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ALMANSI FUNCTIONS IN $\mathbf{R}^n$

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Abstract: Every polyharmonic function in a star domain has a representation given by Almansi [2]. Generalizing this representation, we introduce a class of functions called Almansi functions in an open set in $\mathbf{R}^n$, $n \geq 2$; and investigate their properties. In particular, every polyharmonic function in a star domain is an Almansi function.

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1. Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. In this paper we consider ω to be a star domain in $\mathbf{R}^n$ with center 0. If $u \in C^2(\omega)$, define the Laplace of u to be $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$. Consequently, if $u \in C^{2p}(\omega)$, we say that u is *p-harmonic* (or *polyharmonic* of order p) if $\Delta^p u = \Delta^{p-1}(\Delta u) = 0$. For any positive r, the open ball $B_r(z)$ with center 0 and radius r is defined by

$$B_r(z) = \{x \in \omega \text{ such that } \|x - z\| < r\}.$$

It is well-known that a p-harmonic function $u(x)$ in $B_r(z)$ can be expressed in the form $u(x) = \sum_{i=1}^{m} \|x - z\|^{2i-2} h_i(x)$, where $h_i(x)$ is harmonic on $B_r(z)$. We generalize this situation as follows: Recall that ω is said to be a star domain with center 0, if ω is an open connected set containing 0 such that for any $x \in \omega$, the line joining 0 and x lies in ω.

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Definition. A continuous function u on a star domain ω with center 0 in $\mathbf{R}^n$, $n \geq 2$, is said to be an *Almansi function* of order m, if there exist $m+1$ harmonic functions h_i on ω such that $u(x) = \sum_{i=0}^{m} |x|^i h_i(x)$ for $x \in \omega$.

Similar functions were considered in [5] in a different fashion as a consequence of studying polyharmonic functions in the whole of $\mathbf{R}^n$.

2. Main Results

The following result is well known, see [4] for more details.

Theorem 1. *Every bounded harmonic function on $\mathbf{R}^n$ is a constant function.*

We shall prove the following theorem.

Theorem 2. *Every bounded Almansi function on $\mathbf{R}^n$ is a constant function.*

Proof. Let $u = \sum_{i=0}^{n} r^i h_i(x)$ be a bounded Almansi function on $\mathbf{R}^n$. Then there exists a positive constant M such that $-M \leq u(x) \leq M$. So $u + M = r^m h_m + r^{m-1} h_{m-1} + \cdots + h_0 + M \geq 0$. Divide the inequality by r^m, so that $h_m + r^{-1} h_{m-1} + \cdots + r^{-m}(h_0 + M) \geq 0$. Let $z \in \mathbf{R}^n$ and x_0 be an arbitrary point in the ball $B_r(z)$. Now integrating the previous inequality with respect to the harmonic measure $\rho_{x_0}^r$, we get $h_m(x_0) + r^{-1} h_{m-1}(x_0) + \cdots + r^{-m}(h_0(x_0) + M) \geq 0$. Thus $h_m(x_0) \geq 0$, by taking the limit as r tends to infinity. Since x_0 is arbitrary, we have $h_m \geq 0$ on $\mathbf{R}^n$. Therefore, h_m is a non-negative constant function. Now we apply the same integration to the inequality $u - M \leq 0$ to get h_m is a constant and $h_m \leq 0$. Therefore, $h_m = 0$ on $\mathbf{R}^n$. We use the same technique for $h_{m-1}, \ldots, h_1$ to get $0 = h_{m-1} = \cdots = h_1$. This implies that $u = h_0$ and so we have $-M \leq h_0 \leq M$. Therefore, by Theorem 1, u is a constant function. □

Theorem 3. *Let s be an Almansi function on a star domain ω containing 0. If $s = 0$ in a neighborhood of 0 in ω, then $s \equiv 0$ in ω.*

Proof. Let the Almansi function s be represented in the form $s(x) = \sum_{i=0}^{m} r^i h_i(x)$, where h_i is harmonic on ω for all i and $r = |x|$. Let $z \in \omega$ and N be a neighborhood of z in ω such that $\sum_{i=0}^{m} r^i h_i = 0$ on N. Choose $m+1$ positive numbers $r_1 < r_2 < \cdots < r_{m+1}$ such that the open ball $B_{r_{m+1}}(z)$ is contained in N. Take any arbitrary point $x_0 \in B_{r_1}(z)$ and integrate the equation $\sum_{i=0}^{m} r^i h_i = 0$ with respect to the harmonic measure $\rho_{x_0}^{r_i}$, for all $i = 1, \ldots, m+1$.

Then we have

$$r_i^m h_m(x_0) + r_i^{m-1} h_{m-1}(x_0) + \cdots + r_i h_1(x_0) + h_0(x_0) = 0.$$

The equation $r^m h_m(x_0) + \cdots + r h_1(x_0) + h_0(x_0)$ has at most m distinct roots. Since we have $m+1$ values for r_i, we must have $h_i(x_0) = 0$ for every i. Since x_0 is arbitrary in $B_{r_1}(z)$, $h_i \equiv 0$ on $B_{r_1}(z)$ for every i. Since h_i is harmonic in ω, we must have $h_i \equiv 0$ in ω and therefore $s \equiv 0$ in ω. □

Remark. The above theorem gives us the uniqueness of the representation of Almansi functions.

Theorem 4. *Let $s = \sum_{i=0}^m |x|^i h_i(x)$ be an Almansi function of order m in a star domain ω with center 0. Then the harmonic functions h_i are uniquely determined in ω.*

Proof. Suppose $s = \sum_{i=0}^n |x|^i H_i(x)$ is another such representation. Assume $n \geq m$. Write $h_j(x) = 0$ for $n \geq j \geq m+1$ if such an index j exists. Then $s = \sum_{i=0}^n |x|^i [H_i(x) - h_i(x)] = 0$ is an Almansi function of order n. Consequently (as in the proof of Theorem 3) we prove that $H_i - h_i \equiv 0$ for every i in ω. This proves the uniqueness of the representation. □

Corollary 5. *Let ω be a star domain with center 0 and $s = \sum_{i=0}^m r^i h_i$ be an Almansi function of order m on ω. If s is harmonic on ω, then $s \equiv h_0$.*

Proof. For, if s is harmonic on ω, s can be considered as an Almansi function of order m with the representation $s = \sum_{i=0}^m r^i H_i$ with $H_i \equiv 0$ for $1 \leq i \leq m$ and $H_0 \equiv s$. Hence by the uniqueness of representation of s (Theorem 3), we have $h_i \equiv 0$ for $1 \leq i \leq m$ and $h_0 \equiv H_0 \equiv s$ in ω. □

The following result follows immediately by taking $h_0 = h_1 = \cdots = h_{m-1} = 0$.

Corollary 6. *Let H be a harmonic function on a star domain ω containing 0. If $r^m H$ is harmonic, then $H \equiv 0$, for any positive integer m.*

We can prove a more general result than the above corollary.

Lemma 7. *Let H be a harmonic function on a domain ω (not necessarily a star domain). If for any positive real number α, $r^\alpha H$ is harmonic, then $H \equiv 0$.*

Proof. Let $h = r^\alpha H$ be harmonic. Choose a neighborhood N in ω such that for each $x_0 \in N$, we can find $B_{r_1}(x_0)$ and $B_{r_2}(x_0)$ contained in N for some r_1 and r_2 and $r_1 > r_2$. Now we integrate the equation $h = r^\alpha H$ with respect to the harmonic measures $\rho_{x_0}^{r_1}$ and $\rho_{x_0}^{r_2}$. This implies that $h(x_0) = r_1^\alpha H(x_0)$ and $h(x_0) = r_2^\alpha H(x_0)$. Therefore, $H(x_0) = 0$. Since x_0 is arbitrary in N, we conclude that $H \equiv 0$ on N. Since ω is a domain $H \equiv 0$ on ω. □

Note that in the above result the number α has to be positive, otherwise h is not harmonic.

Lemma 8. *Let H be a harmonic function on a domain ω containing 0. Then there exists a harmonic function h such that $\Delta(r^\alpha H) = r^{\alpha-2}h$, for any number α.*

Proof. If H is a harmonic function on a domain ω, then for any number α, one can show that

$$\Delta(r^\alpha H) = \Delta(r^\alpha)H + 2\,(\mathrm{Grad}\,(r^\alpha), \mathrm{Grad}\,(H)) + r^\alpha \Delta H. \tag{1}$$

Using simple calculations, we find that

$$\begin{aligned}\Delta r^\alpha = \frac{\partial^2 r^\alpha}{\partial r^2} + \frac{n-1}{r}\frac{\partial r^\alpha}{\partial r} &= \alpha(\alpha-1)r^{\alpha-2} + \frac{n-1}{r}\alpha r^{\alpha-1} \\ &= r^{\alpha-2}[\alpha(\alpha-1) + (n-1)\alpha],\end{aligned}$$

$$\begin{aligned}\mathrm{Grad}\,(r^\alpha) &= \left(\frac{\partial r^\alpha}{\partial x_1}, \frac{\partial r^\alpha}{\partial x_2}, \ldots, \frac{\partial r^\alpha}{\partial x_n}\right) \\ &= \alpha r^{\alpha-1}\left(\frac{\partial r}{\partial x_1}, \ldots, \frac{\partial r}{\partial x_n}\right) = \alpha r^{\alpha-2}\,(x_1, \ldots, x_n)\,,\end{aligned}$$

$$\mathrm{Grad}\,(H) = \frac{1}{r}\frac{\partial H}{\partial r}\,(x_1, \ldots, x_n)\,.$$

Therefore,

$$\begin{aligned}\Delta(r^\alpha H) &= r^{\alpha-2}[\alpha(\alpha-1) + (n-1)\alpha]H + 2\alpha r^{\alpha-1}\frac{\partial H}{\partial r} \\ &= r^{\alpha-2}\{[\alpha(\alpha-1) + (n-1)\alpha]H + 2\alpha r\frac{\partial H}{\partial r}\}. \end{aligned}\tag{2}$$

Consider the function

$$h = [\alpha(\alpha-1) + (n-1)\alpha]H + 2\alpha r\frac{\partial H}{\partial r}. \tag{3}$$

Note that $r\frac{\partial H}{\partial r}$ is harmonic and hence h is the required harmonic function on ω. □

Theorem 9. *Let h be a harmonic function on ω and α be a non-zero element such that $\alpha > 2 - n$. Then there exists a unique harmonic function H such that $\Delta(r^\alpha H) = r^{\alpha-2}h$.*

Proof. In equation (2) we have calculated $\Delta(r^\alpha H)$ and we have got

$$\Delta(r^\alpha H) = r^{\alpha-2}\{[\alpha(\alpha-1)+(n-1)\alpha]H + 2\alpha r\frac{\partial H}{\partial r}\},$$

for any harmonic function H. Now we consider the differential equation $h = [\alpha(\alpha-1)+(n-1)\alpha]H + 2\alpha r\frac{\partial H}{\partial r}$ and we solve it for H. $\frac{\partial H}{\partial r} + \frac{(\alpha+n-2)}{2r}H = \frac{h}{2\alpha r}$, for $\alpha \neq 0$. The integrating factor of this differential equation is

$$\mu(r) = e^{\int \frac{(\alpha+n-2)}{2r}dr} = e^{\frac{(\alpha+n-2)}{2}\ln r} = r^{\frac{(\alpha+n-2)}{2}}.$$

The solution of this differential equation is

$$Hr^{\frac{(\alpha+n-2)}{2}} = \int_0^r \frac{h(\phi,\rho)}{2\alpha\rho}\rho^{\frac{(\alpha+n-2)}{2}}d\rho = \int_0^r \frac{h(\phi,\rho)}{2\alpha}\rho^{\frac{(\alpha+n)}{2}-2}d\rho.$$

Therefore,

$$H = r^{\frac{(2-\alpha-n)}{2}}\int_0^r \frac{h(\phi,\rho)}{2\alpha}\rho^{\frac{(\alpha+n)}{2}-2}d\rho. \tag{4}$$

This integral is defined if $\frac{(\alpha+n)}{2} - 2 > -1$ or $\alpha > 2-n$.

Finally, we show that H is unique. Suppose that there exists a harmonic function H_1 such that $\Delta(r^\alpha H_1) = r^{\alpha-2}h$. This implies that $\Delta(r^\alpha H_1) = \Delta(r^\alpha H)$ and thus $\Delta(r^\alpha(H_1 - H)) = 0$. Hence, by Lemma 8, $H_1 = H$ and so H is unique. □

The following result can be proved easily by taking $\alpha = 2$ in the above result.

Corollary 10. *Let h be a harmonic function on ω. Then there exists a unique harmonic function H on ω such that $\Delta(r^2H) = h$.*

Corollary 11. *Let h be a harmonic function on ω. Then there exists a unique harmonic function H such that $h = \Delta^p(r^{2p}H)$, for all positive integers p.*

Proof. Since h is harmonic, by Corollary 10, there exists a unique harmonic function h_1 such that $h = \Delta(r^2h_1)$. Applying the same result again on h_1, gives us a unique harmonic function h_2 such that $\Delta(r^4h_2) = r^2h_1$ and thus $\Delta^2(r^4h_2) = \Delta(r^2h_1) = h$. Consequently, we can find a unique harmonic function $H = h_p$ such that $\Delta(r^{2p}h_p) = r^{2p-2}h_{p-1}$. Therefore, $\Delta^p(r^{2p}h_p) = h$. □

The following result is known as Almansi expansion for polyharmonic functions [3].

Theorem 12. *Let h be a p-harmonic function on a star domain ω with center 0. Then there exist unique harmonic functions $h_{p-1},\dots,h_0$ such that $h = r^{2(p-1)}h_{p-1} + \cdots + r^2h_1 + h_0$.*

Proof. We will use proof by induction on p. Firstly, we prove the result for $p = 2$. Let h be a biharmonic function. Then $\Delta^2 h = 0$ and thus Δh is harmonic. By Theorem 9, there exists a unique harmonic function h_1 such that $\Delta h = \Delta(r^2 h_1)$. This implies that $h_2 = h - r^2 h_1$ is harmonic. Therefore, $h = r^2 h_1 + h_2$. This is called the Almansi expansion for biharmonic functions [1]. Now suppose that the result is true for $p-1$; that is, if H is a $(p-1)$-harmonic function, then there exist unique harmonic functions $f_{p-1}, \dots, f_0$ such that $H = r^{2(p-2)} f_{p-1} + \cdots + r^2 f_2 + f_1$. Now let h be p-harmonic function, with $p > 1$. Then we have $\Delta^p(h) = \Delta(\Delta^{p-1} h) = 0$. This means that $\Delta^{p-1} h$ is a harmonic function. Then, by Corollary 10, there exist a unique harmonic function h_{p-1} such that $\Delta^{p-1} h = \Delta^{p-1}(r^{2p-2} h_{p-1})$. So $h - r^{2p-2} h_{p-1}$ is a $(p-1)$-harmonic function. Then by induction on $p-1$, we can write $h - r^{2p-2} h_{p-1}$ in the form

$$h - r^{2p-2} h_{p-1} = \sum_{i=0}^{p-2} r^{2i} h_i,$$

for unique harmonic functions $h_0, \dots, h_{p-2}$. Therefore, $h = \sum_{i=0}^{p-1} r^{2i} h_i$. □

Remark. The above result is still valid if the center of the star domain ω is any point x_0. In this case, the representation of the p-harmonic function h takes the form

$$h = \|x - x_0\|^{2p-2} h_{p-1}(x) + \cdots + \|x - x_0\| h_1 + h_0. \tag{5}$$

The following result is well known. For more details see [4].

Theorem 13. *Let ω be an open set containing 0 in $\mathbf{R}^n$ and K be a compact subset of ω. If h is a harmonic function on $\omega \setminus K$, then h has a unique decomposition of the form $h = u + v$, where u is a harmonic function on $\mathbf{R}^n \setminus K$ and v is a harmonic function on ω such that if $n = 2$, $\lim_{x\to\infty} u(x) - c \log|x| = 0$ for some real number c, and if $n > 2$, then $u(x) = \mathcal{O}(|x|^{2-n})$ when $|x| \to \infty$.*

Theorem 14. *Let ω be an open set containing 0 in $\mathbf{R}^n$ and K be a compact subset of ω. If u is an Almansi function on $\omega \setminus K$, then there exists an Almansi function s on $\mathbf{R}^n \setminus K$ and a harmonic function v on ω such that $u = s + v$. Moreover, the decomposition is unique by choosing $s(x) = \mathcal{O}(|x|^m \log|x|)$ for $n = 2$ and $s(x) = \mathcal{O}(|x|^{m-n+2})$, for $n > 2$ when $|x| \to \infty$.*

Proof. Let $u = \sum_{i=0}^{m} r^i h_i$ be an Almansi function on $\omega \setminus K$, where h_i is harmonic on $\omega \setminus K$. Now we apply Theorem 13 on the harmonic functions h_i. So h_i has the decomposition $h_i = s_i + v_i$, where s_i is harmonic on $\mathbf{R}^n \setminus K$ and v_i is harmonic on ω, for all i. So we have $u = \sum_{i=0}^{m} r^i h_i = \sum_{i=0}^{m} r^i (s_i + v_i) =$

$\sum_{i=0}^m r^i s_i + \sum_{i=0}^m r^i v_i$. The Almansi functions $s = \sum_{i=0}^m r^i s_i$ and $v = \sum_{i=0}^m r^i v_i$ are the required ones.

The uniqueness of the decomposition follows by the conditions that are given in Theorem 13. For $n = 2$, we can choose a constant A such that $|s_i(x)| \leq A \log |x|$, for all i. Now $|s(x)| = |\sum_{i=0}^m r^i s_i(x)| \leq \sum_{i=0}^m r^i |s_i(x)| \leq \sum_{i=0}^m r^i A \log r \leq m r^m \log r$, for $r > 1$. Therefore, $s(x) = \mathcal{O}(|x|^m \log |x|)$. Also, if $n > 2$, we have $|s_i(x)| \leq A|x|^{2-n}$, for all i and some constant A. □

Corollary 15. *Let K be a compact subset of $\mathbf{R}^n$. If u is an Almansi function on $\mathbf{R}^n \setminus K$, then there exists an Almansi function b on $\mathbf{R}^n$ such that $b - u = \mathcal{O}(|x|^m \log |x|)$ for $n = 2$ and $b - u = \mathcal{O}(|x|^{m-n+2})$, for $n > 2$.*

Proof. Let ω be an open subset of $\mathbf{R}^n$ that contains K. Now by Theorem 14, u has the unique decomposition $u = s + v$, where s and v are Almansi functions on $\mathbf{R}^n \setminus K$ and ω respectively. Now define the function

$$b = \begin{cases} u - s & \text{on} \quad \mathbf{R}^n \setminus K\,, \\ v & \text{on} \quad \omega\,. \end{cases}$$

The function b is well-defined; for if $x \in \omega \setminus K$, then we have $b(x) = u(x) - s(x) = s(x) + v(x) - s(x) = v(x)$. For $n = 2$ the last part of the result follows easily. We prove the result for $n = 2$ and a similar argument holds for $n > 2$. By Theorem 14, $|b(x) - u(x)| = |v(x) - u(x)| = |-s(x)| = \mathcal{O}(|x|^m \log |x|)$. □

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