International Journal of Pure and Applied Mathematics
Volume 28 No. 3 2006, 363-368

SOME RESULTS FOR SEMI DERIVATIONS OF PRIME RINGS

Alev Firat

Department of Mathematics
Faculty of Science
Ege University
Bornova, Izmir, 35100, TURKEY
e-mail: alev.firat@ege.edu.tr

Abstract: Let R be a prime ring. By a semi-derivation associated with a function $g : R \to R$, we mean an additive mapping $f : R \to R$ such that $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and $f(g(x)) = g(f(x))$ for all $x, y \in R$. In this paper we try to generalize some properties of prime rings with derivations to the prime rings with semi-derivations.

AMS Subject Classification: 16A70, 16A72, 16A10, 46H99
Key Words: semi derivation, prime ring, commuting mapping

1. Introduction

Let R be a ring with center $C(R)$. For any $x, y \in R$;$[x, y]$, (x, y) will denote $xy-yx$, $xy+yx$ respectively, and we make extensive use of the basic commutator identities: $(xy, z) = (x, z)y + x[y, z] = [x, z]y + x(y, z)$, $[xy, z] = [x, z]y + x[y, z]$.

A ring R is to be $n - torsionfree$ if $nx = 0, x \in R$ implies $x = 0$.

Recall that a ring R is *prime* if $aRb = 0$ implies that $a = 0$ or $b = 0$.

A mapping F from R to R is said to be *commuting* on R if $[F(x), x] = 0$ holds for all $x \in R$, and is said to be *centralizing* on R if $[F(x), x] \in C(R)$ holds for all $x \in R$.

Received: April 4, 2006

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$.

We introduce the notion of a semi-derivation which is a generalization of derivation in Bergen [1].

Definition. Let R be an associative ring. An additive mapping $f : R \to R$ is called a *semi-derivation* associated with a function $g : R \to R$ if, for all $x, y \in R$:
(i) $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$;
(ii) $f(g(x)) = g(f(x))$.

If $g = I$, i.e., an identity mapping of R, then all semi derivations associated with g are merely ordinary derivations. If g is any endomorphism of R, then other examples of semi-derivations are of the form $f(x) = x - g(x)$. For an example of a semi-derivation which is not a derivation, let $R = R_1 \bigoplus R_2$ where R_1 and R_2 are any rings. Let $\alpha_1 : R_1 \to R_1$ be an additive map $\alpha_2 : R_2 \to R_2$ be a left and right R_2 -module which is not a derivation. Define$f : R \to R$ such that $f((r_1, r_2)) = (0, \alpha_2(r_2))$ and $g : R \to R$ such that $g((r_1, r_2)) = (\alpha_1(r_1), 0)$, $r_1 \in R_1$, $r_2 \in R_2$. Then it can be easily seen that f is a semi-derivation of R (with associated map g)which is not a derivation. In case R is prime and $f \neq 0$, it has been shown that by Chang [3], Theorem 1, that g must necessarily be a ring endomorphism.

In [4], I.S. Chang, K.W. Jun and Y.S. Jung prove that if there exist a derivation D on a non-commutative 2-torsion free prime ring R such that the mapping $x \to [aD(x), x]$ is commuting on R, then $a = 0$ or $D = 0$. We proved that this conclusion for semi-derivations of prime rings as follows in Theorem 1.

In [5], K. Kaya, Ö. Golbasi, N. Aydin proved that if R is a prime ring of characteristic different from 2, d is a nonzero derivation of R, then $(d(R), a) = 0$, if and only if, $((R, a)) = 0$. We also proved that this conclusion for semi-derivations of prime rings as follows in Theorem 2.

Theorem 1. *Let R be a non-commutative 2-torsion free prime ring and f is a semi-derivation of R with $g : R \to R$ is an onto endomorphism. If the mapping $x \to [af(x), x]$ is commuting on R, then $a = 0$ or $f = 0$.*

Proof. Firstly, we assume that a be a non zero element of R. Then by [2, Theorem 1], the mapping $x \to af(x)$ is commuting on R. Thus we have

$$[af(x), x] = 0, \text{ for all } x \in R. \tag{1}$$

By linearizing (1), we have

$$[af(x), y] + [af(y), x] = 0, \text{ for all } x, y \in R. \tag{2}$$

From this relation it follows that

$$a[f(x),y]+[a,y]f(x)+a[f(y),x]+[a,x]f(y)=0,\ \text{for all } x,y\in R\,. \tag{3}$$

Replacing y by yx in (2) and using (1), we get

$$\begin{aligned}0=a[f(x),y]x+[a,y]f(x)x+a[f(y),x]x+[a,x]f(y)x+ag(y)[f(x),x]\\ +a[g(y),x]f(x)+[a,x]g(y)f(x),\ \text{for all } x,y\in R\,.\end{aligned} \tag{4}$$

Right multiplication of (3) by x gives

$$a[f(x),y]x+[a,y]f(x)x+a[f(y),x]x+[a,x]f(y)x=0,\ \text{for all } x,y\in R\,. \tag{5}$$

Subtracting (5) from (4), we obtain

$$ag(y)[f(x),x]+a[g(y),x]f(x)+[a,x]g(y)f(x)=0,\ \text{for all } x,y\in R\,. \tag{6}$$

Taking $ag(y)$ instead of $g(y)$ in (6), we have

$$\begin{aligned}0=a^2g(y)[f(x),x]+a^2[g(y),x]f(x)+a[a,x]g(y)f(x)+[a,x]ag(y)f(x),\\ \text{for all } x,y\in R\,.\end{aligned} \tag{7}$$

Left multiplication of (6) by a leads to

$$a^2g(y)[f(x),x]+a^2[g(y),x]f(x)+a[a,x]g(y)f(x)=0,\ \text{for all } x,y\in R\,. \tag{8}$$

Subtracting (8) from (7), we get

$$[a,x]ag(y)f(x)=0,\ \text{for all } x,y\in R\,.$$

Since R is prime, we obtain that for any $x\in R$ either $f(x)=0$ or $[a,x]=0$. It means that R is the union of its additive subgroups $P=\{x\in R:f(x)=0\}$ and $Q=\{x\in R:[a,x]a=0\}$. Since a group cannot be the union of two proper subgroups, we find that either $P=R$ or $Q=R$.

If $P=R$, then $f=0$. If $Q=R$, then this implies that:

$$[a,x]a=0,\ \text{for all } x\in R.$$

Let us take xy instead of x in this relation. Then we get

$$[a,x]ya=0,\ \text{for all } x\in R.$$

Since $a \in R$ is nonzero and R is prime, we obtain $a \in C(R)$. Thus by this and (1), the relation (6) reduces to

$$a[g(y), x]f(x) = 0, \text{ for all } x, y \in R.$$

Since g is onto, we see that

$$az[u, x]f(x) = 0, \text{ for all } x, u, z \in R.$$

Now by primeness of R, we obtain that

$$[u, x]f(x) = 0, \text{ for all } x, u \in R.$$

Replacing u by uw, we get

$$[u, x]wf(x) = 0, \text{ for all } x, u, w \in R.$$

Again using the fact that a group cannot be the union of two proper subgroups, it follows that $f = 0$, since R is noncommutative. Hence we see that, in any case, $f = 0$. This completes the proof. □

Theorem 2. *Let R be a prime ring of characteristic different from two, f is a nonzero semi-derivation of R, with associated endomorphism g and $a \in R$. If $g \neq \mp I$ (I is an identity map of R), then $(f(R), a) = 0$ if, and only, if $f((R, a)) = 0$.*

Proof. Suppose $(f(R), a) = 0$. Firstly, we will prove that $f(a) = 0$. If $a = 0$ then $f(a) = 0$. So we assume that $a \neq 0$. By our hypothesis, we have

$$(f(x), a) = 0, \text{ for all } x \in R.$$

From this relation, we get

$$\begin{aligned} 0 = (f(xa), a) &= (f(x)g(a) + xf(a), a) \\ &= f(x)[g(a), a] + (f(x), a)a + x(f(a), a) - [x, a]\,, \end{aligned}$$

and so,

$$[x, a]f(a) = 0, \text{ for all } x \in R. \tag{9}$$

Now, replacing x by xy in (9), we get

$$[R, a]Rf(a) = (0). \tag{10}$$

The primeness of R implies that $a \in C(R)$ or $f(a) = 0$. Now suppose that $a \in c(R)$. Then we obtain that

$$0 = (f(a), a) = f(a)a + af(a) = 2af(a)\,.$$

Since the characteristic of R is different from 2, $af(a) = 0$. Since we assumed that $0 \neq a$ and R is prime ring, we get $f(a) = 0$. Hence we have

$$f((r,a)) = f(ra + ar) = (f(r), a) + (g(r), f(a)),\ \text{for all } x \in R.$$

This yields that

$$f((R,a)) = 0.$$

Conversely, for all $x \in R$,

$$\begin{aligned} 0 = f((ax,a)) &= f(a(x,a) + [a,a]x) \\ &= f(a(x,a)) = f(a)(x,a) + g(a)f((x),a)\,. \end{aligned}$$

By hypothesis we have

$$f(a)(x,a) = 0,\ \text{for all } x \in R. \tag{11}$$

Replacing x by xy in (11), we get

$$0 = f(a)(xy,a) = f(a)x[y,a] + f(a)(x,a)y = f(a)x[y,a].$$

This implies that

$$f(a)R[R,a] = 0.$$

For the primeness of R, we have either $f(a) = 0$ or $a \in C(R)$. If $f(a) = 0$, then we have

$$0 = f((r),a) = (f(r),a) + (f(a), g(r)) = (f(r),a),\ \text{for all } r \in R.$$

This yields that $(f(R),a) = 0$. If $a \in C(R)$, then we have

$$0 = f((a,a)) = 2f(a)(a + g(a)).$$

Since the characteristic of R is different from 2, we obtain $f(a)(a+g(a)) = 0$. Since R is prime we have $f(a) = 0$ or $a + g(a) = 0$. But since g is different from $\mp I$ we find that $f(a) = 0$. Finally, $(f(R),a) = 0$ implies the required result.

References

[1] J. Bergen, Derivations in prime rings, *Canad. Math. Bull.*, **26** (1983), 267-270.

[2] M. Bresar, On a generalization of the notation of centralizing mappings, *Proc. Amer. Math. Soc.*, **114** (1992), 641-649.

[3] J.-C. Chang, On semi-derivations on prime rings, *Chinese J. Math.*, **12** (1984), 255-262.

[4] I.-S. Chang, K. W. Jun, Y-S. Jung, On derivations in Banach algebras, *Bull. Korean Math. Soc.*, **39**, No. 4 (2002), 635-643.

[5] K. Kaya, Ö. Golbasi, N. Aydin, Some results for generalized Lie ideals in prime rings with derivation II, *Applied Mathematics, E-Notes*, **1** (2001), 24-30.