 \\ \lambda_{c_0} & \lambda_{c_1} & \lambda_{c_2} & 0 & 0 & 0 \\ \lambda_{h_0} & \lambda_{h_1} & \lambda_{h_2} & 0 & 0 & 0 \end{pmatrix}.$$

Take state space X as follows.

$$X = \left\{ p \in R \times R \times R \times L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \;\middle|\; \|p\| = \sum_{i=0}^{2} |p_i| + \sum_{i=3}^{5} \|p_i\|_{L^1[0,\infty)} \right\}.$$

It is obvious that X is a Banach space. Define operator A and its domain as follows:

$$D(A) = \left\{ p \in X \;\middle|\; \begin{array}{l} \frac{dp_i(x)}{dx} \in L^1[0,\infty),\ p_i(x) \text{ is absolutely continuous} \\ (i = 3, 4, 5) \text{ and } p(0) = \Gamma p(x) \end{array} \right\},$$

$$A\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} =$$

$$\begin{pmatrix} -(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta) & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{dx}-\mu_3(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{d}{dx}-\mu_4(x) & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{d}{dx}-\mu_5(x) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix}.$$

For $\forall p \in X$ we define B and E as follows:

$$B\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix},$$

$$E\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix} = \begin{pmatrix} \sum_{i=3}^{5}\int_0^\infty \mu_i(x)p_i(x)dx \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \mu_1 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3(x) \\ p_4(x) \\ p_5(x) \end{pmatrix}.$$

Then the system of equations (1)–(8) can be written as an abstract Cauchy problem in the Banach space X

$$\frac{dp(t)}{dt} = (A+B+E)p(t), \quad t \in [0,\infty), \tag{9}$$

$$p(0) = (1,0,0,0,0,0). \tag{10}$$

In [1], the author obtained the following result.

Proposition 1. *$A+B$ generates a positive contraction C_0-semigroup $S(t)$.*

Theorem 1. *$A+B+E$ generates a positive contraction C_0-semigroup $T(t)$.*

2. Main Results

Lemma 1. *If $p(x,t)=S(t)\phi(x)$ is the solution of the following system*

$$\frac{dp(t)}{dt}=(A+B)p(t), \quad t\in[0,\infty), \tag{11}$$

$$p(0)=\phi(x), \tag{12}$$

then

$$p(x,t)=$$

$$\begin{cases} \begin{pmatrix} \phi_0 e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t} \\ \phi_1 e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}+\frac{\lambda\phi_0[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t}-e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}]}{\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta} \\ \phi_2 e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t}+\frac{\eta\phi_0[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t}-e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t}]}{\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta} \\ p_3(0,t-x)e^{-\int_0^x\mu_3(\tau)d\tau} \\ p_4(0,t-x)e^{-\int_0^x\mu_4(\tau)d\tau} \\ p_5(0,t-x)e^{-\int_0^x\mu_5(\tau)d\tau} \end{pmatrix} & \text{as } x<t, \\ \begin{pmatrix} \phi_0 e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t} \\ \phi_1 e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}+\frac{\lambda\phi_0[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t}-e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}]}{\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta} \\ \phi_2 e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t}+\frac{\eta\phi_0[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t}-e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t}]}{\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta} \\ \phi_3(x-t)e^{-\int_{x-t}^x\mu_3(\tau)d\tau} \\ \phi_4(x-t)e^{-\int_{x-t}^x\mu_4(\tau)d\tau} \\ \phi_5(x-t)e^{-\int_{x-t}^x\mu_5(\tau)d\tau} \end{pmatrix} & \text{as } x>t, \end{cases}$$

where $p_3(0,t-x),p_4(0,t-x),p_5(0,t-x)$ are given by (5), (6), (7).

Proof. Since $p(x,t)=S(t)\phi(x)$ is a solution of the system (11)–(12), $p(x,t)$ satisfies

$$\frac{dp_0(t)}{dt}=-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)p_0(t), \tag{13}$$

$$\frac{dp_1(t)}{dt} = \lambda p_0(t) - (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)p_1(t), \tag{14}$$

$$\frac{dp_2(t)}{dt} = \eta p_0(t) - (\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)p_2(t), \tag{15}$$

$$\frac{\partial p_i(x,t)}{\partial t} + \frac{\partial p_i(x,t)}{\partial x} = -\mu_i(x)p_i(x,t), \quad i = 3,4,5, \tag{16}$$

$$p_3(0,t) = \lambda[p_1(t) + p_2(t)], \tag{17}$$

$$p_4(0,t) = \lambda_{c_0}p_0(t) + \lambda_{c_1}p_1(t) + \lambda_{c_2}p_2(t), \tag{18}$$

$$p_5(0,t) = \lambda_{h_0}p_0(t) + \lambda_{h_1}p_1(t) + \lambda_{h_2}p_2(t), \tag{19}$$

$$p_i(0) = \phi_i, \quad i = 0,1,2, \; p_j(x,0) = \phi_j(x), \quad j = 3,4,5. \tag{20}$$

If define $\xi = x - t$ and $Q_i(t) = p_i(\xi + t, t), \; i = 3,4,5$, then from (16) we deduce

$$\frac{dQ_i(t)}{dt} = -\mu_i(\xi + t)Q_i(t), \quad i = 3,4,5. \tag{21}$$

If $\xi < 0$ (i.e., $x < t$), then by integrating (21) from $-\xi$ to t and using $Q_i(-\xi) = p_i(0,-\xi) = p_i(0,t-x), \; i = 3,4,5$ we obtain

$$\begin{aligned} p_i(x,t) &= Q_i(t) = Q_i(-\xi)e^{-\int_{-\xi}^{t}\mu_i(\xi+\tau)d\tau} \\ &= p_i(0,t-x)e^{-\int_0^{\xi+t}\mu_i(\tau)d\tau} = p_i(0,t-x)e^{-\int_0^{x}\mu_i(\tau)d\tau}, \quad i = 3,4,5\,.. \end{aligned} \tag{22}$$

By solving (13)–(15) we have

$$p_0(t) = \phi_0 e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t}, \tag{23}$$

$$\begin{aligned} p_1(t) &= \phi_1 e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t} \\ &\quad + e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}\int_0^t \lambda p_0(\tau)e^{(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)\tau}d\tau = \phi_1 e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t} \\ &\qquad + e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}\int_0^t \lambda\phi_0 e^{(\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta)\tau}d\tau \\ &= \phi_1 e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t} + \frac{\lambda\phi_0\left[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}\right]}{\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta}, \end{aligned} \tag{24}$$

$$\begin{aligned} p_2(t) &= \phi_2 e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t} \\ &\qquad + e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t}\int_0^t \eta p_0(\tau)e^{(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)\tau}d\tau \end{aligned}$$

$$= \phi_2 e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t} + \frac{\eta\phi_0 \left[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t}\right]}{\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta}. \quad (25)$$

If $\xi > 0$ (i.e., $x > t$), then by integrating (21) from 0 to t and using $Q_i(0) = p_i(\xi, 0) = \phi_i(\xi) = \phi_i(x-t),\ i = 3,4,5$ we derive

$$\begin{aligned} p_i(x,t) = Q_i(t) &= Q_i(0)e^{-\int_0^t \mu_i(\xi+\tau)d\tau} = \phi_i(x-t)e^{-\int_0^t \mu_i(\xi+\tau)d\tau} \\ &= \phi_i(x-t)e^{-\int_0^t \mu_i(x-t+\tau)d\tau} = \phi_i(x-t)e^{-\int_{x-t}^x \mu_i(\tau)d\tau}, \quad i = 3,4,5. \quad (26) \end{aligned}$$

By combining (22), (23), (24) and (25) with (26) we know that the result of this lemma is right. □

For $\phi \in X$ we define two operators as follows.

$$(U(t)\phi)(x) = \begin{cases} 0 & x \in [0,t)\,, \\ (S(t)\phi)(x) & x \in [t,\infty)\,, \end{cases} \quad (27)$$

$$(V(t)\phi)(x) = \begin{cases} (S(t)\phi)(x) & x \in [0,t)\,, \\ 0 & x \in [t,\infty)\,. \end{cases} \quad (28)$$

Then $S(t)\phi = U(t)\phi + V(t)\phi, \quad \phi \in X$.

In [2], the author proved the following result.

Theorem 2. *A bounded and closed subset Y of $X = R \times R \times R \times L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty)$ is compact if and only if the following two conditions hold simultaneously.*

(1) $\sum_{i=3}^{5} \lim_{h\to 0} \int_0^\infty |\phi_i(x+h) - \phi_i(x)|dx = 0$, *uniformly for* $\phi = (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in Y$.

(2) $\sum_{i=3}^{5} \lim_{h\to\infty} \int_h^\infty |\phi_i(x)|dx = 0$, *uniformly for* $\phi = (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in Y$.

Theorem 3. *$V(t)$ is a compact operator in X.*

Proof. From the definition of $V(t)$ and Theorem 2 we know that it is sufficient to prove the condition (1) in Theorem 2. By Lemma 1 we have

$$\begin{aligned} &\sum_{i=3}^{5} \int_0^t |p_i(x+h,t) - p_i(x,t)|dx \\ &\quad = \sum_{i=3}^{5} \int_0^t \left| p_i(0,t-x-h)e^{-\int_0^{x+h} \mu_i(\tau)d\tau} - p_i(0,t-x)e^{-\int_0^x \mu_i(\tau)d\tau} \right| dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=3}^{5} \int_0^t \Big| p_i(0, t-x-h) e^{-\int_0^{x+h} \mu_i(\tau) d\tau} - p_i(0, t-x-h) e^{-\int_0^{x} \mu_i(\tau) d\tau} \\
&\quad + p_i(0, t-x-h) e^{-\int_0^{x} \mu_i(\tau) d\tau} - p_i(0, t-x) e^{-\int_0^{x} \mu_i(\tau) d\tau} \Big| dx \\
&\le \sum_{i=3}^{5} \int_0^t |p_i(0, t-x-h)| \left| e^{-\int_0^{x+h} \mu_i(\tau) d\tau} - e^{-\int_0^{x} \mu_i(\tau) d\tau} \right| dx \\
&\quad + \sum_{i=3}^{5} \int_0^t |p_i(0, t-x-h) - p_i(0, t-x)| e^{-\int_0^{x} \mu_i(\tau) d\tau}. \qquad (29)
\end{aligned}
$$

In the following we will estimate the first term in (29). By noting (17) and using Lemma 1 we have, for $t - x - h \ge 0$,

$$
\begin{aligned}
&|p_3(0, t-x-h)| = |\lambda(p_1(t-x-h) + p_2(t-x-h))| \\
&\le \lambda(|p_1(t-x-h)| + |p_2(t-x-h)|) \le \lambda |\phi_1| e^{-(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)(t-x-h)} \\
&\quad + \frac{\lambda^2 |\phi_0|}{|\mu_1 + \lambda_{c_1} + \lambda_{h_1} - \lambda_{c_0} - \lambda_{h_0} - \eta|} \\
&\quad \times \left[e^{-(\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta)(t-x-h)} + e^{-(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)(t-x-h)} \right] \\
&\quad + \lambda |\phi_2| e^{-(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)(t-x-h)} \\
&\quad + \frac{\lambda \eta |\phi_0|}{|\mu_2 + \lambda_{c_2} + \lambda_{h_2} - \lambda_{c_0} - \lambda_{h_0} - \eta|} \\
&\quad \times \left[e^{-(\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta)(t-x-h)} + e^{-(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)(t-x-h)} \right] \\
&\le \lambda |\phi_1| + \frac{2\lambda^2}{|\mu_1 + \lambda_{c_1} + \lambda_{h_1} - \lambda_{c_0} - \lambda_{h_0} - \eta|} |\phi_0| \\
&\quad + \lambda |\phi_2| + \frac{2\lambda\eta}{|\mu_2 + \lambda_{c_2} + \lambda_{h_2} - \lambda_{c_0} - \lambda_{h_0} - \eta|} |\phi_0| \\
&\le \Bigg\{ 2\lambda + \frac{2\lambda^2}{|\mu_1 + \lambda_{c_1} + \lambda_{h_1} - \lambda_{c_0} - \lambda_{h_0} - \eta|} \\
&\quad + \frac{2\lambda\eta}{|\mu_2 + \lambda_{c_2} + \lambda_{h_2} - \lambda_{c_0} - \lambda_{h_0} - \eta|} \Bigg\} \|\phi\|_X. \qquad (30)
\end{aligned}
$$

It is the same as (30), by using (18), (19) and Lemma 1 we deduce, for $t-x-h \ge 0$,

$$
|p_4(0, t-x-h)| = \left| \sum_{i=0}^{2} \lambda_{c_i} p_i(t-x-h) \right| \le \sum_{i=0}^{2} \lambda_{c_i} |p_i(t-x-h)|
$$

$$\leq \left\{ \sum_{i=0}^{2} \lambda_{c_i} + \frac{2\lambda\lambda_{c_1}}{|\mu_1 + \lambda_{c_1} + \lambda_{h_1} - \lambda_{c_0} - \lambda_{h_0} - \eta|} + \frac{2\eta\lambda_{c_0}}{|\mu_2 + \lambda_{c_2} + \lambda_{\eta_2} - \lambda_{c_0} - \lambda_{h_0} - \eta|} \right\} \|\phi\|_X. \quad (31)$$

$$|p_5(0, t-x-h)| = \left| \sum_{i=0}^{2} \lambda_{h_i} p_i(t-x-h) \right|$$
$$\leq \left\{ \sum_{i=0}^{2} \lambda_{h_i} + \frac{2\lambda\lambda_{h_1}}{|\mu_2 + \lambda_{c_2} + \lambda_{h_2} - \lambda_{c_0} - \lambda_{h_0} - \eta|} + \frac{2\eta\lambda_{h_2}}{|\mu_2 + \lambda_{c_2} + \lambda_{h_2} - \lambda_{c_0} - \lambda_{h_0} - \eta|} \right\} \|\phi\|_X. \quad (32)$$

By combining (30) and (31) with (32) we derive

$$\sum_{i=3}^{5} \int_0^t |p_i(0, t-x-h)| \left| e^{-\int_0^{x+h} \mu_i(\tau)d\tau} - e^{-\int_0^{x} \mu_i(\tau)d\tau} \right| dx$$
$$\leq \left\{ 2\lambda + \sum_{i=0}^{2} (\lambda_{c_i} + \lambda_{h_i}) + \frac{2(\lambda^2 + \lambda\lambda_{c_1} + \lambda\lambda_{h_1})}{|\mu_1 + \lambda_{c_1} + \lambda_{h_1} - \lambda_{c_0} - \lambda_{h_0} - \eta|} + \frac{2(\lambda\eta + \eta\lambda_{c_2} + \eta\lambda_{h_2})}{|\mu_2 + \lambda_{c_2} + \lambda_{h_2} - \lambda_{c_0} - \lambda_{h_0} - \eta|} \right\} \|\phi\|_X$$
$$\times \sum_{i=3}^{5} \int_0^t \left| e^{-\int_0^{x+h} \mu_i(\tau)d\tau} - e^{-\int_0^{x} \mu_i(\tau)d\tau} \right| dx$$
$$\longrightarrow 0, \text{ as } \quad h \to 0, \quad \text{uniformly for } \phi. \quad (33)$$

In the following we will estimate the second term in (29). By using (17) and Lemma 1 we calculate

$$|p_3(0, t-x-h) - p_3(0, t-x)| = |\lambda[p_1(t-x-h) + p_2(t-x-h)] - \lambda[p_1(t-x) + p_2(t-x)]|$$
$$\leq \lambda|p_1(t-x-h) - p_1(t-x)| + \lambda|p_2(t-x-h) - p_2(t-x)|$$
$$= \lambda \left| \phi_1 e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)} + \frac{\lambda\phi_0[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)}]}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} - \lambda_{c_0} - \lambda_{h_0} - \eta} \right.$$
$$\left. - \phi_1 e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)} - \frac{\lambda\phi_0[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)}]}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} - \lambda_{c_0} - \lambda_{h_0} - \eta} \right|$$
$$+ \lambda \left| \phi_2 e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x-h)} + \frac{\eta\phi_0[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x-h)}]}{\mu_2 + \lambda_{c_2} + \lambda_{h_2} - \lambda_{c_0} - \lambda_{h_0} - \eta} \right.$$

$$- \phi_2 e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x)} - \frac{\eta\phi_0[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x)}]}{\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta}\Bigg|$$

$$\leq \lambda|\phi_1|\left|e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)}\right|$$
$$+ \frac{\lambda^2|\phi_0|}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\left|e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)}\right|$$
$$+ \frac{\lambda^2|\phi_0|}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\left|e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)}\right|$$
$$+ \lambda|\phi_2|\left|e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x)}\right|$$
$$+ \frac{\lambda\eta|\phi_0|}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\left|e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)}\right|$$
$$+ \frac{\lambda\eta|\phi_0|}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\left|e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x)}\right|$$
$$\leq \Bigg\{\left|e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)}\right|$$
$$+ \frac{\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\left|e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)}\right|$$
$$+ \frac{\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\left|e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)}\right|$$
$$+ \left|e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\eta)(t-x)}\right|$$
$$+ \frac{\eta}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\left|e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\eta)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\eta)(t-x)}\right|$$
$$+ \frac{\eta\left|e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x)}\right|}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\Bigg\}\lambda\|\phi\|_X$$
$$\longrightarrow 0 \quad \text{as } h \to 0, \quad \text{uniformly for } \phi. \quad (34)$$

Similar to (34), from (18), (19) and Lemma 1 it follows that

$$|p_4(0,t-x-h) - p_4(0,t-x)| = \left|\sum_{i=0}^{2}\lambda_{c_i}p_i(t-x-h) - \sum_{i=0}^{2}\lambda_{c_i}p_i(t-x)\right|$$
$$\leq \Bigg\{\lambda_{c_0}\left|e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)}\right|$$
$$+ \lambda_{c_1}\left|e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)}\right|$$
$$+ \frac{\lambda\lambda_{c_1}\left|e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)}\right|}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}$$
$$+ \frac{\lambda\lambda_{c_1}\left|e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)}\right|}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}$$
$$+ \lambda_{c_2}\left|e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x)}\right|$$

$$+ \frac{\lambda_{c_2}\eta \left| e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)} \right|}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}$$

$$+ \left. \frac{\lambda_{c_2}\eta \left| e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\eta)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\eta)(t-x)} \right|}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|} \right\} \|\phi\|_X$$

$$\longrightarrow 0, \text{ as } h \to 0, \text{ uniformly for } \phi. \quad (35)$$

$$|p_5(0,t-x-h) - p_5(0,t-x)| \leq \sum_{i=0}^{2} \lambda_{h_i} |p_i(t-x-h) - p_i(t-x)|$$

$$\leq \left\{ \lambda_{h_0} \left| e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)} \right| \right.$$

$$+ \lambda_{h_1} \left| e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)} \right|$$

$$+ \frac{\lambda\lambda_{h_1} \left| e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)} \right|}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}$$

$$+ \frac{\lambda\lambda_{h_1} \left| e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x-h)} - e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)(t-x)} \right|}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}$$

$$+ \lambda_{h_2} \left| e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\eta)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\eta)(t-x)} \right|$$

$$+ \frac{\lambda_{h_2}\eta \left| e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x-h)} - e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)(t-x)} \right|}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}$$

$$+ \left. \frac{\lambda_{h_2}\eta \left| e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x-h)} - e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)(t-x)} \right|}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|} \right\} \|\phi\|_X$$

$$\longrightarrow 0, \text{ as } h \to 0, \text{ uniformly for } \quad \phi. \quad (36)$$

From (34), (35) and (36) we derive

$$\sum_{i=3}^{5}\int_0^t |p_i(0,t-x-h)-p_i(0,t-x)|e^{-\int_0^x \mu_i(\tau)d\tau}dx \to 0,$$
$$\text{as } h\to 0, \text{ uniformly for } \phi. \quad (37)$$

When $x\in(0,t),\ x+h\in(0,t),$ by (33), (37) and (29) we know that

$$\sum_{i=3}^{5}\int_0^t |p_i(x+h,t)-p_i(x,t)|dx \to 0, \text{ as } h\to 0, \quad \text{uniformly for } \phi. \quad (38)$$

If $h\in(-t,0),\ x\in[0,t)$, then from $p_i(x+h,0)=0$ for $x+h<0,\ i=3,4,5,$ we deduce

$$\begin{aligned}&\sum_{i=3}^{5}\int_0^t |p_i(x+h,t)-p_i(x,t)|dx\\ &=\sum_{i=3}^{5}\left\{\int_{-h}^t |p_i(x+h,t)-p_i(x,t)|dx+\int_0^{-h}|p_i(x+h,t)-p_i(x,t)|dx\right\}\\ &=\sum_{i=3}^{5}\int_{-h}^t|p_i(x+h,t)-p_i(x,t)|dx+\sum_{i=3}^{5}\int_0^{-h}|p_i(x+h,t)-p_i(x,t)|dx. \quad (39)\end{aligned}$$

Since $x+h\in[0,t)$ for $x\in[-h,t),\ h\in[-t,0)$, by (30)–(37) we derive

$$\sum_{i=3}^{5}\int_{-h}^t |p_i(x+h,t)-p_i(x,t)|dx \to 0, \text{ as } |h|\to 0, \quad \text{uniformly for } \phi. \quad (40)$$

By using Lemma 1, from (30), (31) and (32) we estimate

$$\begin{aligned}&\int_0^{-h}|p_3(x,t)|dx=\int_0^{-h}|p_3(0,t-x)|e^{-\int_0^x\mu_3(\tau)d\tau}dx\le\lambda\\ &\times\left\{2\lambda+\frac{2\lambda^2}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}+\frac{2\lambda\eta}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\}\\ &\qquad\times\|\phi\|_X\int_0^{-h}e^{-\int_0^x\mu(\tau)d\tau}dx\to 0, \text{ as } |h|\to 0, \text{ uniformly for } \phi. \quad (41)\end{aligned}$$

$$\int_0^{-h}|p_4(x,t)|dx=\int_0^{-h}|p_4(0,t-x)|e^{-\int_0^x\mu_4(\tau)d\tau}dx\le\lambda\times$$
$$\left\{\sum_{i=0}^{2}\lambda_{c_i}+\frac{2\lambda\lambda_{c_1}}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}+\frac{2\eta\lambda_{c_2}}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\}$$
$$\times\|\phi\|_X\int_0^{-h}e^{-\int_0^x\mu(\tau)d\tau}dx\to 0,\text{ as }|h|\to 0,\text{ uniformly for }\phi. \quad (42)$$

$$\int_0^{-h}|p_5(x,t)|dx=\int_0^{-h}|p_5(0,t-x)|e^{-\int_0^x\mu_5(\tau)d\tau}dx\le\lambda$$
$$\times\left\{\sum_{i=1}^{2}\lambda_{h_i}+\frac{2\lambda\lambda_{h_1}}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}+\frac{2\eta\lambda_{h_2}}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\}$$
$$\times\|\phi\|_X\int_0^{-h}e^{-\int_0^x\mu(\tau)d\tau}dx\to 0,\text{ as }|h|\to 0,\text{ uniformly for }\phi. \quad (43)$$

By summing up (39), (40), (41), (42) and (43) it follows that, for $x\in(0,t),\ h\in(-t,0)$

$$\sum_{i=3}^{5}\int_0^t|p_i(x+h,t)-p_i(x+h,t)|dx\to 0\quad\text{as }|h|\to 0,\text{ uniformly for }\phi. \quad (44)$$

From (44) and (38) we know that the result of this theorem is right. □

Theorem 4. *Assume that there exist two positive constant* $\overline{\mu}$, $\underline{\mu}$ *such that* $0<\underline{\mu}\le\mu(x)\le\overline{\mu}<\infty$, *then*

$$\|U(t)\phi\|_X\le e^{-\min\{\underline{\mu},\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta,\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda,\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}t}\left\{4+\right.$$
$$\left.\frac{2\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}+\frac{2\eta}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\}\|\phi\|_X\,.$$

Proof. For any $\phi\in X$, from the definition of $U(t)$ and Lemma 1 we estimate

$$\|U(t)\phi\|=\sum_{i=0}^{2}|p_i(t)|+\sum_{j=3}^{5}\int_t^{\infty}|p_j(x,t)|dx$$
$$\le|\phi_0|\left\{e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t}\right.$$
$$+\frac{\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\left[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t}+e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}\right]$$

$$+\frac{\eta\left[e^{-(\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta)t}+e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t}\right]}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\Bigg\}$$

$$+|\phi_1|e^{-(\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda)t}+|\phi_2|e^{-(\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda)t}$$

$$+\sum_{j=3}^{5}\int_t^\infty|\phi_j(x-t)|e^{-\int_{x-t}^x\mu_j(\tau)d\tau}dx$$

$$\le e^{-\min\{\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta,\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda,\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}t}$$

$$\times\left\{1+\frac{2\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}+\frac{2\eta}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\}$$

$$\times(|\phi_0|+|\phi_1|+|\phi_2|)+\sum_{j=3}^{5}\int_t^\infty|\phi_j(x-t)|e^{-\int_{x-t}^x\underline{\mu}d\tau}dx$$

$$\le e^{-\min\{\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta,\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda,\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}t}$$

$$\times\left\{1+\frac{2\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}+\frac{2\eta}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\}$$

$$\times(|\phi_0|+|\phi_1|+|\phi_2|)+\sum_{j=3}^{5}\|\phi_j\|_{L^1[0,\infty)}e^{-\underline{\mu}t}$$

$$\le e^{-\min\{\underline{\mu},\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta,\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda,\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}t}$$

$$\times\left\{4+\frac{2\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}+\frac{2\eta}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\}$$

$$\times\|\phi\|_X. \quad (45)$$

(45) shows that the result of this theorem is right. □

From Theorem 4 we derive

$$\|S(t)-V(t)\|$$

$$=\|U(t)\|\le e^{-\min\{\underline{\mu},\lambda+\lambda_{c_0}+\lambda_{h_0}+\eta,\mu_1+\lambda_{c_1}+\lambda_{h_1}+\lambda,\mu_2+\lambda_{c_2}+\lambda_{h_2}+\lambda\}t}$$

$$\times\left\{4+\frac{2\lambda}{|\mu_1+\lambda_{c_1}+\lambda_{h_1}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right.$$

$$\left.+\frac{2\eta}{|\mu_2+\lambda_{c_2}+\lambda_{h_2}-\lambda_{c_0}-\lambda_{h_0}-\eta|}\right\}.$$

From which together with Definition 2.7 in [4] we deduce the following result.

Theorem 5. *Assume that* $\mu_i(x)$ *satisfy* $0<\underline{\mu}\le\mu_i(x)\le\overline{\mu}<\infty$ *for* $i=3,4,5$, *then* $S(t)$ *is a quasi-compact operator in* X.

Since E is compact in X, from Theorem 5 and Proposition 2.9 in [5] we conclude the following corollary.

Corollary 1. *Assume that* $\mu_i(x)$ *satisfy* $0 < \underline{\mu} \leq \mu_i(x) \leq \overline{\mu} < \infty$ *for* $i = 3, 4, 5$, *then* $T(t)$ *is a quasi-compact operator in* X *.*

Lemma 2. *Assume that* $\mu(x)$ *satisfies the conditions in Theorem 4, then* 0 *is an eigenvalue of* $A + B + E$ *with geometric multiplicity one.*

Proof. Consider $(A + B + E)p = 0$, that is,

$$\sum_{i=1}^{2} \mu_i p_i + \sum_{i=3}^{5} \int_0^{\infty} \mu_i(x) p_i(x) dx = (\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta) p_0, \tag{46}$$

$$\lambda p_0 = (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda) p_1, \tag{47}$$

$$\eta p_0 = (\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda) p_2, \tag{48}$$

$$\frac{dp_i(x)}{dx} = -\mu_i(x) p_i(x), \quad i = 3, 4, 5, \tag{49}$$

$$p_3(0) = \lambda(p_1 + p_2), \tag{50}$$

$$p_4(0) = \sum_{i=0}^{2} \lambda_{c_i} p_i, \tag{51}$$

$$p_5(0) = \sum_{i=0}^{2} \lambda_{h_i} p_i. \tag{52}$$

By solving (47), (48) and (49) we have

$$p_1 = \frac{\lambda}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda} p_0, \tag{53}$$

$$p_2 = \frac{\eta}{\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda} p_0, \tag{54}$$

$$p_i(x) = a_i e^{-\int_0^x \mu_i(\tau) d\tau}, \quad i = 3, 4, 5. \tag{55}$$

By substituting (55), (54) and (53) into (50), (51) and (52) we obtain

$$a_3 = \left\{ \frac{\lambda^2}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda} + \frac{\lambda\eta}{\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda} \right\} p_0, \tag{56}$$

$$a_4 = \left\{ \lambda_{c_0} + \frac{\lambda\lambda_{c_1}}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda} + \frac{\eta\lambda_{c_2}}{\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda} \right\} p_0, \tag{57}$$

$$a_5 = \left\{ \lambda_{h_0} + \frac{\lambda\lambda_{h_1}}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda} + \frac{\eta\lambda_{h_2}}{\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda} \right\} p_0. \tag{58}$$

From (53)–(58) we estimate

$$\begin{aligned}
\|p\| &= |p_0| + |p_1| + |p_2| + \sum_{i=3}^{5} \|p_i\|_{L^1[0,\infty)} \\
&\le \left\{1 + \frac{\lambda}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda} + \frac{\eta}{\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda}\right\} |p_0| \\
&\quad + \sum_{i=3}^{5} \int_0^\infty |a_i| e^{-\int_0^x \mu_i(\tau) d\tau} dx \\
&\le \left\{1 + \frac{\lambda}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda} + \frac{\eta}{\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda}\right\} |p_0| + \sum_{i=3}^{5} \frac{1}{\underline{\mu}} |a_i| \\
&= \left\{1 + \frac{\lambda}{\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda} + \frac{\eta}{\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda}\right. \\
&\quad \left. + \frac{\lambda(\lambda + \lambda_{c_1} + \lambda_{h_1})}{\underline{\mu}(\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)} + \frac{\eta(\lambda + \lambda_{c_2} + \lambda_{h_2})}{\underline{\mu}(\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)} + \frac{\lambda_{c_0} + \lambda_{h_0}}{\underline{\mu}}\right\} |p_0| \\
&< \infty. \qquad (59)
\end{aligned}$$

(59) shows that 0 is an eigenvalue of $A+B+E$. Moreover, from (52)–(57) it is easy to see that eigenvector space corresponding to 0 is one dimensional linear space, that is, geometric multiplicity of 0 is one. The proof of this lemma is complete.

X^*, dual space of X, is

$$X^* = \{q^* \in R \times R \times R \times L^\infty[0,\infty) \times L^\infty[0,\infty) \times L^\infty[0,\infty) \mid$$
$$|||q^*||| = \max\{|q_0^*|, |q_1^*|, |q_2^*|, \|q_3^*\|_{L^\infty[0,\infty)}, \|q_4^*\|_{L^\infty[0,\infty)}, \|q_5^*\|_{L^\infty[0,\infty)}\}\}.$$

From the definition of adjoint operator, it is not difficult to prove that $(A + B + E)^*$, the adjoint operator of $A+B+E$, is given by (for simplicity, take $s_i = \mu_i + \lambda_{c_i} + \lambda_{h_i} + \lambda,\ i = 1, 2,\ s = \lambda + \lambda_{c_0} + \lambda_{h_0} + \eta$):

$$(A+B+E)^* q^*$$
$$= \begin{pmatrix} -s & \lambda & \eta & 0 & 0 & 0 \\ \mu_1 & -s_1 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & -s_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{dx} - \mu_3(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{d}{dx} - \mu_4(x) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{dx} - \mu_5(x) \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ q_3^*(x) \\ q_4^*(x) \\ q_5^*(x) \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & 0 & 0 & \lambda_{c_0} & \lambda_{h_0} \\ 0 & 0 & 0 & \lambda & \lambda_{c_1} & \lambda_{h_1} \\ 0 & 0 & 0 & \lambda & \lambda_{c_2} & \lambda_{h_2} \\ \mu_3(x) & 0 & 0 & 0 & 0 & 0 \\ \mu_4(x) & 0 & 0 & 0 & 0 & 0 \\ \mu_5(x) & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_0^* \\ q_1^* \\ q_2^* \\ q_3^*(0) \\ q_4^*(0) \\ q_5^*(0) \end{pmatrix},$$

$$D((A+B+E)^*) = \left\{ \frac{dq_i^*(x)}{dx} \text{ exists and } q_i^*(\infty) = \alpha,\ i = 3,4,5 \right\}.$$

Lemma 3. *0 is an eigenvalue of $(A+B+E)^*$ with geometric multiplicity one.*

Proof. Consider $(A+B+E)^*q^* = 0$. It is equivalent to

$$-(\lambda + \lambda_{c_0} + \lambda_{h_0} + \eta)q_0^* + \lambda q_1^* + \eta q_2^* + \lambda_{c_0} q_4^*(0) + \lambda_{h_0} q_5^*(0) = 0, \tag{60}$$

$$\mu_1 q_0^* - (\mu_1 + \lambda_{c_1} + \lambda_{h_1} + \lambda)q_1^* + \lambda q_3^*(0) + \lambda_{c_1} q_4^*(0) + \lambda_{h_1} q_5^*(0) = 0, \tag{61}$$

$$\mu_2 q_0^* - (\mu_2 + \lambda_{c_2} + \lambda_{h_2} + \lambda)q_2^* + \lambda q_3^*(0) + \lambda_{c_2} q_4^*(0) + \lambda_{h_2} q_5^*(0) = 0, \tag{62}$$

$$\frac{dq_3^*(x)}{dx} = -\mu_3(x) q_3^*(x) + \mu_3(x) q_0^*, \tag{63}$$

$$\frac{dq_4^*(x)}{dx} = -\mu_4(x) q_4^*(x) + \mu_4(x) q_0^*, \tag{64}$$

$$\frac{dq_5^*(x)}{dx} = -\mu_5(x) q_5^*(x) + \mu_5(x) q_0^*, \tag{65}$$

$$q_i^*(\infty) = \alpha, \quad i = 3,4,5. \tag{66}$$

By solving (63)–(65) we have

$$q_3^*(x) = b_3 e^{\int_0^x \mu_3(\tau)d\tau} + e^{\int_0^x \mu_3(\tau)d\tau} \int_0^x q_0^* \mu_3(\tau) e^{-\int_0^\tau \mu_3(\xi)d\xi} d\tau, \tag{67}$$

$$q_4^*(x) = b_4 e^{\int_0^x \mu_4(\tau)d\tau} + e^{\int_0^x \mu_4(\tau)d\tau} \int_0^x q_0^* \mu_4(\tau) e^{-\int_0^\tau \mu_4(\xi)d\xi} d\tau, \tag{68}$$

$$q_4^*(x) = b_5 e^{\int_0^x \mu_5(\tau)d\tau} + e^{\int_0^x \mu_5(\tau)d\tau} \int_0^x q_0^* \mu_5(\tau) e^{-\int_0^\tau \mu_5(\xi)d\xi} d\tau. \tag{69}$$

By combining (67)–(69) with (66) we deduce

$$b_3 = q_0^* \int_0^\infty \mu_3(\tau) e^{-\int_0^\tau \mu_3(\xi)d\xi} d\tau = q_0^*, \tag{70}$$

$$b_4 = q_0^* \int_0^\infty \mu_4(\tau) e^{-\int_0^\tau \mu_4(\xi)d\xi} d\tau = q_0^*, \tag{71}$$

$$b_5 = q_0^* \int_0^\infty \mu_5(\tau) e^{-\int_0^\tau \mu_5(\xi)d\xi} d\tau = q_0^*. \tag{72}$$

By substituting (70)–(72) into (67)–(69) we obtain

$$q_3^*(x) = q_0^* e^{\int_0^x \mu_3(\tau)d\tau} \int_x^\infty \mu_3(\tau) e^{-\int_0^\tau \mu_3(\xi)d\xi} d\tau, \tag{73}$$

$$q_4^*(x) = q_0^* e^{\int_0^x \mu_4(\tau)d\tau} \int_x^\infty \mu_4(\tau) e^{-\int_0^\tau \mu_4(\xi)d\xi} d\tau, \tag{74}$$

$$q_3^*(x) = q_0^* e^{\int_0^x \mu_5(\tau)d\tau} \int_x^\infty \mu_5(\tau) e^{-\int_0^\tau \mu_5(\xi)d\xi} d\tau. \tag{75}$$

From $b_i = p_i(0) = q_0^*,\ i = 3,4,5$ and (61), (62) it follows that

$$q_1^* = q_0^*, \quad q_2^* = q_0^*. \tag{76}$$

From (73)–(75) we estimate

$$\begin{aligned}\|q_i^*\|_{L^\infty[0,\infty)} &= \sup_{x\in[0,\infty)} \left| q_0^* e^{\int_0^x \mu_i(\tau)d\tau} \int_x^\infty \mu_i(\tau) e^{-\int_0^\tau \mu_3(\xi)d\xi} d\tau \right| \\ &= |q_0^*| \sup_{x\in[0,\infty)} \left| \int_x^\infty \mu_i(\tau) e^{-\int_x^\tau \mu_i(\xi)d\xi} d\tau \right| \\ &\le |q_0^*| \int_0^\infty \mu_i(\tau) e^{-\int_0^\tau \mu_i(\xi)d\xi} d\tau = -|q_0^*| e^{-\int_0^\tau \mu_i(\xi)d\xi}\Big|_0^\infty = |q_0^*|, \\ &\qquad i = 3,4,5.\end{aligned} \tag{77}$$

By combining (76) with (77) we calculate

$$\begin{aligned}\||q^*\|| &= \max\{|q_0^*|, |q_1^*|, |q_2^*|, \|q_3^*\|_{L^\infty[0,\infty)}, \|q_4^*\|_{L^\infty[0,\infty)}, \|q_5^*\|_{L^\infty[0,\infty)}\} \\ &\le |q_0^*| < \infty.\end{aligned} \tag{78}$$

(78) shows that 0 is an eigenvalue of $(A+B+E)^*$. Moreover, from (76) and (73)–(75) it is easy to see geometric multiplicity of 0 is one. □

From Lemma 2 and Lemma 3 we know that algebraic multiplicity of 0 is one (see [3]).

By combining Lemma 2, Corollary 1 and Theorem 1 we deduce that spectral bound of $A+B+E$ is zero, that is, $s(A+B+E) = 0$. Therefore, from Lemma 2, Lemma 3, Theorem 1, Corollary 1, and Theorem 2.1 and Remark 2.2(c) in [5, p. 343] we conclude the following result.

Theorem 6. *If $\mu_j(x)$ satisfy $0 < \underline{\mu} \leq \mu_j(x) \leq \overline{\mu} < \infty$ for $j = 3, 4, 5,$ then there exist a positive projection operator P with rank one and suitable positive constants $\delta > 0,\ M \geq 0$ such that*

$$\|T(t) - P\| \leq Me^{-\delta t},$$

where $P = \frac{1}{2\pi i}\int_{\overline{\Gamma}}(zI - A - B - E)^{-1}dz$, $\overline{\Gamma}$ is a circle with center 0 and sufficiently small radius.

By Lemma 2, Lemma 3, Theorem 1, Corollary 1, and Proposition 2.9 and Theorem 2.10 in [5, p. 302] we deduce

$$\{\gamma \in \sigma(A + U + E) \mid \mathrm{Re}\gamma = 0\} = \{0\}.$$

From which together with Theorem 14 in [3] we derive the following result.

Theorem 7. *If $\mu_j(x)$ satisfy $0 < \underline{\mu} \leq \mu_j(x) \leq \overline{\mu} < \infty$ for $j = 3, 4, 5,$ then the time-dependent solution of the system (9)-(10) converges strongly to the steady-state solution of the system (9)-(10) as time tends to infinite, that is,*

$$\lim_{t\to\infty} p(x, t) = \alpha p(x),$$

where $p(x)$ is an eigenvector corresponding to 0 (see Lemma 2).

Acknowledgments

The authors was supported by the Major Project of the Ministry of Education of China (No: 205180), Excellent Youth Reward Foundation of the Higher Education Institution of Xinjiang (No: XJEDU 2004E05), the NSFC (No: 10371105) and Xinjiang University Science Foundation.

References

[1] Geni Gupur, Well-posedness of the model describing a repairable, standby human and machine system, *Journal of Systems Science and Complexity*, **16** (2003), 483-493.

[2] Geni Gupur, Description of relatively compact subset of a Banach space, *Journal of Xinjiang University*, Natural Science Edition, **22** (2005), 389-392.

[3] Geni Gupur, Xue-zhi Li, Guang-tian Zhu, *Functional Analysis Method in Queueing Theory*, Research Information Ltd., Herdfortshire (2001).

[4] Rainer Nagel, *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Mathematics, **1184**, Springer, Berlin (1986).

[5] V. Sridharan, P. Mohanavadivu, Some statistical characteristics of repairable, human and machine system, *IEEE Transactions on Reliability*, **47** (1998), 431-435.

[6] Xu Li-guang, Asymptotic property of the solution of the model describing a repairable, standby, human and machine system, *Journal of Xinjiang University*, Natural Science Edition, **22** (2005), 416-424.