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ANALYTICAL ASSESSMENT OF MOLTEN METAL FLOW PATTERNS IN ALUMINUM REDUCTION CELLS

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Abstract: Determining the electromagnetically driven molten metal flow pattern and velocity values in aluminum reduction cells is a crucial study that requires the solution of Navier-Stokes equations. Solving Navier-Stokes equations using an innovative 2D analytical technique is proposed. A 2D numerical solution adopting a finite difference strategy using the vorticity-stream function approach subject to a time marching technique was performed, with the purpose of qualitatively assessing the proposed analytical technique. The proposed analytical technique rendered qualitatively reasonable values when compared to the numerical solution. This technique could be extended to 3D analysis. Suggestions for ramifications to take the time saving advantage of such an analytical solution were proposed. In view of reviewed literature for similar 2D problems, this analytical solution is applied for the first time, on our point of view.

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1. Introduction

The Hall-Heroult process is the only process commercially practiced for the

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extraction of aluminum. Within this process liquid aluminum is produced by electrolytic reduction of alumina (the aluminum oxide powder found in nature) inside reduction cells. One of the crucial studies for this process is determining the electromagnetically driven molten metal flow pattern and velocity values. Although a reasonable molten metal flow is favorable for a proper electrolysis process, the circulation of the molten metal causes carbon-lining erosion that may decrease cell lifetime [5]. Deterioration of carbon lining is not the only consequence of this circulation as it also affects the current efficiency and stability of the cell (refer, for instance, to [8] and [4]). In addition, the generated motion affects the dissolution of alumina fed and heat transfer within the cell [11]. It is clear, therefore, that the molten metal flow has great effect on the process of aluminum reduction and studying such phenomenon appears to be a major step in studying the cell performance. Determining the molten metal flow pattern involves solving Navier-Stokes equations while the electromagnetic driving force being known a priori [5].

The purpose of this paper is to present an innovative 2D analytical technique using which Navier-Stokes equations in aluminum reduction cells may be solved. This technique attempts to give an estimate of velocities profile and values. It is time saving, compared to any numerical solution, and its accuracy may be easily improved provided that powerful symbolic execution programs and super computers become available. Results obtained using this proposed technique were compared to (and gave qualitatively reasonable values with) 2D numerical solution adopting a to a time marching finite difference strategy using the vorticity-stream function approach. Details of the proposed technique and sample computation results are give in the following sections.

2. Analysis Approach

In this section typical 2D flow patterns and velocity values computed for *EGYPTALUM 208 KA* pre-backed cells are demonstrated. Details of the suggested analytical technique and the 2D numerical solution are presented below. It should be pointed out that the 3D electromagnetic current density and force distributions driving the flow were previously determined using a combination of analytical and time saving numerical computation approaches (please refer to [1] and [6]). Since such work is beyond the scope of this paper, driving forces of the 2D Navier-Stokes equations are taken here as known inputs.

The 2D vorticity-stream function approach subject to a time marching technique has been utilized. Viscosity was assumed to be laminar in order to simplify the analysis. This assumption is supported by the fact that the magnetic field has a damping effect and tends to stabilize the turbulence [2]. Moreover, the diffusive terms are typically small in case of Hall-Heroult cells [8]. However, the technique followed can account easily for turbulence if constant viscosity model is used as in [9].

For a 2D flow the vorticity function ζ can be defined as follows (see [10]):

$$\overline{\zeta} = \nabla \times \overline{V}, \tag{1}$$

where its magnitude is given by,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2}$$

In equation (1) $\overline{V}$ is the velocity vector, while u and v are the velocities in x and y directions, respectively.

By defining the stream function Ψ by $\overline{V} = \nabla \times \Psi$ the continuity equation $\nabla \cdot \overline{V}$ will always be guaranteed.

It is well known that the 2D incompressible Navier-Stokes equations may be expressed as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}f_x + \frac{\mu}{\rho}\nabla^2 u, \tag{3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{1}{\rho}f_y + \frac{\mu}{\rho}\nabla^2 v, \tag{4}$$

where f_x and f_y are the electromagnetic forces in x and y directions respectively, while P, μ and ρ are the pressure, molten metal viscosity and density, respectively.

Differentiating equation (3) with respect to y and equation (4) with respect to x then subtracting we get,

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] + u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} - u\frac{\partial^2 v}{\partial v^2} - v\frac{\partial^2 v}{\partial x \partial y}$$
$$= \frac{\mu}{\rho}\left[\frac{\partial}{\partial y}\nabla^2 u - \frac{\partial}{\partial x}\nabla^2 v\right] + \frac{1}{\rho}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right). \tag{5}$$

By making use of the above definitions of ζ and Ψ the following equations may be obtained:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \frac{\mu}{\rho}\nabla^2\zeta + \frac{1}{\rho}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right), \tag{6}$$

$$\nabla^2\Psi = -\zeta. \tag{7}$$

Equation (6) is known as the vorticity transport equation, while equation (7) is the Poisson equation. A time marching procedure and a finite difference representation [10] were used to solve equations (6) and (7), in order to determine the flow pattern and velocity values.

The procedure adopted in solving these equations starts by specifying initial values for ζ and Ψ at time $t = 0$. The vorticity transport equation (3) is then solved for ζ at each point at a time $t + \Delta t$. Afterwards, iterations for new Ψ values at each point are carried out by solving the Poisson equation (4) using new ζ values at each point. From equation (2), the velocity components u, v are thus obtained. In the case when the solution is not converged, the vorticity transport equation (3) is then re-solved for ζ and the above described procedure is repeated until convergence is achieved.

The solution was performed for a grid of 125 by 81 points in x and y directions, respectively, for a cell whose approximate dimensions are 10 m by 4 m. Velocities at contact with the cell carbon lining and cathode are considered to be zeros in fulfillment of the no slip conditions of the viscous flow. Initial guesses for Ψ, ζ, u and v are given by zeros. The second step in the time marching technique involves calculating ζ at all points. Since the solution involves a finite difference approach representing the derivatives at any location as a difference of surrounding points, ζ is evaluated at interior points after skipping a boundary layer. The same is done while calculating Ψ, u and v. This means that the velocities are calculated just near the cell edge but not at the cell edge. Within every iteration, the largest possible time step that does not cause the solution to diverge is chosen to decrease computation time.

Obtaining an analytical solution for the Navier-Stokes equations has always been an attractive, yet difficult, topic for researchers (see, for instance, [12], [3]). There is no doubt that the computation time of Navier-Stokes equations in aluminum reduction cells can be dramatically reduced if an analytical solution to this problem is found. The proposed analytical technique gives an estimate of molten metal velocity profiles and values. Obviously, it is a time saver compared to any numerical solution.

As previously mentioned, for a 2D flow continuity is always guaranteed by considering the stream function formulation $\overline{V} = \nabla \times \Psi$. From this stream function formulation, the velocities u and v may be expressed as:

$$u = +\frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{8}$$

By substituting (8) into (5) the steady state molten metal flow patterns may be described by the following expression:

$$\frac{\partial \Psi}{\partial y}\left[\frac{\partial^3 \Psi}{\partial x^3} + \frac{\partial}{\partial x}\frac{\partial^2 \Psi}{\partial y^2}\right] - \frac{\partial \Psi}{\partial x}\left[\frac{\partial^3 \Psi}{\partial y^3} + \frac{\partial}{\partial y}\frac{\partial^2 \Psi}{\partial x^2}\right]$$

$$-\frac{\mu}{\rho}\left[\frac{\partial^2}{\partial x^2}\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^4\Psi}{\partial x^4}+\frac{\partial^2}{\partial y^2}\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^4\Psi}{\partial y^4}\right]=\frac{1}{\rho}\left[\frac{\partial f_x}{\partial y}-\frac{\partial f_y}{\partial x}\right]. \tag{9}$$

The main idea of the proposed analytical technique is to assume a function, such as a polynomial of two variables for example, that can represent Ψ. The coefficients of that polynomial are to be determined using a special fitting technique to match the left and right hand sides (LHS and RHS) of equation (9) in a grid that represents a 2D plane in the cell. The RHS of equation (9) values at each grid point are the derivatives of the already known electromagnetic forces [6]. Having a polynomial representing Ψ, velocities may then be calculated using the relation $\overline{V}=\nabla\times\Psi$.

Suitable choice of Ψ is critical for the success of this technique. The polynomial chosen to represent Ψ should be able to express the physical conditions of the problem. Two physical conditions have been taken into consideration while choosing the structure of the polynomial representing Ψ. Firstly, normal component of velocities vanishes at the sides of the cell (to ensure continuity). Secondly, existence of an even symmetry with respect to the x-axis for u velocities (velocities along the x direction) as a consequence of the symmetric physical configuration of the reduction cells and surrounding feeding busbars.

Whence, a possible choice of the polynomial may be given by:

$$\Psi=\left[x^2-\left(\frac{C_l}{2}\right)^2\right]\left[y^2-\left(\frac{C_w}{2}\right)^2\right](xC_{11}+C_{01})\,y, \tag{10}$$

where, C_l is the cell length and C_w is its width, while C_{01} and C_{11} are unknown polynomial coefficients. It is clear that this suggested Ψ formulation satisfies all of the aforementioned physical conditions. The first two brackets ensure that u velocities vanish at the edges $\left(x=\pm\frac{C_l}{2}\right)$ and v velocities vanish at edges $\left(y=\pm\frac{C_w}{2}\right)$, respectively. Moreover, the y term multiplied by the polynomial enforces even symmetry with respect to x axis for u velocities.

In order to find the equations that may be simultaneously solved to find the coefficients C_{01} and C_{11} which best fit both sides equation (9), very complicated symbolic operations are carried out. Substituting (10) into (9) we obtain:

$$\begin{aligned}LHS_{i,j}=a_0(x_i,y_j)C_{11}^2+a_1(x_i,y_j)C_{11}C_{01}\\+a_2(x_i,y_j)C_{11}+a_3(x_i,y_j)C_{01}+a_4(x_i,y_j)C_{01}^2,\end{aligned} \tag{11}$$

$$RHS_{i,j}=\frac{1}{\rho}\left[\frac{\partial f_x(i,j)}{\partial y}-\frac{\partial f_y(i,j)}{\partial x}\right], \tag{12}$$

where, a_0, a_1, a_2, a_3 and a_4 are polynomials of variables x and y whose coefficients may be dependent on μ and ρ.

The square error ε may be defined by the expression:

$$\varepsilon = \sum_{i=1}^{m} \sum_{j=1}^{n} (LHS_{i,j} - RHS_{i,j})^2 , \tag{13}$$

where m and n are the numbers of grid points along x and y directions respectively. Hence, coefficients C_{01} and C_{11} that would yield the minimum error may be obtained by solving the following two simultaneous equations:

$$\frac{\partial \varepsilon}{\partial C_{01}} = \sum_{i=1}^{m} \sum_{j=1}^{n} 2 \left(LHS_{i,j} - RHS_{i,j}\right) \frac{\partial LHS_{i,j}}{\partial C_{01}} = 0, \tag{14}$$

$$\frac{\partial \varepsilon}{\partial C_{11}} = \sum_{i=1}^{m} \sum_{j=1}^{n} 2 \left(LHS_{i,j} - RHS_{i,j}\right) \frac{\partial LHS_{i,j}}{\partial C_{11}} = 0. \tag{15}$$

It should be pointed out that the complexity of these operations increases with the increase of number of coefficients (i.e, number of unknowns). It is also clear that the presence of a powerful symbolic execution program could have allowed a choice of a polynomial, with more coeffecients thus, improving the solution by allowing a better fitting of LHS to RHS of equation (9). Thus, the presence of a powerful symbolic execution program becomes inevitable if the number of coefficients of a polynomial representing Ψ is to be increased.

3. Computation Results

The sample results shown in Figure 1 have been computed using the proposed 2D analytical technique for a horizontal layer in the *EGYPTALUM 208 KA* reduction cells. Corresponding results for the same layer using the presented 2D vorticity-stream function numerical approach are given in Figure 2. In specific, both results correspond to a layer 12 cm from the bottom in a cell whose molten aluminium depth is 30 cm.

It can be seen that the general profile of the velocities in both cases, (numerically and analytically calculated values) has the shape of four eddies. In both cases, there are high velocities at the corners. However, in the analytical technique there are high velocities at the center that are not encountered in the numerical solution. This could be interpreted as the effect of using two variables only in equation (5), the fact that restricts revealing all the flow information. In other words, for the fitted LHS of equation (5) to its RHS the polynomial under consideration accounted for edge velocities but also generated relatively high velocities at the cell center. Figure 3 and Figure 4 show Ψ values deduced from numerical and analytical solutions. It should be mentioned here that the

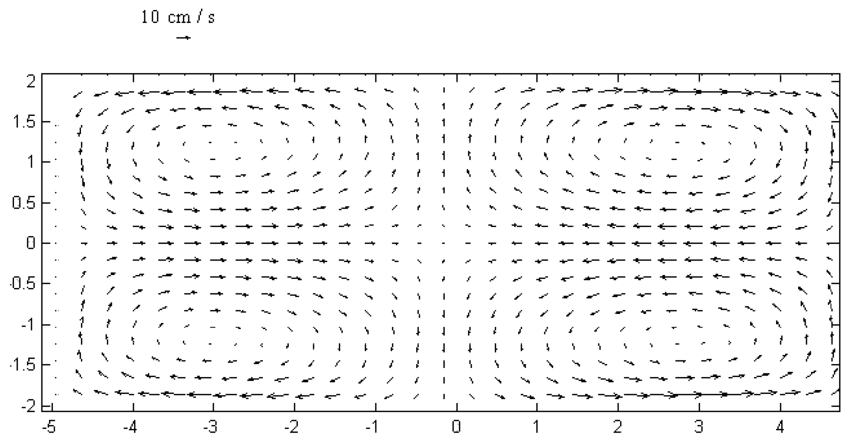

Figure 1: Velocities calculated using the presented vorticity-stream function numerical approach at a layer 12 cm from the bottom in a 208 KA cell

numerical Ψ plotted can be very useful in guessing the proper polynomial required to represent it. This is a very critical step in the success of the proposed analytical technique.

From Figure 1 - Figure 4, it can be observed that the overall velocity and Ψ patterns computed using the 2D numerical solution agree qualitatively with those computed using the proposed analytical technique. On the other hand, numerically computed results agree qualitatively and quantitatively with the preliminary measurements as well as the erosion patterns experienced in *EGYPTALUM* cells (please refer to [7]).

4. Conclusions

It can be concluded that the proposed 2D analytical solution technique may be utilized to solve Navier-Stokes equations for molten aluminum in reduction cells. This non-iterative technique is indeed promising as far as its computational time efficiency is concerned. Moreover, provided that a powerful symbolic execution program is available, the solution accuracy could be enhanced by assuming a more sophisticated Ψ function. Naturally, this assumed Ψ can be of any form other than a polynomial (such as sinusoidal functions) to tailor the shape of the expected real Ψ. However, cases with sinusoidal functions are almost intractable given our available symbolic computing engines. Even

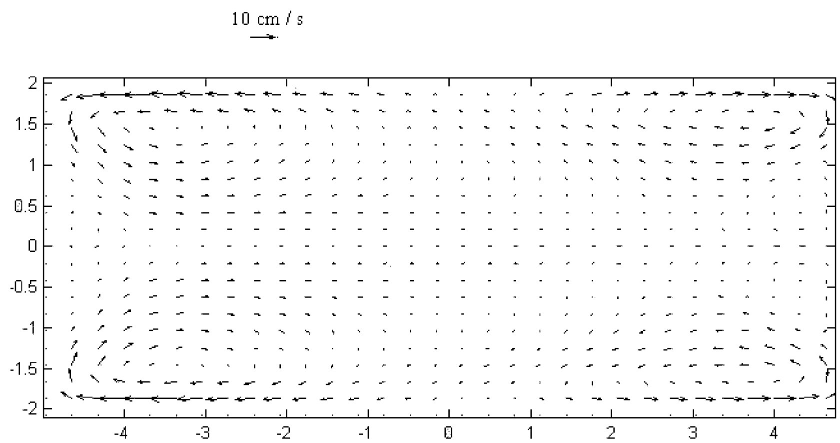

Figure 2: Velocities calculated using the proposed analytical technique at a layer 12 cm from the bottom in a 208 KA cell

without enhancing its accuracy, the proposed analytical technique may be used to provide initial values for the vorticity-stream function iterative numerical technique, thus reducing its computational time. Obviously, the proposed 2D analytical technique can be easily extended to solve similar 3D Navier-Stokes problems. This is currently being considered as a future work.

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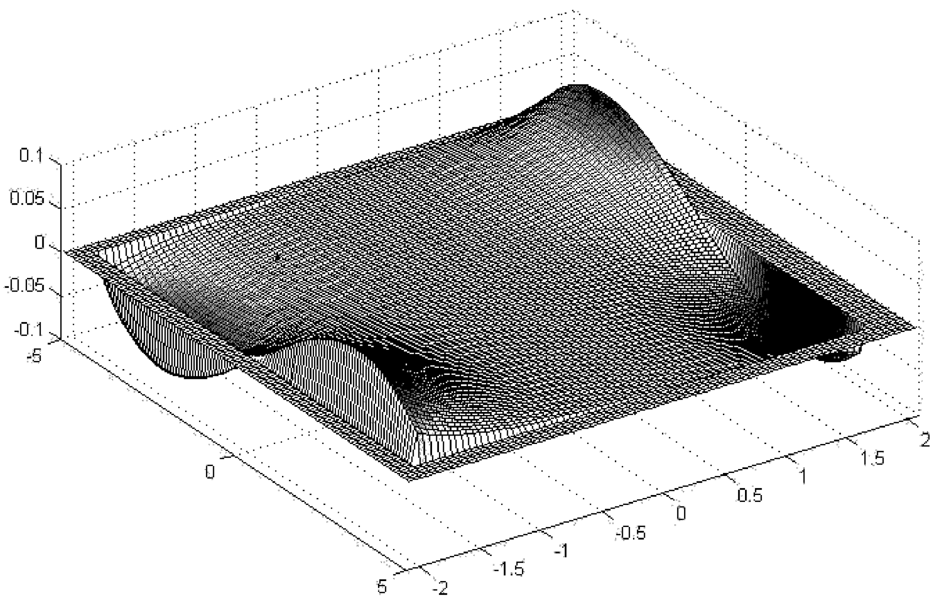

Figure 3: Computed Ψ using the proposed analytical technique

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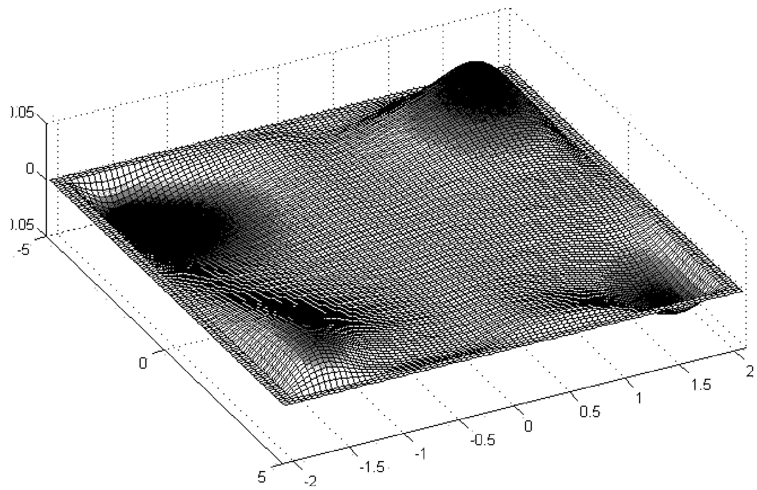

Figure 4: Computed Ψ using the presented numerical vorticity-stream function approach

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