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COHERENT SYSTEMS ON $\mathbf{P}^1_{\mathbb{F}_q}$

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Abstract: Here we study the stability of coherent systems defined over $\mathbb{F}_q$ on genus 0 curves over $\mathbb{F}_q$.

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1. Genus 0 Coherent Systems over a Finite Field

Let K be a non-algebraically closed field and $\mathbb{K}$ an algebraic closure of K. For any algebraic K-scheme Y and any field L such that $K \subseteq L \subseteq \mathbb{K}$ set $Y_L := Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. Let $Y(L)$ or $Y_L(L)$ denote the set of all equivalence classes of L-morphism $\mathrm{Spec}(L) \to Y$ with L as their residue field. For any vector bundle E on Y, i.e. for any locally free coherent algebraic sheaf E on Y, let E_L (or just E) the induced locally free sheaf on Y. Let X be a smooth and geometrically connected projective curve defined over K. A coherent pair on X over K is a pair (E, V) such that E is a vector bundle on X defined over K and $V \subseteq H^0(X, E)$ is a linear subspace. The pair (E, V) is of type (n, d, k) if $\mathrm{rank}(E) = n$, $\deg(E) = d$ and $\dim(V) = k$. Hence (E, V) is uniquely determined by E and by a map of $\mathcal{O}_X$-sheaves $\mathcal{O}_X{}^{\oplus k} \to E$ which induces an injection of global sections. Fix $\alpha \in \mathbb{R}$.Let $\mu(E) := d/n$ denote the slope

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of E. Set $\mu_\alpha(E,V) := \mu(E) + \alpha k/n$. The real number μ_α is called the α-slope of the pair (E,V). A subcoherent system $(F,W) \subseteq (E,V)$ is a coherent system such that $F \subseteq E$ and $W \subseteq V \cap H^0(X,F)$. The pair (E,V) is said to be α-stable (resp. α-semistable) if $\mu_\alpha(F,W) < \mu_\alpha(E,V)$ (resp. $\mu_\alpha(F,W) \le \mu_\alpha(E,V)$) for all subcoherent systems (F,W) of (E,V). Let $G(X;\alpha : n,d,k)_K$ (resp. $\tilde{G}(X;\alpha : n,d,k)_K$) denote the moduli scheme of all α-stable (resp. equivalence classes of α-semistable) coherent systems of type (n,d,k) on X. For the general theory of coherent systems and several results on the moduli schemes of α-coherent systems over $\mathbb{K}$, see [8], [5], [3], [6], [7] and [4] (at least in characteristic zero). Here we will give some results in the case $K = \mathbb{F}_q$ and $g = 0$. Notice that even when $K = \mathbb{C}$ the non-emptiness of most $G(X;\alpha : n,d,k)_\mathbb{C}$ is not known (see [6]). We recall that a smooth and geometrically connected genus 0 curve X defined over $\mathbb{F}_q$ is isomorphic over $\mathbb{F}_q$ to $\mathbf{P}^1$ (e.g. use Hasse-Weil, or that the anticanonical divisor is defined over $\mathbb{F}_q$ and that a plane conic with coefficients in $\mathbb{F}_q$ has at least a point defined over $\mathbb{F}_q$ by Chevalley-Warning). Hence $\sharp(X(\mathbb{F}_q)) = q+1$. More generally, any smooth and geometrically connected genus 0 curve defined over a field K and such that $X(K) \neq \emptyset$ is isomorphic over K to $\mathbf{P}^1$, because the anticanonical divisor is defined over K, it is very ample and any smooth conic with coefficients in K and with a point P with coordinates in K is isomorphic over K to $\mathbf{P}^1$ (use the stereographic projection from P). Hence in both cases all vector bundles on $\mathbf{P}^1_\mathbb{K}$ are defined over K and every indecomposable vector bundle over K is gemoetrically indecomposable: just use the Harder-Narasimhan filtration, $h^0(\mathbf{P}^1,E) = h^0(\mathbf{P}^1_\mathbb{K},E_\mathbb{K})$ and hence E is trivial if $E_\mathbb{K}$ is trivial) and hence it is a line bundle. If K is infinite, then $h^0(\mathbf{P}^1,E)$ is a Zariski dense subset of the affine space $H^0(\mathbf{P}^1_\mathbb{K},E_\mathbb{K})$. By the openness of α-stability we get the following result, the latter part following from the fact that the set of all K-points of any Grassmannian is Zariski dense in the set of all $\mathbb{K}$-points of that Grassmannian.

Theorem 1. *Fix $\alpha \in \mathbb{R}$ and positive integers d,n,k. Let K be an infinite field and $\mathbb{K}$ its algebraic closure. Then $G(\mathbf{P}^1_K;\alpha : n,d,k)_K$ (resp. $\tilde{G}(\mathbf{P}^1_K;\alpha : n,d,k)_K$) is non-empty if and only if $G(\mathbf{P}^1_\mathbb{K};\alpha : n,d,k)_\mathbb{K}$ (resp. $\tilde{G}(\mathbf{P}^1_\mathbb{K};\alpha : n,d,k)_\mathbb{K}$) is non-empty. Let E be a vector bundle on $\mathbf{P}^1_K$ defined over K. There is a k-dimensional linear subspace V of the K-vector space $H^0(\mathbf{P}^1_K,E)$ such that the coherent system (E,V) is geometrically α-stable if and only if there is a k-dimensional linear subspace W of the $\mathbb{K}$-vector space $H^0(\mathbf{P}^1_\mathbb{K},E_\mathbb{K})$ such that the coherent system $(E_\mathbb{K},W)$ is α-stable.*

Theorem 1 explains why from now on we only consider the case $K = \mathbb{F}_q$. From now on set $\mathbf{P}^1 := \mathbf{P}^1_{\mathbb{F}_q}$. Here we prove the following results.

Theorem 2. *Fix a prime-power q and integers $n > t \ge 0$ and $a \ge 2$. Set $E := \mathcal{O}_{\mathbf{P}^1}(a)^{n-t} \otimes \mathcal{O}_{\mathbf{P}}(a-1)^t$. Then there exists an $(n+1)$-dimensional $\mathbb{F}_q$-linear subspace V of $H^0(\mathbf{P}^1, E)$ such that the coherent system (E, V) is geometrically α-stable for all $\alpha > t$.*

Theorem 3. *Fix a prime-power q and integers $n > t \ge 0$ and $a >$ $)n+1 = t/n$. Set $E := \mathcal{O}_{\mathbf{P}^1}(a)^{n-t} \otimes \mathcal{O}_{\mathbf{P}}(a-1)^t$. Then there exists an $(n-1)$-dimensional $\mathbb{F}_q$-linear subspace V of $H^0(\mathbf{P}^1, E)$ such that the coherent system (E, V) is geometrically α-stable for all α such that $n - t < \alpha < a - t/n$.*

The following result was proved in [1] over an algebraically closed field; its truth for an arbitrary base field K follows from its truth over some algebraically closed field containing K.

Lemma 1. *Fix an integer $n > 0$ and a smooth and connected projective curve X. Let E be a rank n vector bundle on X and $V \subseteq H^0(X, E)$ such that $\dim(V) = n+1$ and V spans E. Assume that $\mathcal{O}_X$ is not a direct factor of E. Then for all integers $1 \le r \le n-1$ and all linear subspaces $W \subset V$ such that $\dim(W) = r+1$ the evaluation map $e_{W,E} : W \otimes \mathcal{O}_X \to E$ has image with rank $r+1$.*

Lemma 2. *Fix a rank n spanned vector bundle E on $\mathbf{P}^1$ and an integer r such that $1 \le r < n$. Then there exists an r-dimensional $\mathbb{F}_q$-linear subspace V of $H^0(\mathbf{P}^1, E)$ such that the evaluation map $e_{E,V} : V \otimes \mathcal{O}_{\mathbf{P}^1} \to E$ is injective and with locally free cokernel.*

Proof. Taking out trivial factors of E (if any) we reduce to the case in which E is ample, say with splitting type $a_1 \ge \cdots \ge a_n > 0$. First assume $r = 1$. Since $\sharp(\mathbf{P}^1(\mathbb{F}_q)) = q+1 \ge 2$, we may fix $P, Q \in \mathbf{P}^1(\mathbb{F}_q)$ such that $P \ne Q$. Take $(a_1 P, \ldots, a_{n-1}P, a_n Q)$ has non-zero section of E. Now assume $r \ge 2$ and that the lemma is true for all integers $r' < r$. Fix an $(r-1)$-dimensional $\mathbb{F}_q$-linear subspace W of $H^0(\mathbf{P}^1, E)$ such that the evaluation map $e_{E,W}$ is injective and with locally free cokernel. Then apply the case $r = 1$ to the vector bundle $\mathrm{Coker}(e_{E,W})$ and use that $h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}^{\oplus(r-1)}) = 0$. □

Lemma 3. *Fix a spanned vector bundle E on $\mathbf{P}^1$ such that $\deg(E) > 0$. Then there exists an $(n+1)$-dimensional $\mathbb{F}_q$-linear subspace V of $H^0(\mathbf{P}^1, E)$ spanning E at each point of $\mathbf{P}^1(\bar{\mathbb{F}}_q)$.*

Proof. Taking out trivial factor of E we reduce to the case in which E is ample. Set $n := \mathrm{rank}(E)$ and $d := \deg(E) \ge n$. First assume $n = 1$. Fix $P, Q \in \mathbf{P}^1(\mathbb{F}_q)$ such that $P \ne Q$ and take the pencil V spanned by the two effective divisors dP and dQ. Now assume $n \geq 2$. Apply the case $r := n-1$ of Lemma 2 and use that $h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}^{\oplus(r-1)}) = 0$. □

Proof of Theorem 2. By Lemma 2 there is a $(n+1)$-dimensional linear subspace V of $H^0(\mathbf{P}^1, E)$ spanning E. We fix any such linear subspace V and show (exactly as in the proof of [6], Proposition 6.4) that the coherent system (E, V) is geometrically α-stable for all $\alpha > t$. Fix an integer r such that $1 \leq r, n$ and a coherent subsystem (F, W) of (E, V) such that $\mathrm{rank}(F) = r$. Since $F \subset E$ we have $\mu(F) \leq a$. By Lemma 1 we have $\dim(W) \leq r$ and hence $\mu_\alpha \leq a + \alpha$. Since $\mu_\alpha(E, V) = a - t/n + (n+1)\alpha/n$, we are done. □

Proof of Theorem 3. By Lemma 2 there is an $(n-1)$-dimensional linear subspace V of $H^0(\mathbf{P}^1, E)$ defined over $\mathbb{F}_q$ such that the evaluation map $e_{E,V}$ is injective and with locally free cokernel. Fix an integer r such that $1 \leq i \leq n-1$ and assume the existence of a rank r subsheaf F of E (defined over $\bar{\mathbb{F}}_q$) and $W \subseteq V_{\bar{\mathbb{F}}_q} \cap H^0(\mathbf{P}^1_{\bar{\mathbb{F}}_q}, F)$ such that $\mu_\alpha(F, W) \geq \mu_\alpha(E, V) = a - t/n + (n-1)\alpha/n$. First assume G trivial. Thus $\mu_\alpha(G, V) \leq \alpha$, contradicting the asssumption $\alpha < a - t/n$. Now assume F non-trivial. Since $F \subset E$, we have $\mu(F) \leq a$. Since $\mathrm{Im}(e_{E,V})$ is saturated in E, we have $\dim(F) \leq r-1$. Thus $\mu_\alpha(F, W) \leq a + (r-1)\alpha/r$. First assume $r \leq n-t$. Since $\alpha \geq 1$, we get $\mu_\alpha(F, W) \leq a(n-t-1)\alpha/(n-t)$. Now assume $r > n-t$. Thus $\mu(F) \leq (a(n-t) + (a-1)(r-n+t))/r = a - (r-n+t)/r$. Hence $\mu_\alpha(F, W) \leq a - (r-n+t)/r + (r-1)\alpha/r$. □

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