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ON BOOLEAN'S ALGEBRA COMPACTIFICATION IN BITOPOLOGICAL SPACES

B.M. Taher
Department of Mathematics
Faculty of Science
Tanta University
Tanta, EGYPT
e-mail: Scare86@yahoo.com

Abstract: We construct a Boolean's algebra compactification in bitopological spaces in terms of $(i,\ j)\lambda_\gamma$ open and $(i,\ j)\lambda_\gamma$ closed sets and study its relation with Stone-Čech compactification in $\beta_i X$ bitopological spaces.

We define the concepts of bizero-dimensional and bitotally disconnected bispace and give some of its properties and characterizations.

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1. Introduction

For brevity we refer a bitopological space (see [6]) as bispace throughout this paper $(X,\ \tau_1,\ \tau_2)$ and $(Y,\ \sigma_1, \sigma_2\ )$ (or simply X and Y) always mean bispaces. We denote the closure (interior) operator with respect the topological τ_i $(i = 1,\ 2)$ by cl_{τ_i} respectively.

In 1979, Kasahara [4] defined an operation on a topology τ on a non empty set X to be a function on τ onto the power set $P(X)$ such that $G \subseteq G^\alpha$, for every $G \in \tau$, where G^α denoted the value of α at G. The family of all operation α is denoted by $O_\tau(X)$. In 1983, Abd El-Monsef et al [3] generalized Kasahara's

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operation on the powers set $P\ (X)$ of a topological space $(X,\ \tau)$. A function $\Delta : P\ (X) \to P\ (X)$ (resp. $\delta : P \to P\ (X)$) is said to be an operation on $P\ (X)$ of type I [3] (resp. of type II [3]), if $\mathrm{int}_\tau(A) \subseteq A^\Delta$) (resp. $\mathrm{cl}_\tau(A) \subseteq A^\delta$); for every $A \in P\ (X)$, where $A^\Delta(A^\delta)$ denotes the values of $\Delta(\delta)$ at A. In 1995 Kozae et al [5] introduced an $(i,\ j)\lambda_\gamma$ operation $\lambda_\gamma : P\ (X) \to P\ (X)$ on $P\ (X)$ of a bispace $(X,\ \tau_1,\ \tau_2\ )$, if λ_γ is an operation of type I w.r.t. $(X,\ \tau_i\ )$ and an operation of type II w.r.t $(X,\ \tau_i)$ (i.e. $\mathrm{int}_{\tau_i}\ A \subseteq A^{\lambda_\gamma}$ and $\mathrm{cl}_{\tau_i}(A) \supseteq A^{\lambda_\gamma}$ for every $A \in P\ (X)$, $\{i,\ j = 1,\ 2,\ \ i \neq j\}$). A subset A of a bispace X is called $(i,\ j)\lambda_\gamma$ open, see [5], if $A \subseteq A^{\lambda_\gamma}$, its complement is called $(i,\ j)\lambda_\gamma$ closed, see [5]. A Boolean's algebra $\mathfrak{B}$ [1] consists of a non empty set B and three functions (operations) such that $\wedge,\ \vee : B \times B \to B,' : B \to B$ such that

$$a \wedge b = b \wedge a\ , \qquad a \vee b = b \vee \mathfrak{a}\,,$$
$$a \wedge (b \wedge c) = (b \wedge a) \wedge c\ , \qquad a \vee (b \vee c) = (b \vee a) \vee c\,,$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\ , \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\,,$$
$$(a \vee a') \wedge b = b\ , \qquad (a \wedge a') \vee b = b, \text{ for } a,\ b \in B.$$

A field of sets $\wp$ on a subset A is a family of subsets of A such that $a \cap b,\ a \cup b,\ A \backslash \mathfrak{a} \in \wp$ for $a,\ b \in \wp$, see [1] (by the set theory means $\cap$ and $\cup$ for $\wedge$ and $\vee$ and $A \backslash a = a\forall\forall'$). A bispace $(X,\ \tau_1,\ \tau_2\ )$ is to be jointly "P" [7], if $\tau_1 \vee \tau_2$ has property "P", where $\tau_1 \vee \tau_2$ is a supremum topology generated by $\tau_1 \cup \tau_2$. The concepts of i-open, i-base and so on, must be an open in topology τ_i, base for τ_i and so on. A map $f : (X,\ \tau_1,\ \tau_2) \to (Y,\ \sigma_1,\ \sigma_2\ )$ is said to be bi-open (bicontinuous, bihomeomorphism, see [8]) if the induced maps $f_i : (X,\ \tau_i) \to (Y,\ \sigma_i)$ are open (continuous, homeomorphism). A compactification of a bispace $(X,\ \tau_1,\ \tau_2)$, see [2], is apair ($(Y,\ p_1,\ p_2),\ f$), where $(Y,\ p_1,\ p_2)$ is compact, f is a bihomeomorphism of $(X,\ \tau_1,\ \tau_2)$ onto the bispace $f\ (X) \subseteq Y,\ [f\ (X)]_Y = Y$.

2. Some Properties of Boolean's Algebra in Bitopological Space

We develop here the usual finitary properties of (Boolean's algebra involving the concepts of homeomorphism and filters in terms of $(i,\ j)\lambda_\gamma$-open sets in bitopological spaces (bispaces.).

in the following we use the convention $\{i,\ j = 1,\ 2,\ i \neq j\}$.

Definition 2.1. Let $(X,\ \tau_1,\ \tau_2)$ be a bispace. the set of $(i,\ j)\lambda_\gamma$-open and $(i,\ j)\lambda_\gamma$ closed subset of X and is denoted by $\mathfrak{B}_i(X)$.

Definition 2.2. The family of subsets $\wp$ of a subset X and is called a field of sets, if $A \cap B,\ A \cup B,\ X \backslash A \in \wp$ for $A,\ B \in \wp$.

Remark 2.1. (i) A field of sets $\wp$ on X is a Boolean's algebra such that

$\emptyset = \emptyset,\ 1 = X$ and by the set theory operations $\cap$ and $\cup$ for $\wedge$ and $\vee$ with $X \setminus A = A'$ and $A \leq B$ iff $A \subseteq B$ for $A,\ B \in \wp$.

(ii) $\mathfrak{B}_i(X)$ is a field of sets on X and $\mathfrak{B}_i(\alpha) = P\ (\alpha)$ for cardinal α (with the discrete topology).

(iii) We denote by $\mathbf{2}$ =the field of sets $P\ (1)$; thus $\mathbf{2}$ is the unique up to isomorphic to a Boolean's algebra $\mathfrak{B}_i(X)$ such that $|\mathfrak{B}_i(X)| = \mathbf{2}$.

Definition 2.3. Let $u_i\ (X)$ and $\mathfrak{B}_i(X)$ be two Boolean's algebra on a bispace X. The function $\Phi_i : u_i\ (X) \to \mathfrak{B}_i(X)$ is called a Boolean's algebra homomorphism (B.a. homomorphism) if:

$$\Phi_i\ (a \wedge b) = \Phi_i\ (a) \wedge \Phi_i\ (b), \quad \Phi_i\ (a \vee b) = \Phi_i\ (a) \vee \Phi_i\ (b),$$
$$\Phi_i\ (a') = (\Phi_i\ (a))' \text{ for } a,\ b \in u_i\ (X).$$

(i) If Φ_i is an one-to-one function and a B.a.-homomorphism, then Φ_i, is called Boolean's algebra embedding (B.a.-embedding).

(ii) If Φ_i is a B.a.-embedding and onto function then Φ_i is a Boolean's algebra-isomorphism (B.a.-isomorphism).

Remark 2.2. If $b \in \mathfrak{B}_i(X),\ n < \omega$ and $b_k \in \mathfrak{B}_i(X)$ for $k \leq n$, then:

(i) $b \setminus \vee_{k\ \leq n}\, b_k = \wedge_{k\ \leq n}(b\ \setminus b_k)$.

(ii) $b \setminus \wedge_{k\ \leq n}\, b_k = \vee_{k\ \leq n}(b\ \setminus b_k)$.

Lemma 2.1. *Let $\mathfrak{B}_i(X)$ be a Boolean's algebra on a bispace X, let $n \leq \omega,\ r_k \leq \omega$ for $k \leq n$ and $b_{k,\ l} \in \mathfrak{B}_i(X)$ for $k \leq n$. Set $K = \Pi_{k\ \leq n}(r_k + 1)$.*

Then the following equalities hold:

(i) $\vee_{k\ \leq n} \wedge_{l\ \leq r_k} b_{k,\ l} = \wedge_{a\ \in k} \vee_{k\ \leq n} b_{k,\ a_k}$.

(ii) $\wedge_{k\ \leq n} \wedge_{l\ \leq r_k} b_{k,\ l} = \vee_{a\ \in k} \wedge_{k\ \leq n} b_{k,\ a_k}$.

Proof. Obvious by using Remark 2.2. □

Lemma 2.2. *Let $\mathfrak{B}_i(X)$ be a Boolean's algebra on a bispace X generated by a non empty subset C, let $u_i(X)$ be a Boolean's algebra on bispace X and let $f : C \to u_i\ (X)$ such that if $\wedge_{k\ \leq n}\varepsilon_k c_k = \emptyset$, then $\wedge_{k\ \leq n}\varepsilon_k f\ (c_k) = \emptyset$, for $n \geq \omega$ and $c_k \in C,\ \varepsilon_k \in \{-1,\ 1\}$ for $k \leq n$. Moreover there is a unique B. a. homomorphism $\Phi_i : \mathfrak{B}_i(X) \to u_i\ (X)$ such that $f \subset \Phi_i$.*

Proof. We claim that if $\vee_{k\ \leq n} \wedge_{l\ \leq r_k} \varepsilon_{k,\ l}\ c_{k,\ l} = \vee_{\bar{k}\ \leq \bar{n}} \wedge_{\bar{l}\ \leq \bar{r}_{\bar{k}}} \varepsilon_{\bar{k},\ \bar{l}}\ c_{\bar{k},\ \bar{l}}$, then $\vee_{k\ \leq n} \wedge_{l\ \leq r_k} \varepsilon_{k,\ l}\ f\ (c_{k,\ l}) = \vee_{\bar{k}\ \leq \bar{n}} \wedge_{\bar{l}\ \leq \bar{r}_{\bar{k}}} \varepsilon_{\bar{k},\ \bar{l}}\ f\ (c_{\bar{k},\ \bar{l}})$, for $n,\ \bar{n} \leq \omega$, $r_k,\ \bar{r}_k < \omega,\ k \leq n,\ \bar{k} \leq \bar{n}$, where $c_{k,\ l} \in C,\ \varepsilon_{k,\ l} \in \{-1,\ 1\}$ for $k \leq n,\ l \leq r_k$ and $\bar{c}_{\bar{k},\ \bar{l}} \in C,\ \bar{\varepsilon}_{\bar{k},\ \bar{l}} \in \{-1,\ 1\}$ for $\bar{k} \leq \bar{n},\ \bar{l} \leq \bar{r}_{\bar{k}}$. Set $K = \Pi_{\bar{k}\ \leq \bar{n}}(\bar{r}_{\bar{k}} + 1)$ and using Lemma 2.1 and Remark 2.2, we have:

$$\emptyset = \vee_{k\ \leq n} \wedge_{l\ \leq r_k} \varepsilon_{k,\ l}\ c_{k,\ l} \setminus \vee_{\bar{k}\ \leq \bar{n}} \wedge_{\bar{l} \leq \bar{r}_{\bar{k}}} \bar{\varepsilon}_{\bar{k},\ \bar{l}}\ \bar{c}_{\bar{k},\ \bar{l}}$$

$$= \vee_{k \leq n} \wedge_{l \leq r_k} (\varepsilon_{k,\ l}\ c_{k,\ l}) \setminus \wedge_{a,\ k} \vee_{\bar{k} \leq \bar{n}} \bar{\varepsilon}_{\bar{k},\ \bar{a}_{\bar{k}}}\ \bar{c}_{\bar{k},\ \bar{a}_{\bar{k}}}$$
$$= \vee_{a,\ k} \wedge_{l \leq r_k} (\varepsilon_{k,\ l}\ c_{k,\ l}) \setminus \wedge_{a,\ k} \vee_{\bar{k} \leq \bar{n}} \bar{\varepsilon}_{\bar{k},\ \bar{a}_{\bar{k}}}\ \bar{c}_{\bar{k},\ \bar{a}_{\bar{k}}}.$$

Thus

$$\wedge_{\bar{k} \leq \bar{n}} \wedge_{l \leq \bar{r}_{\bar{k}}} (\varepsilon_{k,\ l}\ c_{k,\ l}) \wedge \setminus (\bar{\varepsilon}_{\bar{k},\ \mathfrak{a}_{\bar{k}}}\ \bar{c}_{\bar{k},\ \bar{a}_{\bar{k}}}) = \emptyset,$$

for $a \in K$ and $k \leq n$. The assumption on f clearly implies that:

$$\wedge_{\bar{k} \leq \bar{n}} \wedge_{l \leq \bar{r}_{\bar{k}}} (\varepsilon_{k,\ l}\ f\ (c_{k,\ l})) \wedge \setminus \bar{\varepsilon}_{\bar{k},\ a_{\bar{k}}}\ f\ (\bar{c}_{\bar{k},\ \bar{a}_{\bar{k}}}) = \emptyset,$$

for $a \in K$ and $k \leq n$ and using Remark 2.2 and Lemma 2.1 that:

$$\vee_{k \leq n} \wedge_{l \leq r_k} f\ (c_{k,\ l}) \leq \vee_{\bar{k},\ \bar{n}}\ f\ (\bar{c}_{\bar{k},\ l}) = \emptyset\,.$$

A symmetric argument proves the reverse inequality and establishes the claim. Now let $b \in \mathfrak{B}_i(X)$ and since $\emptyset \neq C \subset B \in \mathfrak{B}_i(X)$ so there are $n < \omega,\ r_k < \omega$ for $k \leq n$ and $c_{k,l} \in C,\ \varepsilon_{k,\ l} \in \{-1,\ 1\}$, for $k \leq n,\ l \leq r_k$ such that $b = \vee_{k \leq n} \vee_{l \leq r_k} \varepsilon_{k,\ l}\ f\ (C_{k,\ l})$

Thus the function $\Phi_i : \mathfrak{B}_i(X) \to u_i(X)$ is well defined and it is easy to verify that Φ_i, is the unique B.a. homomorphism such that $f \subset \Phi_i$. □

Definition 2.4. Let $\mathfrak{B}_i(X)$ be a Boolean's algebra on a bispace X and $\emptyset \neq \mathcal{F} \subset \mathfrak{B}_i(X)$.

(a) $\mathcal{F}$ has the finite intersection property if $\wedge_{k \leq n} b_k = \emptyset,\ n < \omega,\ b_k \in \mathcal{F},\ k \leq n$.

(b) $\mathcal{F}$ is called a Boolean's algebra filter (B.a.-filter) if:

(i) $a \wedge b \in \mathcal{F},$ for $a,\ b \in \mathcal{F}$.

(ii) $a \in \mathcal{F},\ b \in \mathfrak{B}_i(X)$ and $a \leq b$, then $b \in \mathcal{F}$.

(iii) A Boolean's algebra filter $\mathcal{F}$ is proper if $\mathcal{F} \neq \mathfrak{B}_i(X)$ (i.e. if $\emptyset \notin \mathcal{F}$); the improper filter of $\mathfrak{B}_i(X)$ is $\mathfrak{B}_i(X)$.

(iv) The intersection of the family of all Boolean's algebra filters of $\mathfrak{B}_i(X)$ that contains $\mathcal{F}$ is called a Boolean's algebra filter of $\mathfrak{B}_i(X)$ generated by $\mathcal{F}$ is denoted by $< \mathcal{F} >$. i.e. $< \mathcal{F} >= \{b \in \mathfrak{B}_i(X) : \exists n < \omega$ and $b_k \in \mathcal{F}$, for $k \leq n$ such that $\{b_0 \wedge b_1 \wedge \cdots \wedge b_n < b\}$ and $\mathcal{F}$ has the finite intersection property.

(v) A Boolean's algebra filter (B.a.-filter) $\mathcal{F}$ of $\mathfrak{B}_i(X)$ is a principle B.a.-filter if there is $a \in \mathfrak{B}_i(X)$ such that $\mathcal{F}$ is generated by $\{a\}$.

(vi) A proper Boolean's algebra filter in $\mathfrak{B}_i(X)$ which does not properly contained in any other Boolean's algebra filter of $\mathfrak{B}_i(X)$is called a Boolean's algebra-ultra filter (B.a.-ultra filter).

Theorem 2.1. *Let $\mathfrak{B}_i(X)$ be a Boolean's algebra on a bispace X, let $\mathcal{F} \subset \mathfrak{B}_i(X)$. If $\mathcal{F}$ has the finite intersection then there is a B.a.-ultra-filter P of $\mathfrak{B}_i(X)$ such that $\mathcal{F} \subset P$ (in particular every proper B.a.-filter of $\mathfrak{B}_i(X)$ is* *contained in a B.a> ultra-filter).*

Proof. Obvious from the definition of B.a.-ultra-filter.

Definition 2.5. if $\Phi_i : u_i\ (X) \to \mathfrak{B}_i(X)$ is a B.a.-homomorphism we define:

$$\mathcal{F} = \{a \in u_i\ (X) : \Phi_i\ (a) = \mathbf{1}\}.$$

Remark 2.3. $\mathcal{F}\{a \in u_i\ (X) : \Phi_i(a) = \mathbf{1}\}$ is a B.a.-filter of $\mathfrak{B}_i(X)$, it is called the B.a.-filter associated with Φ_i.

Definition 2.6. Let $\mathfrak{B}_i(X)$ be a Boolean's algebra on a bispace X, let $\mathcal{F}$ be a B.a.-filter of $\mathfrak{B}_i(X)$, for $a,\ b \in \mathfrak{B}_i(X)$, we define $a \underset{\mathcal{F}}{=} b$ if there is $c \in \mathcal{F}$ such that $a \wedge c = b \wedge c$.

Lemma 2.3. *The relation $\underset{\mathcal{F}}{=}$ is an equivalence relation on $\mathfrak{B}_i(X)$.*

Proof. Let $a, b, c \in \mathfrak{B}_i(X)$. Then:

(i) Since $1 \in \mathcal{F} a \underset{\mathcal{F}}{=} a$ where $(a \wedge 1 = a \wedge 1)$.

(ii) If $a \underset{\mathcal{F}}{=} b \Longrightarrow a \wedge c = b \wedge c = a \wedge c \Longrightarrow b \underset{\mathcal{F}}{=} a$.

(iii) If $a \underset{\mathcal{F}}{=} b,\ b \underset{\mathcal{F}}{=} c$, then there are $d,\ e \in \mathcal{F}$ such that $a \wedge d = b \wedge d$ and $b \wedge e = c \wedge e$ such that $d \wedge e \in \mathcal{F}$ and $a \wedge (d \wedge e) = c \wedge (d \wedge e) \Longrightarrow a \underset{\mathcal{F}}{=} c$, thus the relation $\underset{\mathcal{F}}{=}$ is an equivalence relation on $\mathfrak{B}_i(X)$. □

Remark 2.4. We denote by $\mathfrak{B}_i(X)/\mathcal{F}$ the set of $\underset{\mathcal{F}}{=}$ equivalence classes of $\mathfrak{B}_i(X)$ and denoted by $a\ /\mathcal{F}$ the $\underset{\mathcal{F}}{=}$ equivalence class of a (for $a \in \mathfrak{B}_i(X)$).

Lemma 2.4. *If $a_0,\ b_0,\ a_1,\ b_2 \in \mathfrak{B}_i(X)$ and $a_0 \underset{\mathcal{F}}{=} b_0$ and $a_1 \underset{\mathcal{F}}{=} b_1$, then $a_0 \wedge a_1 \underset{\mathcal{F}}{=} b_0 \wedge b_1,\ a_0 \vee a_1 \underset{\mathcal{F}}{=} b_0 \vee b_1$ and $a'_0 \underset{\mathcal{F}}{=} b'_0$.*

Proof. For $c_0,\ c_1 \in \mathcal{F} \Longrightarrow c_0 \wedge c_1 \in \mathcal{F}$ but $a_0 \underset{\mathcal{F}}{=} b_0,\ a_0 \wedge c_0$ and also

$$a_1 \wedge c_1 = b_1 \wedge c_1 \Longrightarrow (a_0 \wedge a_1) \wedge (c_0 \wedge c_1) = (b_0 \wedge b_1) \wedge (c_0 \wedge c_1) \Longrightarrow a_0 \wedge a_1 \underset{\mathcal{F}}{=} b_0 \wedge b_1,$$

also

$$(a_0 \vee a_1) \wedge (c_0 \wedge c_1) = (b_0 \vee b_1) \wedge (c_0 \wedge c_1) \Longrightarrow (a_0 \vee a_1) \underset{\mathcal{F}}{=} b_0 \vee b_1,$$

and

$$a'_0 \wedge c_0 = c_0 \backslash (a_0 \wedge c_0) = c_0 \backslash (b_0 \wedge c_0) = b'_0 \wedge c_0 \Longrightarrow a'_0 \underset{\mathcal{F}}{=} b'_0.$$ □

Remark 2.5. (i) The set $\mathfrak{B}_i(X)/\mathcal{F}$ together with the (well-defined operations given by $a_0/\mathcal{F} \wedge a_1/\mathcal{F} = (a_0 \wedge a_1)/\mathcal{F},\ a_0/\mathcal{F} \vee a_1/\mathcal{F} = (a_0 \vee a_1)/\mathcal{F}$ and $(a_0/\mathcal{F})' = a'_0/\mathcal{F}$ is a Boolean's algebra on a bispace X. This is called the quotion Boolean's algebra of $\mathfrak{B}_i(X)$ modulo $\mathcal{F}$.

(ii) The quotion function $\Pi_{\mathcal{F}} : \mathfrak{B}_i(X) \to \mathfrak{B}_i(X)/\mathcal{F}$ which is clearly B.a.-homomorphism is called the Boolean's algebra canonical homomorphism associated with $\mathcal{F}$.

Lemma 2.5. *(a) Let $\mathfrak{B}_i(X)$ be a Boolean's algebra on a bispace X and let $\mathcal{F}, Q$ be B.a.-filters of $\mathfrak{B}_i(X)$. Then $\mathcal{F} \subset Q$ iff there is a B.a.-homomorphism $\psi_i : \mathfrak{B}_i(X)/\mathcal{F} \to \mathfrak{B}_i(X)/Q$ such that $\Psi \circ \Pi_{\mathcal{F}} = \Pi_Q$.*

(b) Let $u_i(X)$ and $\mathfrak{B}_i(X)$ be Boolean algebra on a bispace X let $\Phi_i : u_i(X) \to \mathfrak{B}_i(X)$ be a B.a. homomorphism associated with a B.a.-filter $\mathcal{F}$. Then there is a unique B.a.-isomorphism $\psi_i : u_i(X)/\mathcal{F} \to \mathfrak{B}_i(X)$ such that $\psi_i \circ \Pi_{\mathcal{F}} = \Phi_i$, where $\Pi_{\mathcal{F}} : u_i(X) \to u_i(X)/\mathcal{F}$.

Theorem 2.2. *Let $\mathcal{F}$ be a proper B.a.-filter of $\mathfrak{B}_i(X)$, then the following statements are equivalent.*

(i) $\mathcal{F}$ is a B.a.-ultra-filter.

(ii) If $a \vee b \in \mathcal{F},\ a,\ b \in \mathfrak{B}_i(X)$, then $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

(iii) If $a \in \mathfrak{B}_i(X)$, then $a \in \mathcal{F}$ or $a' \in \mathcal{F}$.

(iv) $\mathfrak{B}_i(X)/\mathcal{F}$ is B.a.-isomorphism to **2**.

Proof. (i) $\Longrightarrow$ (ii). If $a \vee b \in \mathcal{F}$ and $a \notin \mathcal{F}$, then $< \mathcal{F} \cup \{a\} >= \mathfrak{B}_i(X)$, since $\mathcal{F}$ is a B.a.-ultra-filter and there is $c \in \mathcal{F}$ such that $a \wedge c = \emptyset$ we have $b \geq b \wedge c = (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge c \in \mathcal{F}$ i. e. $b \in \mathcal{F}$.

(ii) $\Longrightarrow$ (iii). For $a \in \mathfrak{B}_i(X)$ we have $a \vee a' = 1 \in \mathcal{F}$ and hence $a \in \mathcal{F}$ or $a' \in \mathcal{F}$ by (i).

(iii) $\Longrightarrow$ (iv). Let $\Pi_{\mathcal{F}} : \mathfrak{B}_i(X) \to \mathfrak{B}_i(X)/\mathcal{F}$ be the B.a.-canonical homomorphism. Since $\mathcal{F}$ is a proper B.a.-filter and the Boolean's algebra $\mathfrak{B}_i(X)/\mathcal{F}$ is proper. If $a \in \mathfrak{B}_i(X)$ and $\Pi_{\mathcal{F}}(a) = 1$, then $a \notin \mathcal{F}$ and hence $a' \in \mathcal{F}$. By (iii) it follows that $\Pi_{\mathcal{F}}(a) = \Pi_{\mathcal{F}}(a'') = (\Pi_{\mathcal{F}}(a'))' = 1' = \emptyset$. Hence $\mathfrak{B}_i(X)/\mathcal{F}$ is a B.a.-isomorphism to **2**. □

3. A Boolean's Algebra Compactification in Bispaces

Remark 3.1. For a Boolean algebra $\mathfrak{B}_i(X)$ we denoted by $S\ (\mathfrak{B}_i(X))$ the set of all B.a.-ultra filters of $\mathfrak{B}_i(X)$. It is clear $|S\ (\mathfrak{B}_i(X))|\ \leq 2\ |\mathbf{B}_i(X)|$ and the set $(S\ (\ \mathfrak{B}_i(X)) = \emptyset$ iff $\mathfrak{B}_i(X)$ is the improper Boolean's algebra.

Lemma 3.1. *Let $\mathfrak{B}_i(X)$be Boolean's algebra and let $\emptyset \neq T \subset S\ (\mathfrak{B}_i(X))$. Define $\Phi_i : \mathfrak{B}_i(X) \to P\ (T)$ by $\Phi_i(a) = \{p \in T : a \in P\ \}$. Then:*

(a) Φ_i is a Boolean's algebra homomorphism.

(b) If for every $a \in \mathfrak{B}_i(X)$ such that $a \neq \emptyset$ there is $p \in T$ such that $a \in p$, then Φ_i is a B.a.-embedding.

Proof. (a) If $a,\ b \in \mathfrak{B}_i(X)$, then

$$\Phi_i\ (a \wedge b) = \{p \in T : a \wedge b \in p\} = \{p \in T : a \in p\} \cap \{p \in T : b \in p\} = \Phi_i\ (a) \cap \Phi_i\ (b)\,.$$

Also

$$(\ \Phi_i\ (a)\ )' \ =\ \Phi_i(a') = \{p \in T : a' \in p\} = \{p \in T : a \notin p\} = P\ (T)\backslash\Phi_i\ (a)\,.$$

(b) If $a,\ b \in \mathfrak{B}_i(X)$ and $a \neq b$ then $\emptyset = (a\ \backslash b) \vee (b\ \backslash a)$, there is $p \in T$ such that $(a\backslash b) \vee (b\backslash a) \in p$, then from $p \in (\Phi_i\ (a)\backslash\Phi_i\ (b)) \cup (\Phi_i\ (b)\backslash\Phi_i\ (a))$ it follows that $\Phi_i\ (a) \neq \Phi_i\ (b)$. □

Remark 3.2. We can define $\psi_i : \mathfrak{B}_i(X) \to P\ (S\ (\mathfrak{B}_i(X)\ ))$ such that $\psi_i(a) = \{p \in S\ (\mathfrak{B}_i(X)) : a \in p\}$, since the two topologies σ_i on $S\ (\mathfrak{B}_i(X))$ are two topologies determined by subbases of $\psi_i\ [\mathfrak{B}_i(X)]$ and the set $S\ (\mathfrak{B}_i(X))$ with two topologies σ_i is called a Boolean's algebra bispace.

Definition 3.1. A bispace X is called **bitotally disconnected** if for $p, q \in X,\ p \neq q$, there is an $(i,j)\lambda_j$-open subset of X containing p which does not contain q.

Definition 3.2. Let $(X,\ \tau_1, \tau_2)$ be bispace. Then $(Y,\ \tau_1^*,\ \tau_2^*)$ is called a Boolean's algebra compactification of $(X,\ \tau_1, \tau_2)$ if Y is both a Boolean's algebra and a jointly compact bispace and if there is a B.a.-embedding function on X onto a jointly dense subset of Y.

Theorem 3.1. *Let $\mathfrak{B}_i(X)$ be a Boolean's algebra on bispace X, then:*

(a) $\psi_i\ [\mathfrak{B}_i(X)]$ are two i-bases for σ_i.

(b) $\psi_i\ [\mathfrak{B}_i(X)] \subset \mathfrak{B}_i\ (S\ (\mathfrak{B}_i(X)))$.

(c) $S\ (\mathfrak{B}_i(X))$ is a jointly compact bitotally disconnected space.

Proof. We note that if $\mathfrak{B}_i(X)$ is the improper Boolean's algebra, then $S\ (\mathfrak{B}_i(X)) = \emptyset$, $\psi_i\ (a)$ is B. a. isomorphism from $\mathfrak{B}_i(X)$ onto $\mathfrak{B}_i(S\ (\mathfrak{B}_i(X)))$ $= \{\emptyset\}$ and the statements are obvious. Thus we assume that $\mathfrak{B}_i(X)$ is a proper Boolean's algebra on bispace X.

(a) We have $\psi_i\ (1) = S\ (\mathfrak{B}_i(X))$ and $\psi_i\ (a) \cap \psi_i\ (b) = \psi_i\ (a \wedge b)$, for $a,\ b \in \mathfrak{B}_i(X)$ by Lemma 3.1.

(b) From the definition of the two topologies σ_i on $S\ (\mathfrak{B}_i(X))$, every element of $\psi_i[\ \mathfrak{B}_i(X)\ ]$ is σ_i -open. Then from Lemma 3.1 (a) it follows that $S\ (\mathfrak{B}_i(X))\backslash\psi_i\ (a) = \psi_i\ (a') \in S\ (\ \mathfrak{B}_i(X))$ for $a \in \mathfrak{B}_i(X)$, so that $\psi_i\ [\mathfrak{B}_i(X)] \subset \mathfrak{B}_i\ (S\ (\mathfrak{B}_i(X)))$.

(c) If $p,\ q \in S\ (\ \mathfrak{B}_i(X)\ (X))$ and $p \neq q$, then $p\ /q \neq \emptyset$ and there is $a \in p\ /q$ follows from (b) that $\psi_i\ (a)$ is σ_i-open and σ_i-closed subset of $S\ (\mathfrak{B}_i(X))$ containing p and does not contain q. Thus $S\ (\mathfrak{B}_i(X))$ is bitotally disconnected. We must prove that $S\ (\mathfrak{B}_i(X))$ is jointly compact, thus, let $A \subset \mathfrak{B}_i(X)$ and suppose that $\{\psi_i\ (a) : a \in A\}$ is a σ_i-open cover of $S\ (\mathfrak{B}_i(X))$which has no finite subcover. Then the family $\mathcal{F} = \{a' : a \in A\}$ has the finite intersection property and hence there is $p \in S\ (\mathfrak{B}_i(X))$ such that $\mathcal{F} \subset p$. Since $\{\psi_i\ (a) : a \in A\}$ is a cover of $S\ (\mathfrak{B}_i(X))$, there is $a \in A$ such that $p \in \psi_i\ (a)$ i.e. $a \in p$. It follows that $\emptyset = a \wedge a' \in p$, this is a contradiction for the definition of p thus $S\ (\mathfrak{B}_i(X))$ is jointly compact.

(d) From (b) and Lemma 3.1 (b) ψ_i is a B.a.-embedding of $\mathfrak{B}_i(X)$ into $\mathfrak{B}_i\ (S\ (\mathfrak{B}_i(X)))$ i.e. $\mathfrak{B}_i(X) \subset \mathfrak{B}_i\ (S\ (\mathfrak{B}_i(X)))$. We verify that $\psi_i\ [\mathfrak{B}_i(X)] = \mathfrak{B}_i\ (S\ (\mathfrak{B}_i(X)))$, so, let $A \in \mathfrak{B}_i\ (S\ (\mathfrak{B}_i(X)))$. Then by (a), there is $\{a_k : k \in I\} \subset \mathfrak{B}_i(X)$ such that:

$$A = \cup\{\psi_i\ (a_k) : k \in I\}.$$

Since A is jointly compact there are $n < \omega$ and $l \in I$ for such that:

$$A = \psi_i\ (a_0) \cup \cdots \cup \psi_i\ (a_n) = \psi_i\ (a_0 \vee a_1 \vee \cdots \vee a_n) \in \psi_i\ [\mathfrak{B}_i(X)].$$

So, $\mathfrak{B}_i\ (S\ (\mathfrak{B}_i(X))) \subset \mathfrak{B}_i(X)$. The proof of this theorem is complete. □

Lemma 3.2. *If X is a bitotally disconnected bispace $p \in X$, then*

$$\{\ a \in \mathfrak{B}_i(X) : p \in a\ \} \in S\ (\mathfrak{B}_i(X)\ ).$$

Theorem 3.2. *Let X be a bitotally disconnected bispace, then the following statements arse satisfied:*

(a) $h_i\ [X] \cap \psi_i\ (A) = h_i\ [A]$, for $A \in \mathfrak{B}_i(X)$, where $h_i : X \to S\ (\mathfrak{B}_i(X))$ such that $h_i\ (p) = \{a \in \mathfrak{B}_i(X) : p \in a\}$.

(b) h_i is a bicontinuous function.

(c) $cl_{\sigma i}\ h_i\ [A] = \psi_i\ (A)$, $A \in \mathfrak{B}_i(X)$.

(d) $h_i\ (X)$ is a jointly dense set in $S\ (\mathfrak{B}_i(X))$.

(e) $p = \{A \in \mathfrak{B}_i(X) : p \in cl_{\sigma_i}\ h_i\ [A]\}$, for $p \in S\ (\mathfrak{B}_i(X))$.

(f) $cl_{\sigma_i}\ h_i\ [A] \cap cl_{\sigma_i}\ h_i\ [B] = cl_{\sigma_i}\ h_i\ [A \cap B]$, for $A,\ B \in \mathfrak{B}_i(X)$.

(g) If $\mathfrak{B}_i(X)$ are two subbases for two topologies τ_i in X, then h_i is a topological biembedding.

Proof. Obvious:

(a) It is clear from the definition of ψ_i and h_i.

(b) The family $\{h_i\ [X] \cap \psi_i\ (A) : A \in \mathfrak{B}_i(X)\}$ is i-base for $h_i\ (X)$ and from (a) and the fact h_i is one-to-one we have:

$$h_i^{-1}\ (h_i\ [X] \cap \psi_i\ (A)) = A \text{ for } A \in \mathfrak{B}_i(X).$$

(c) For $A \in \mathfrak{B}_i(X)$ we have $\psi_i\ (A) \in \mathfrak{B}_i\ (S\ (\mathfrak{B}_i(X)))$ and by Theorem 3.1. (b). $cl_{\sigma_i}\ h_i\ [A] \subset \psi_i\ (A)$ by part (a). Conversely, let, $p \in S\ (\mathfrak{B}_i(X))|cl_{\sigma_i}\ h_i\ [A]$ by Theorem 3.1 (a), there is $B \in \mathfrak{B}_i(X)$ such that $h_i\ [A] \subset \psi_i\ (B)$ and $p \notin \psi_i\ (B)$, we have $h_i\ [A] \subset h_i\ (B)$ by part (a), so from the fact h_i is one-to-one it follows that $A \subset B$ and hence: $\psi_i\ (A) \subset \psi_i\ (B)$. Thus $p \notin \psi_i\ (A)$.

(d) From the part (c) we have; $cl_{\sigma_i}\ h_i\ [X] = \psi_i\ (X) = S\ (\mathfrak{B}_i(X))$.

(e) Is equivalent to (c).

(f) Follows from (e).

(g) Since $\{\psi_i\ (A) \cap h_i\ [X] : A \in \mathfrak{B}_i(X)\}$ is an i-base for $h_i\ [X]$ and is a one-to-one function and from the part (a). Thus h_i is topological biembedding. □

4. Bizero Dimensional Spaces and the Stone-Čech Compactification in Bispace

Definition 4.1. The set of the inverse image of $\{0\}$ in the bicontinuous function $C_i\ (X) = \{f : X \to R\}$ is called a bizero set and the set of all bizero sets is denoted by $Z_i\ (X)$.

Definition 4.2. (see [9]) The Stone-Čech compactification $\beta_i\ X$ of a bispace X is the set all bi-Z_i-ultrafilters on X with the topologies ω_i determined by the subbase:

$$\mathfrak{B}_i = \{p_i \in \beta_i X : A \notin p_i : A \in Z_i\ (X)\}.$$

Definition 4.3. A bispace X is called bizero-dimensional if for A, $B \in Z_i\ (X)$ with $A \cap B = \emptyset$ there is $C \in \mathfrak{B}_i(X)$ such that $A \subset C$ and $B \cap C = \emptyset$.

Lemma 4.1. *Let X be a bitotally disconnected space and let $\bar{h}_i : \beta_i\ X \to S\ (\ \mathfrak{B}_i(X))$ be a extension of h_i. Then:*

(a) $\bar{h}_i\ (p) = p \cap \mathfrak{B}_i(X)$ for $p \in \beta_i\ (X)$.

(b) If X is bizero-dimensional, then $\bar{h}_i$ is a B.a.-homomorphism from $\beta_i X$ onto $S\ (\mathfrak{B}_i(X))$.

Proof. (a) It is clear that if $p \in \beta_i\ X$, then $p \cap \mathfrak{B}_i(X) \in S\ (\mathfrak{B}_i(X))$. Further if $A \in P \cap \mathfrak{B}_i(X)$, then $p \in cl_{\omega_i} A$ and from bicontinuity of $\bar{h}_i$, we have $\bar{h}_i\ (p) \in cl_{\sigma_i}\ h\ [A] = \psi_i\ (A)$ by Theorem 3.1, hence $A \in \bar{h}_i\ (p)$. Thus $P \cap \mathfrak{B}_i(X) \subset \bar{h}_i$ and since $P \cap \mathfrak{B}_i(X)$ is a B.a. ultra filter of $\mathfrak{B}_i(X)$ it follows that $P \cap \mathfrak{B}_i(X) = \bar{h}_i\ (p)$.

(b) Since $\beta_i X$ is jointly compact we have $\bar{h}_i\ [\beta_i X] = S\ (\mathfrak{B}_i(X))$ by Theorem 3.1 and it enough to prove that $\bar{h}_i$ is one-to-one. If $p,\ q \in \beta_i X$ and $p \neq q$, then there are $A,\ B \in Z_i\ (X)$ such that $A \in p$ and $B \in q$ and $A \cap B = \emptyset$. Since X is bizero dimensional there are $C,\ D \in \mathfrak{B}_i(X)$ such that $A \subset C$, $B \subset D$ and $C \cap D = \emptyset$ and by part (a) we have $\bar{h}_i\ (p) \neq \bar{h}_i\ (q)$. □

Lemma 4.2. *If X is jointly compact, then the following statements are equivalent:*

(a) X is bizero dimensional.

(b) $\mathfrak{B}_i(X)$ is an i-base for X.

(c) X is bitotally disconnected.

Proof. $a \Longrightarrow b$ is obvious. $b \Longrightarrow c$ is clear. To prove $c \Longrightarrow a$ Let $A,\ B \in Z_i\ (X)$, such that $A \cap B =$ from (c), there is $A_{p,\ q} \in \mathfrak{B}_i(X)$ such that $p \in A,\ q \in B$. Then family $\{A_{p,\ q} : p \in A\}$ is τ_i-open cover of a jointly compact set A. Thus there are $n\ (q) < \omega$ and $p_0,\ \cdots,\ p_{(n)_q} \in A$ such that $A \subset A_{p_0,\ q} \cup \cdots U\ p_{n,\ q}$, for $q \in B$. We set $B_q = X \backslash \{A_{p_0,\ q},\ \cdots,\ A_{p_{n\ (q),\ q}}\}$ so that $B_q \in \mathfrak{B}(X)$, $\alpha \in B$. The family $\{\ B_q : q \in B\ \}$ is τ_i-open cover of B and thus there are $n < \omega$ and $q_0,\ q_1,\ ...,\ q_n \in B$ such that $B \subset B_{q_0} \cup \cdots \cup\ B_{q_n}$. Set $C = B_{q_0} \cup \cdots \cup\ B_{q_n}$. Then $C \in \mathfrak{B}_i(X)$. $B \subset C$, $A \cap C = \emptyset$. Hence X is bitotally disconnected. □

Corollary 4.1. *If X is a jointly compact, bitotally disconnected space, then $h_i : X \to S\ (\ \mathfrak{B}_i(X))$ is a B.a. homomorphism.*

Proof. This follows from Lemma 4.1 and Lemma 4.2. □

Theorem 4.1. *Let X be a bispace. Then X is bizero dimensional iff $\beta_i X$ is bizero dimensional.*

Proof. Let X is bizero dimensional and $A,\ B \in Z_i\ (X)$ and $A \cap B = \emptyset$, then there is a bicontinuous function $f : \beta_i X \to [0,\ 1]$ such that $f\ (p) = 0$ for $p \in A$ and $f\ (p) = 1$ and if $p \in B$. Consider $E = \{p \in X : f\ (p) \leq 1/3\}$, $F = \{p \in X : f\ (p) \leq 2/3\}$ since $E,\ F \in Z_i\ (X)$ and $E \cap F = \emptyset$, there is Let $C \in \mathfrak{B}_i(X)$ such that $F \cap C = \emptyset$. It is clear that $\mathrm{cl}_{\omega_i}\ C \in \mathfrak{B}_i(\beta_i\ X)$ and that $A \subset cl_{\omega_i}\ C$ and $B \cap \mathrm{cl}_{\omega_i}\ C = \emptyset$. By Lemma 4.1 (b) $\beta_i\ X$ is B.a.-homomorphic to $S\ (\mathfrak{B}_i(X))$ and so that $\beta_i X$ is bizero dimensional by Lemma 4.2 and Theorem 3.1. Conversely. Let $\beta_i\ X$ be bizero dimensional and let $A,\ B \in Z_i$ and $A \cap B = \emptyset$. Then $\mathrm{cl}_{\omega_i}\ A \cap cl_{\omega_i}\ B = \emptyset$, there are $C,\ D \in Z_i\ (\beta_i\ X)$ such that $\mathrm{cl}_{\omega_i}\ A \subset C$, $\mathrm{cl}_{\omega_i}\ B \subset D$ and $C \cap D = \emptyset$. Since $\beta_i\ X$ is bizero dimensional, there is $E \in \mathfrak{B}_i(\beta_i\ X)$ such that $C \subset E$ and $E \cap D = \emptyset$. Thus $E \cap X \in \mathfrak{B}_i(X)$,

$A \subset E \cap X$ and $B \cap (E \cap X) = \emptyset$. Hence X is bizero dimensional. □

Remark 4.1. (see [5]) If $\lambda_\gamma = \mathrm{cl}_{\tau_i}\ \mathrm{int}_{\tau_i}$, then:

(i) $(i,\ j)\lambda_\gamma$ open is called $(i,\ j)$ semi-open.

(ii) The Boolean's algebra $\mathfrak{B}_i(X)$ is the collection of $(i,\ j)$ semi-open sets and $(i,\ j)$ semi-closed sets.

Example 4.1. A bispace $X = [0,\ 1]$ with two topologies τ_1 = discrete topology, $\tau_2 = \{X,\ \emptyset,\ (a,\ 1] : a \in X\}$ is not jointly semi-compact because the cover $\{\ \{1\},\ [0,\ a) : a \in X\}$ has not a finite subcover. Thus, if we take $S\ (\mathfrak{B}_i(X))$ the collection of the Boolean algebra-ultrafiters on $S\ (\mathfrak{B}_i(X))$ with two topologies σ_i on $S\ (\mathfrak{B}_i(X))$ generated by subbase, $\psi_i\ (S\ (\mathfrak{B}_i(X)))$ where $\psi_i : \mathfrak{B}_i(X) \to P\ (S\ (\mathfrak{B}_i(X)))$ is defined as $\psi_i\ (a) = \{p \in S\ (\mathfrak{B}_i(X)) : a \in P\}$. we get the bispace $S\ (\mathfrak{B}_i(X),\ \sigma_1, \sigma_2)$ is a Boolean's algebra compactification of the bispace $(X, \tau_1,\ \tau_2)$.

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