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DEGREE OF SHEAVES RELATED TO THE DUALIZING SHEAF OF AN INTEGRAL PROJECTIVE CURVE

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1. Introduction

Let X be an integral projective curve defined over an algebraically closed field $\mathbb{K}$ and $\pi : Y \to X$ its normalization. Here we study the integers $\deg(\omega_X^{**})$ and $\deg(\pi^*(\omega_X^{**})/\mathrm{Tors}(\pi^*((\omega_X^{**}))))$ when every non-Gorenstein point of X has a modulus in the sense of [7]. For each $P \in \mathrm{Sing}(X)$ set $\widetilde{\mathcal{O}}_{X,P} := \bigcap_{Q\in\pi^{-1}(P)} \mathcal{O}_{Y,Q}$. Thus the semilocal ring $\widetilde{\mathcal{O}}_{X,P}$ is the normalization of the local ring $\mathcal{O}_{X,P}$ in its total ring of fractions. There is a natural bijection between the finite set $\pi^{-1}(P)$ and the formal branches of X at P, i.e. the maximal ideals of the semi-local ring $\widetilde{\mathcal{O}}_{X,P}$. Set $\delta(X,P) := \dim_{\mathbb{K}}(\widetilde{\mathcal{O}}_{X,P}/\mathcal{O}_{X,P})$. Thus $\delta(X,P) > 0$ for all $P \in \mathrm{Sing}(X)$ and $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) + \sum_{P\in\mathrm{Sing}(X)} \delta(X,P)$. Let $\mathcal{C}_{X,P}$ denote

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the conductor of $\widetilde{\mathcal{O}}_{X,P}$ in $\mathcal{O}_{X,P}$. Set $\widetilde{\delta}(X,P) := \dim_{\mathbb{K}}(\widetilde{\mathcal{O}}_{X,P}/\mathcal{C}_{X,P})$. We have $\delta(X,P)+1 \le \widetilde{\delta}(X,P) \le 2\delta(X,P)$ for all $P \in \mathrm{Sing}(X)$ and $\widetilde{\delta}(X,P) = 2\delta(X,P)$ if and only if the local ring $\mathcal{O}_{X,P}$ is Gorenstein, i.e. if and only if ω_X is locally free at P. We recall that $\omega_{X,P}$ is reflexive if and only if $\mathcal{O}_{X,P}$ is Gorenstein (see [4], Part c, Corollary 3.4). For any rank one torsion free sheaf A on X and any $P \in \mathrm{Sing}(X)$ let $\ell(A,P)$ denote the minimal integer $t \ge 0$ such that the germ A_P of A at P contains a free $\mathcal{O}_{X,P}$-module B such that $\dim_{\mathbb{K}}(A_P/B) = t$ ([2], Definition 2.2.3). Thus $\ell(A,P) = 0$ if and only if A is locally free at P. Set $\widetilde{A} := \pi^*(A)/\mathrm{Tors}(\pi^*(A))$ and let $\widetilde{A}_P$ the germ of $\widetilde{A}$ in the semilocal ring $\widetilde{A}$. Let $\delta(A_P)$ be the minimal integer t such that there is a free rank one $\widetilde{A}$-module B such that $\widetilde{A}_P \subseteq B$; we have $\delta(A_P) = \delta(A,P) - \ell(A_P)$ ([3], [6], Remark 1.2, or [2], p.18). Set $R := \widehat{\mathcal{O}}_{X,P}$. Let $\widetilde{R}$ be the normalization of R. Set $\delta := \dim_{\mathbb{K}}(\widetilde{R}/R)$. Hence $\delta = \delta(X,P)$. A_P (resp. $\widetilde{A}_P$ induces a rank one torsion free R-module (resp. $\widetilde{R}$-module). We may define in the same way the integers $\ell(A)$ and $\delta(A)$ for any fractional ideal A of R. There is a fractional ideal M of R such that $M \cong A$ and $R \subseteq M \subseteq \widetilde{A}$. Notice that the integers $\ell(A)$ and $\delta(A)$ only depend on the isomorphic class of A. Thus $\ell(A) = \ell(M)$ and $\delta(A) = \delta(M)$. We have $\delta(M) = \dim_{\mathbb{K}}(\widetilde{R}/M)$ ([6], Remark 1.2). Thus $\ell(A) = \dim_{\mathbb{K}}(M/R)$ and $\delta(A) + \ell(A) = \delta$. Let $\mathcal{C}$ denote the conductor of $\widetilde{R}$ in R and ω the canonical fractional ideal, i.e ω is a canonical module of R and $R \subseteq \omega \subseteq \widetilde{R}$; furthermore, $\omega : \widetilde{R} = \mathcal{C}$ and these properties are true even in the non-unibranch case (see [4], p. 31, or [1], Lemma 3). Its dual $\theta := R : \omega$ is called the Dedekind different (see [5] for its study in the unibranch case). Set $\delta := \delta_{X,P}$ and $\widetilde{\delta} := \widetilde{\delta}(X,P)$. Let $\mathfrak{m}$ be the maximal ideal of R. Assume that X is not Gorenstein at P. We have $\mathcal{C} \subseteq \theta \subseteq \mathfrak{m}$. The case $\mathcal{C} = \theta$ is the case in which $\dim_{\mathbb{K}}(\omega^{**}/\omega)$ is maximal. Recall that the singular point P of X or the local ring $\mathcal{O}_{X,P}$ or the local ring R is said to have a modulus in the sense of [7] if $\mathcal{C} = \mathfrak{m}$. Hence $\omega^{**} = \widetilde{R}$ if R has a modulus, but it is not Gorenstein. Since $\theta = \theta^{**} = \omega^{***}$, if $\omega^{**} = \widetilde{R}$, then $\theta = \mathcal{C}$.

Theorem 1. *Let X be an integral projective curve and $\pi : Y \to X$ its normalization. Set $\Sigma := \{P \in \mathrm{Sing}(X) : X$ is not Gorenstein at $P\}$, $\delta_\Sigma := \sum_{P\in\Sigma} \delta_P$ and $\widetilde{\delta}_\Sigma := \sum_{P\in\Sigma} \widetilde{\delta}_P$. Assume that $\mathcal{C}_P = \theta_P$ for all $P \in \Sigma$. Then $\deg(\omega_X^{**}) = 2p_a(X) - 2 + \widetilde{\delta}_\Sigma - \delta_\Sigma$ and $\deg(\pi^*(\omega_X^{**})/\mathrm{Tors}(\pi^*(\omega_X^{**}))) = 2p_a(X) - 2 + \widetilde{\delta}_\Sigma - 2\delta_\Sigma$.*

As an immediate corollary of Theorem 1 we get the following result.

Corollary 1. *Let X be an integral projective curve and $\pi : Y \to X$ its normalization. Set $\Sigma := \{P \in \mathrm{Sing}(X) : X$ is not Gorenstein at $P\}$,*

$\delta_\Sigma := \sum_{P\in\Sigma} \delta_P$ *and* $\widetilde{\delta}_\Sigma := \sum_{P\in\Sigma} \widetilde{\delta}_P$. *Assume that every non-Gorenstein singularity of* X *has a modulus. Then* $\deg(\omega_X^{**}) = 2p_a(X) - 2 + \widetilde{\delta}_\Sigma - \delta_\Sigma$ *and* $\deg(\pi^*(\omega_X^{**})/\mathrm{Tors}(\pi^*(\omega_X^{**}))) = 2p_a(X) - 2 + \widetilde{\delta}_\Sigma - 2\delta_\Sigma$.

Remark 1. Let $f : Z \to X$ be a birational morphism between integral projective curves. Fix $L \in \mathrm{Pic}(Z)$. We have $\deg(L) = \chi(L) + p_a(Z) - 1$ and $\deg(f_*(L)) = \chi(f_*(L)) + p_a(X) - 1$. Since f is finite, we have $R^j f_*(L) = 0$ for all $j \geq 1$. Thus $\chi(f_*(L)) = \chi(L)$ by the Leray spectral sequence of f. Hence $\deg(f_*(L)) = \deg(L) + p_a(X) - p_a(Z)$.

Proof of Theorem 1. We have $\deg(\omega_X) = 2p_a(X) - 2$ even if X is not Gorenstein (apply part (b) of [2], Lemma 3.1.7, to the sheaf $\mathcal{F} := \mathcal{O}_X$). We have $\ell(\omega_{X,P}) = 2\delta_P - \widetilde{\delta}_P$ for all $P \in \Sigma$ (([2], Lemma 2.1.5). Since $\omega^{**} = \widetilde{R}$ at every non-Gorenstein singular point of X, we get

$$\begin{aligned}
&\deg(\omega_X^{**}) = 2p_a(X) - 2 + \sum_{P\in\Sigma} \dim_{\mathbb{K}}(\omega_{X,P}^{**}/\omega_{X,P}) \\
&= 2p_a(X) - 2 + \sum_{P\in\Sigma} \delta(\omega_{X,P}) = 2p_a(X) - 2 + \delta_\Sigma - \sum_{P\in\Sigma} \ell(\omega_{X,P}) \\
&= 2p_a(X) - 2 + \delta_\Sigma - 2\delta_\Sigma + \widetilde{\delta}_\Sigma = 2p_a(X) - 2 + \widetilde{\delta}_\Sigma - \delta_\Sigma\,.
\end{aligned}$$

Let $f : Z \to X$ be the partial normalization of X in which we only normalize the points of Σ. Hence π factors through f, say $\pi = f \circ h$ with $h : Y \to Z$. We saw the existence of $L \in \mathrm{Pic}(Z)$ such that $f_*(L) = \omega_X^{**}$. Thus $L = f^*(\omega_X^{**}/\mathrm{Tors}(f^*(\omega_X^{**}))$. Thus $\pi^*(\omega_X^{**}/\mathrm{Tors}(\pi^*(\omega_X^{**}) = h^*(L)$. Hence $\deg(\pi^*(\omega_X^{**}/\mathrm{Tors}(\pi^*(\omega_X^{**}) = \deg(L)$. By Remark 1 we have $\deg(L) = \deg(\omega_X^{**}) + p_a(Z) - p_a(X) = \deg(\omega_X^{**}) - \delta_\Sigma$, concluding the proof. □

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References

[1] V. Barucci, R. Fröberg, One-dimensional almost Gorenstein rings, *J. Algebra*, **188**, No. 2 (1997), 418-442,

[2] Ph.R. Cook, *Local and Global Aspects of the Module Theory of Singular Curves*, Ph.D. Thesis, Liverpool (1993).

[3] G.-M. Greuel, G. Pfister, Moduli spaces for torsion free modules on curve singularities I, *J. Algebraic Geom.*, **2**, No. 1 (1993), 81-135.

[4] J. Herzog, E. Kunz, *Der Kanonische Modul Eines Cohen-Macaulay-Rings*, Lect. Notes in Math., **238**, Springer, Berlin (1971).

[5] F. Odetti, A. Oneto, E. Zatini, Dedekind different and type sequence, *Le Mathematiche*, **55**, No. 2 (2000), 499-516.

[6] A. Oneto, E. Zatini, Type-sequences for modules, *J. Pure Appl. Alg.*

[7] J.-P. Serre, *Groupes Algébriques et Corps de Classes*, Hermann, Paris (1959).