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HAMILTON DECOMPOSITION OF COMPLETE 3-UNIFORM HYPERGRAPHS

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Abstract: A $k-$uniform hypergraph H is a pair (V,ε), where $V=\{v_1,v_2,\dots,v_n\}$ is a set of n vertices and ε is a family of k-subset of V called hyperedges. A cycle of length l of H is a sequence of the form $(v_1,e_1,v_2,e_2,\dots,v_l,e_l,v_1)$, where $v_1,v_2,\dots,v_l$ are distinct vertices, and $e_1,e_2,\dots,e_l$ are k-edges of H and $v_i,v_{i+1}\in e_i, 1\le i\le l$, where addition on the subscripts is modulo n, $e_i\neq e_j$ for $i\neq j$. In this paper we give a new method of Hamilton decomposition of the complete 3-uniform hypergraph K_n^3 for prime n.

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1. Introduction

A $k-$uniform hypergraph H is a pair (V,ε), where $V=\{v_1,v_2,\dots,v_n\}$ is a set

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of n vertices and ε is a family of k-subset of V called hyperedges. If ε consists of all k-subsets of V, then H is a complete $k-$uniform hypergraph on n vertices and is denoted by K_n^k. At the same time we may refer to a vertex $v_i \in V$ as v_{i+n}.

A cycle of length l of H is a sequence of the form

$$(v_1, e_1, v_2, e_2, \ldots, v_l, e_l, v_1),$$

where $v_1, v_2, \ldots, v_l$ are distinct vertices, and $e_1, e_2, \ldots, e_l$ are k-edges of H, satisfying:

(i) $v_i, v_{i+1} \in e_i, 1 \leq i \leq l$, where addition on the subscripts is modulo n, and

(ii) $e_i \neq e_j$ for $i \neq j$.

This cycle is known as a Berge cycle, having been introduced by Berge in his book [1].

A Hamilton cycle of a hypergraph H on n vertices is a cycle of length n, and a Hamilton decomposition of H is a partition of the hyperedges of H into Hamilton cycles.

A set of Hamilton cycles $C_1, C_2, \ldots, C_m$ of a complete 3-uniform hypergraph K_n^3 is called a Hamilton decomposition of K_n^3 if $\varepsilon(K_n^3) = \bigcup_{i=1}^m \varepsilon(C_i)$ and $\varepsilon(C_i) \cap \varepsilon(C_j) = \emptyset$ for $i \neq j$. In this paper, we use a combinatorial and group method to give a new method of Hamilton decomposition of a complete hypergragh K_n^3 for n being prime.

2. Main Results

Let n be a positive prime and D denote the set of all positive integer pairs (k, l) such that $D = D_1 \cup D_2$, where $D_1 = \{(k, l) : 2k + l = n\}$ and $D_2 = \{(k, l) : 1 \leq k < l < \frac{n-k}{2}\}$. From the definition of D we immediately obtain the following lemma.

Lemma 1. *Let $n > 3$ be a prime, then*

$$D_1 \cap D_2 = \emptyset, \mid D_1 \mid = \frac{n-1}{2}, \quad \mid D_2 \mid = \frac{n^2 - 6n + 5}{12}, \quad \mid D \mid = \frac{n^2 - 1}{12}.$$

Proof. It is obvious that $D_1 \cap D_2 = \emptyset$ and $\mid D_1 \mid = \frac{n-1}{2}$. For every $(k, l) \in D$, we discuss it two case:

Case 1. Let k be odd and write $k = 2m - 1 (m = 1, 2, \ldots)$.

If $n = 6m-1$, the set of pairs $\{(2m-1,l) : m = 1,2,\dots,\frac{n-5}{6}\}$ is such that

$$|\ \{(2m-1,l) : m = 1,2,\dots,\frac{n-5}{6}\}\ | = \sum_{m=1}^{\frac{n-5}{6}} (\frac{n+1}{2} - 3m). \qquad (1)$$

If $n = 6m+1$, the set of pairs $\{(2m-1,l) : m = 1,2,\dots,\frac{n-1}{6}\}$ is such that

$$|\ \{(2m-1,l) : m = 1,2,\dots,\frac{n-1}{6}\}\ | = \sum_{m=1}^{\frac{n-1}{6}} (\frac{n+1}{2} - 3m). \qquad (2)$$

Case 2. Let k be even and write $k = 2m(m = 1,2,\dots)$

If $n = 6m-1$, the set of pairs $\{(2m,l) : m = 1,2,\dots,\frac{n-5}{6}\}$ is such that

$$|\ \{(2m,l) : m = 1,2,\dots,\frac{n-5}{6}\}\ | = \sum_{m=1}^{\frac{n-5}{6}} (\frac{n-1}{2} - 3m). \qquad (3)$$

If $n = 6m+1$, the set of pairs $\{(2m,l) : m = 1,2,\dots,\frac{n-7}{6}\}$ is such that

$$|\ \{(2m,l) : m = 1,2,\dots,\frac{n-7}{6}\}\ | = \sum_{m=1}^{\frac{n-7}{6}} (\frac{n-1}{2} - 3m). \qquad (4)$$

So, if $n = 6m-1$, from (1) and (2), we have

$$\begin{aligned} |\ D_2\ | = &|\ \{(2m-1,l) : m = 1,2,\dots,\frac{n-5}{6}\}\ | \\ &+ |\ \{(2m,l) m = 1,2,\dots,\frac{n-5}{6}\}\ | = \sum_{m=1}^{\frac{n-5}{6}} (\frac{n+1}{2} - 3m + \frac{n-1}{2} - 3m) \\ &= \frac{n^2-6n+5}{12}. \end{aligned}$$

If $n = 6m+1$, from (2) and (4), we also have

$$\begin{aligned} |\ D_2\ | = &|\ \{(2m-1,l) : m = 1,2,\dots,\frac{n+5}{6}\}\ | \\ &+ |\ \{(2m,l) : m = 1,2,\dots,\frac{n+5}{6}\}\ | = \sum_{m=1}^{\frac{n+5}{6}} (\frac{n+1}{2} - 3m + \frac{n-1}{2} - 3m) \\ &= \frac{n^2-6n+5}{12}. \end{aligned}$$

From the two cases above, we show that

$$| D |= \frac{n^2-1}{12}.$$

We now define two edge sequences:

If $(k,l) \in D$, we define

$$C(k,l) = \{e_j(k,l) : j = 0,1,\ldots,n-1\} \pmod n; \tag{5}$$

if $(k,l) \in D_2$, we define

$$C(k,l)((l,k)) = \{e_j(k,l)((l,k)) : j = 0,1,\ldots,n-1\} \pmod n, \tag{6}$$

where

$$e_j(k,l) = \{(jr, jr+k,\ (j+1)r)\} \pmod n \tag{7}$$

and

$$e_j(l,k) = \{(jr, jr+l, (j+1)r)\} \pmod n, \tag{8}$$

where $r = k + l$. From the definition of D we see that, if $(k,l) \in D_1$, then l is odd, $r = n - k$ and for any two pairs $(k,l), (k',l') \in D_1$, we have $r \neq r'$; if $(k,l) \in D_2$, then $r < n - l$ and for any two pairs $(k,\ l), (k',l') \in D_2$, we also have $r = r'$ or $r \neq r'$.

Lemma 2. *Let $n > 3$ be a prime, then $e_j(k,\ l) = e_{j'}(k,\ l)$ $(e_j(l,\ k) = e_{j'}(l,\ k))$ for every $(k,\ l) \in D$ if and only if $j \equiv j' \pmod n$.*

Proof. By definition it is obvious that $e_{j+n}(k,l) = e_j(k,l)$. We now prove the only "if" part holds by dividing it into two cases.

Case 1. Suppose $e_j(k,\ l) = e_{j'}(k,l)$ with $0 \leq j, j' \leq n-1$. Set $t = j' - j$. Let $(k,l) \in D$, we have

$$\{jr, jr+k, (j+1)r\} \equiv \{j'r, j'r+k, (j'+1)r\} \pmod n,$$

which implies that

$$\{0, k, r\} \equiv \{tr, tr+k, (t+1)r\} \pmod n.$$

If $tr \not\equiv 0 \pmod n$ (equivalently, $tr + k \not\equiv k \pmod n$ and $(t+1)r \not\equiv r \pmod n$), then there are two subcases:

(i) $tr \equiv k \pmod n$, $tr + k \equiv r \pmod n$ $(t+1)r \equiv 0 \pmod n$;

(ii) $tr \equiv r \pmod n$, $tr + k \equiv 0 \pmod n$ and $(t+1)r \equiv k \pmod n$.

The subcase (i) imply that $k + r \equiv 0 \pmod n$ and $2k \equiv r \pmod n$. So we have $3k \equiv 0 \pmod n$, i.e., $k \equiv 0 \pmod n$, a contradiction. The subcase (ii) imply that $k + r \equiv 0 \pmod n$ and $2k \equiv r \pmod n$. So we have $3r \equiv 0$

(mod n), i.e., $r \equiv 0 \pmod{n}$, a contradiction also. They show that $tr \equiv 0 \pmod{n}$. Recall n is prime and $k < r < n$, so we must have $t \equiv 0 \pmod{n}\}$, i.e., $j \equiv j' \pmod{n}$.

Case 2. Suppose $e_j(l,k) = e_{j'}(l,k)$ with $0 \le j, j' \le n-1$, let $t = j' - j$ and $(k,l) \in D_2$, we have

$$\{jr, jr+l, (j+1)r\} \equiv \{j'r, j'r+l, (j'+1)r\} \pmod{n},$$

which implies that

$$\{0, l, r\} \equiv \{tr, tr+l, (t+1)r\} \pmod{n}.$$

If $tr \not\equiv 0 \pmod{n}$ (equivalently, $tr + l \not\equiv l \pmod{n}$ and $(t+1)r \not\equiv r \pmod{n}$), then there are two subcases:

(i) $tr \equiv k \pmod{n}$, $tr + k \equiv r \pmod{n}$ and $(t+1)r \equiv 0 \pmod{n}$;

(ii) $tr \equiv r \pmod{n}$, $tr + k \equiv 0 \pmod{n}$ and $(t+1)r \equiv k \pmod{n}$.

The remainder is similar to the Case 1. The proof is completed. □

Lemma 3. *Let $n > 3$ be a prime and K_n^3 be a complete 3−uniform hypergraph on $V = \{0, 1, \ldots, n-1\}$. Then the edge sequence $C(k,l)$ or $C(l,k)$ defined by* (5) *or* (6)*, respectively, is a Hamilton cycle of K_n^3.*

Proof. By the definition of e_j and Lemma 1, for edges $e_j, e_{j'} \in C(k,l)$, $e_j \ne e_{j'}$ if and only if $j \ne j' (\pmod{n})$ for $j = 0, 1, 2, \ldots, n-1$. So $|C(k,r)| = n$. For any $(k,l) \in D$, by the definition of $C(k,l)$ or $C(l,k)$, we have $e_j \cap e_{j+1} = \{(j+1)r \quad j = 0, 1, 2, \ldots, n-1\}$ and claim they are distinct vertices of V. It satisfies the following two conditions.

(1) $jr, (j+1)r \pmod{n} \in e_j$. $0 \le j \le n-1$,

and

(2) $e_i(k,\ r) \ne e_j(k,r)$ for $i \ne j$.

This shows that $C(k,l)$ or $C(l,k)$ is a Hamilton cycle of K_n^3. □

Lemma 4. *Let (k,l) and (k',l') be two distinct pairs of D. Then the cycles defined in* (5) *or* (6)*, satisfying $C(\,k,\ l\,) \cap C(k',l') = \emptyset$ or $C(k,l) \cap C(l',k') = \emptyset$ or $C(l,k) \cap C(l',k') = \emptyset$.*

Proof. By the definition of $C(k,l)$ (or $C(l,k)$), let us put the reduced residues modulo n equidistantly and clockwise on a circle. Take three of them, say, a, b and c. Then $\{a,b,c\} \in C(k,l)$ for some $(k,l) \in D$ if and only if the spaces among the three elements are in turn k, $r-k$ and $n-r$ ($r = k+l$, $r' = k'+l'$).

Therefore, if $e_j(k,r) = e_{j'}(k', \ r')$ (or $e_j(k,r) = e_{j'}(k',r')$, or $e_j(l,r) = e_{j'}(l',r')$), then the cycle permutations $(k, r-k, n-r)$ and $(k', r'-k', n-r')$ (or $(k, r, n-r)$ and $(l', r'-l', n-r')$ or $(l, r-l, n-r)$ and $(l', r'-l', n-r')$) are identical. We only need to consider one case, if $e_j(k,l) = e_{j'}(k',l')$, then the cycle permutations $(k, r-k, n-r) = (k', r'-k', n-r')$, i.e., $C(k,l) = C(k',l')$. For the other two cases, the discussions are similar. We now prove it in three cases.

Case 1. If k, k' are odd. Then, that there are two subcases:

(1) r and r' are odd,

(2) r and r' are even.

From the subcase (1), we have that $n-r$ and $n-r'$ are even, so we obtain that: (i) $k = k', n-r = n-r'$ and $r-k = r'-k'$, or (ii) $k = k', \ n-r = r'-k'$ and $r-k = n-r'$. By the subcase (i) we have that $(k,r) = (k',r')$, a contradiction. Sameness, the subcases (ii) implies that $(k,l), (k',l') \in D_2$ and $n = k+l+l'$, therefore implies $k+l+l' < n$, a contradiction.

From the subcase (2), we have that $n-r$ and $n-r'$ are odd, so we obtain that:

(i) $k = k', \ r-k = r'-k'$ and $n-r = n-r'$, or

(ii) $k = k', r-k = n-r'$ and $n-r = r'-k'$, or

(iii) $k = r'-k', \ r-k = n-r'$ and $n-r = k'$, or

(iv) $k = r'-k', \ r-k = k'$ and $n-r = n-r'$, or

(v) $k = n-r', r-k = r'-k'$ and $n-r = k'$, or

(vi) $k = n-r', r-k = k'$ and $n-r = r'-k'$.

By the subcase (i) we immediately obtain that $(k,l) = (k',r')$ a contradiction.

By the subcase (ii), we have the following result: if $(k,l), (k',l') \in D_1$, then $(k,r) = (k',r')$, a contradiction. If $(k,r)((k',r')) \in D_2$, and $(k',r')((k,r)) \in D_2$, without loss of generality we assume that $(k,r) \in D_1$ and $(k',r') \in D_2$, we obtain that $r = l$, a contradiction. If $(k,l), (k',l') \in D_2$, then $l > \frac{n-k}{2}$, a contradiction. For (iii), (iv), (v) and (vi), the discussions are similar. Recall k, k are odd, so we must have $e_j(k,l) = e_{j'}(k',l')$, which implies that $(k, r-k, n-r)$=$(k', r'-k', n-r')$, i.e., $C(k,l) = C(k',l')$.

Case 2. If k, k' are even. Then there are three subcases:

(1) r and r' are odd,

(2) r and r' are even,

(3) r (r') is odd and $r'(r)$ is even.

From the subcase (1), we have that $n-r$ and $n-r'$ are even, so we obtain that:

(i) $r-k=r'-k',\ k=k'$ and $n-r=n-r'$, or

(ii)$r-k=r'-k', k=n-r'$ and $n-r=k'$.

By the subcase (i) we have that $(k,r)=(k',r')$, a contradiction. Sameness, the subcases (ii) implies the following result: if $(k,l),(k',l') \in D_1$, then $(k,l)=(k',l')$, a contradiction. If (k,l) (or $(k',l')) \in D_2$, then $2n=r+r'+k+k'$, a contradiction. If (k,r) (or $(k',r')) \in D_2$, and (k',l') (or $(k,l)) \in D_2$, without loss of generality we assume that $(k,l) \in D_1$ and $(k',l') \in D_2$, we obtain that $n=2k'+l'$, a contradiction.

From the subcase (2), we have that $n-r$ and $n-r'$ are odd, so we obtain that:

(i) $n-r=n-r', k=k'$ and $r-k=r'-k'$, or

(ii) $n-r=n-r', k=r'-k'$ and $r-k=k'$.

By the subcase (i) we have that $(k,l)=(k',l',)$, a contradiction.

Sameness, the subcases (ii) implies that $(k,l),(k',l') \in D_2$ and $(k,l)=(k,k'),(k',l')=(l,k)$, therefore implies $k<l,\ l<k$, a contradiction.

By the subcase (3), without loss of generality we assume that r is odd and r' is even, we have that $n-r$ is even and $n-r'$ is odd, so we obtain that:

(i) $r-k=r'-k',\ k=k'$ and $n-r=n-r'$, or

(ii) $r-k=r'-k',\ k=n-r'$ and $n-r=k'$.

The subcase (i) implies that $(\ k',\ l'\ ) \in D_2$ and $(k'=2l')$, a contradiction.

The subcase (ii), imply that $(k,l) \in D_1$, and $n=3k+l$, a contradiction. Recall $k,\ k'$ are even, so we must have $e_j(k,\ l)=e_{j'}(k',l')$, which implies that, $(k,r-k,n-r)$=$(k',r'-k',n-r')$, i.e., $C(k,l)=C(k',l')$.

Case 3. If k (k') is odd and k' (k) is even. Without loss of generality we assume that k is odd and k' is even. Then there are two subcases:

(1) r is odd and r' is even,

(2) r and r' are odd.

From the subcase (1), we have that $n-r$ is odd and $n-r'$ are even, so we obtain that:

(i) $k = n - r'$, $r - k = k'$ and $n - r = r' - k'$, or

(ii) $k = n - r' r - k = r' - k'$ and $n - r = k'$.

By the subcase (i) we have that $(k, l), (k', l') \in D_2$, and $2n = 2k + l + 2k' + l'$, a contradiction.

Sameness, the subcases (ii) implies that $(k, l), (k', l') \in D_2$ and $n = k + l + l'$, therefore implies $(k, l), (k', l') \in D_2$, and $r = k$, a contradiction.

From the subcase (2), we have that $n - r$ and $n - r'$ are even, so we obtain that:

(i) $k = r' - k', r - k = n - r'$ and $n - r = k'$, or

(ii) $k = r' - k'$, $r - k = k'$ and $n - r = n - r'$.

By the subcase (i) we have that $(k, l) \in D_2$, and $2n = 2k + l + 2k' + l'$, a contradiction.

Sameness, the subcases (ii) implies that $(k, l) \in D_2$ and $k = l'$, $k' = l$, a contradiction. Recall $k(k')$ is odd and $k'(k)$ is even, so we must have $e_j(k, l) = e_{j'}(k', l')$, which implies that $(k, r - k, n - r)$=$(k', r' - k', n - r')$, i.e., $C(k, l) = C(k', l')$. The proof is completed. □

Theorem 5. *Let $n > 3$ be a prime, then the decomposition*

$$K_n^3 = \bigcup_{(k,l) \in D} C(k, l) \bigcup_{(k,l) \in D_2} C(l, k)$$

is a Hamiltonian decomposition.

Proof. Let $V = \{0, 1, \ldots, n-1\}$. From Lemma 3, for any $(k, l) \in D$, $C(l, k)$ is a Hamilton cycle of K_n^3. Therefore, we shall prove that

$$\{C(k, l), C(l, k) : (k, l) \in D\}$$

is a decomposition of K_n^3 into Hamiltonian cycles. By Lemma 1, $\mid D_2 \mid= \frac{n^2-6n+5}{12}$, $\mid D \mid= \frac{n^2-1}{12}$ and Lemma 4, let (k, l) and (k', l') be two distinct pairs of D, then the cycle defined in (5) or (6) satisfying $C(k, l) \cap C(k', l') = \emptyset$ or $C(k, l) \cap C(l', k') = \emptyset$ or $C(l, k) \cap C(l', k') = \emptyset$, and because $\mid C(k, l) \mid= n, \mid C(l, k) \mid= n$, so

$$\begin{aligned} &\mid C(k, l) \mid \cdot \mid D \mid + \mid C(l, k) \mid \cdot \mid D_2 \mid \\ &\qquad = n \cdot \frac{n^2 - 1}{12} + n \cdot \frac{n^2 - 6n + 5}{12} = \frac{n(n-1)(n-2)}{3!}, \end{aligned}$$

which equals the size of $|(K_n^3)|$, that is

$$K_n^3 = \bigcup_{(k,l)\in D} C(k,l) \bigcup_{(k,l)\in D_2} C(l,k).$$

The proof is completed. □

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