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DECAY EQUATIONS FOR QUBITS

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Abstract: We investigate the temporal stability of the pure states of one and two qubits interacting with a free Boson field in its vacuum state, via the rotating wave Hamiltonian. The problem reduces to the solution of coupled integro-differential equations of Volterra type. These are investigated via Laplace transforms and Hardy spaces.

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1. Introduction

The genesis of this work, but not its thrust, was an attempt to understand the spontaneous decay of a finite collection of qubits due to their coupling to the electromagnetic field vacuum.

The complete answer to this question lies in the province of quantum electrodynamics and remains an open question. As usual in applied mathematics, interim remedies consist of idealized models emphasizing only some of the complexity of the real problem.

In this paper we begin by considering a more or less standard such model. Only initial states which are pure and where the field is in its vacuum state are considered. The quantum dynamical problem is to determine the time

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evolution of these states. It turns out that this problem is equivalent to the solution of certain coupled integro-differential equations of Volterra type. It is the derivation and solution of these equations which is the principal object of this paper. For convenience, we refer to these equations as *classical*, and their solutions as the *classical solutions*. Knowledge of the classical solutions will be shown to allow construction of the quantum evolution of the qubit states.

In broad outline, the quantum model consists of the qubit and field subsystems, and an interaction between them. The field is simply a non-relativistic scalar Boson field (with an unspecified dispersion law subject only to some regularity conditions) evolving in time in accordance with the standard free evolution. By this choice we avoid many technical problems associated with the electromagnetic field. We note that the particles carrying the field are often termed Bosons, but we feel that term should be reserved for the collective oscillations of a lattice.

A qubit is an atom for which the only internal structure of interest is a chosen pair of energy eigenstates. This circumstance allows a qubit to be modelled as a two-level quantum spin system, although no spin angular momentum is in any way involved. We term the two energy eigenstates *upper* and *lower*; in the spin description, these correspond to spin up and spin down, respectively. Without interaction, each qubit evolves through the Hamiltonian proportional to σ_3 (in the spin formalism). In the qubit system for our model there are k qubits: we will consider only $k = 1$ and $k = 2$; for $k > 2$ the classical system of equations that results is too complicated to justify the effort at present.

It is the choice of interaction Hamiltonian coupling the field and qubits that characterizes the model. We use what is known in quantum optics as the rotating wave approximation. As with the choice of field, this avoids the most serious analytical difficulties. Nonetheless, solving this model rigorously still requires a certain effort.

This Hamiltonian allows us to determine the general form that such states evolve to after a time t by means of a conservation law. There are two conservation laws operating, in fact. The first is obvious: the total number k of qubits does not change in time. The second conservation law is a bit more interesting.

Consider a system with an arbitrary number k of qubits, and a pure state with $n(0)$ of the qubits in their upper state[1] (and zero Bosons). The number of qubits in the upper state at time t, $n(t)$, can be less (but never more) than $n(0)$, and the number $N(t)$ of Bosons can increase from zero. The conservation

[1]Of course $n(0) \leq k$.

law says that the integer $n(t) + N(t) = \eta$ is constant in time, with $\eta = n(0)$ since $N(0) = 0$. We attach the positive integer pair (k, η) as a label to each such state.

This is not to say that the pair (k, η) determines a unique pure state. Hence the conservation law in question tells us that at time t, the initial state vector has evolved into a linear combination of all those vectors which have the same label (k, η).

The classical integro-differential equations emerge from these circumstances as follows. The coefficients in the expansion over all (k, η) vectors are unknown functions of the time. We substitute this as an ansatz into the time dependent Schrödinger equation. Using orthogonality, we shall show that these expansion coefficient functions of time must satisfy certain integro-differential equations, and these are the classical equations in our terminology.

Once we have obtained the classical systems, we focus on their solutions. The solutions can then be considered in light of their physical interpretation as probability amplitudes detailing the underlying quantum evolution. In particular, we can determine the so-called decay law and the lifetime of the initial states. As we will argue further on, the standard lifetime gives only a very rough measure of stability, and so we shall suggest others in the course of the work.

2. The Model

In this section we detail the model outlined above. Our Hilbert space will be the tensor product of a qubit Hilbert space $\mathcal{Q}$ and a Boson Hilbert space $\mathcal{P}$, $\mathcal{H} = \mathcal{Q} \otimes \mathcal{P}$; as $\mathcal{Q}$ is finite dimensional, the algebraic product is complete in the Hilbertian topology.

Using the spin formalism, Mandel [7], $\mathcal{Q} \cong \mathbb{C}^{2k}$, where k is the number of qubits. The algebra of qubit observables is $\mathbb{B}(\mathcal{Q})$, which is linearly isomorphic and homeomorphic to the set of all $2k \times 2k$ complex matrices.

For $k = 1$, the Pauli matrices $\{\, \sigma_j : j = 1, 2, 3 \,\}$ and the unit matrix I constitute a linear basis for $\mathbb{B}(\mathcal{Q})$; we also use the spin flip operators $\sigma_+ = (\sigma_1 + i\sigma_2)/2$ and $\sigma_- = (\sigma_1 - i\sigma_2)/2$. For $k > 1$, the corresponding basis operators are[2]

$$d\Gamma^{(k)}(\sigma_j) = \sigma_j \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes \sigma_j, \tag{2.1}$$

where we are using the 'amplification' map $d\Gamma^{(k)}$ for self-adjoint (Hermitian) operators, familiar from multilinear algebra.

When $k = 1$, a preferred orthonormal basis for $\mathcal{Q}$ consists of the two normalized eigenvectors of σ_3, $e[-] = (0,1)^{\mathrm{T}}$ and $e[+] = (1,0)^{\mathrm{T}}$, corresponding to the eigenvalues -1, $+1$. For general k we introduce the notation

$$e[j] = e_{j_1} \otimes \cdots \otimes e_{j_k}, \qquad j \in \Pi^k, \tag{2.2}$$

for the indicated product vector, where $\Pi = \{ -, + \}$ and Π^k is its k-fold Cartesian product.

The qubit Hamiltonian is[3]

$$H_{\mathcal{Q}} = \xi \, d\Gamma^{(k)}(\sigma_3) \,, \tag{2.3}$$

and the evolution operator is $\exp(-itH_{\mathcal{Q}})$. The strictly positive number ξ is interpreted as twice the energy difference between the atomic levels and is one of the parameters of the model.

The vector $e[j]$ is an eigenvector of $H_{\mathcal{Q}}$,

$$H_{\mathcal{Q}}\, e[j] = \langle j \rangle \; \xi e[j], \qquad j \in \Pi^k \,, \tag{2.4}$$

with the eigenvalue

$$\langle j \rangle = j_1 + \cdots + j_k \,. \tag{2.5}$$

We have identified $\pm \in \Pi$ with the integers ± 1, as we shall do from time to time.

The Boson field is described in the standard formalism of second quantization, Cook [3]. Our notation is that

$$\mathcal{P} = \bigoplus\nolimits_{n \geq 0} L^2_+ \left(\mathbb{R}^n \right) \qquad \text{(Hilbertian direct sum)} \tag{2.6}$$

is the symmetric Fock space over the one particle Hilbert space $L^2(\mathbb{R})$. We choose the following dense subset $\mathcal{V}$ of $\mathcal{P}$ as the domain for the fields. Let

$$V = \left\{ f \in L^2(\mathbb{R}) \,:\, pf \in L^2(\mathbb{R}) \;\; \text{for all polynomials} \;\; p \right\}. \tag{2.7}$$

[2]Even though we only consider the solution for $k = 1$, 2, many formulas can be written for arbitrary k with no additional difficulty.

[3]In principle we should write $H^{(k)}_{\mathcal{Q}}$ here, but no confusion should result from omitting the k label.

Then

$$\mathcal{V} = \sum\nolimits_{n\geq 0}^{\oplus} \otimes^n V \qquad \text{(algebraic tensor product and direct sum)}\ . \tag{2.8}$$

The Fock vacuum is $\Omega = (1, 0, 0, 0, \ldots)$.

The lowering and raising operators $a(f)$ and $a^+(f)$ are defined for functions $f \in V$, and are complex linear maps of $\mathcal{V}$ into itself satisfying the canonical commutation relations there. We choose to work in the momentum representation, so the independent variable for field operators and test functions will be interpreted as the momentum. The Fock condition holds: $a(f)\Omega = 0$ for all $f \in V$.

It is convenient for calculations to introduce the linear maps $\Phi^{(n)} : \otimes^n V \to \mathcal{V}$ whose formula on product vectors is

$$\Phi^{(n)}(f_1 \otimes \cdots \otimes f_n) = a^+(f_1) \cdots a^+(f_n)\Omega\ . \tag{2.9}$$

Hence $\Phi^{(n)}$ is a vector-valued tempered distribution.

By ε we mean the energy-momentum dispersion law for the Bosons. It is required to satisfy the following four conditions:

ε_1: $\varepsilon : [0, \infty) \to [0, \infty)$ is a continuously differentiable and strictly monotonic increasing bijection;

ε_2: $\varepsilon'(k) > 0$ for all $k > 0$;

ε_3: $\varepsilon(k) = \varepsilon(|\, k\,|) \geq 0$ for all $k \in \mathbb{R}$;

ε_4: it is polynomially bounded.

For example $\varepsilon(k) = c\,|\,k\,|$ or $\varepsilon(k) = \hbar^2 k^2/2M$ are of this class.[4]

We will also let the symbol ε stand for the (unbounded) operator of multiplication on $L^2(\mathbb{R})$ by $\varepsilon(k)$. Then the Hamiltonian for the Boson field is

$$H_{\mathcal{P}} = d\Gamma\,(\varepsilon) = \sum_{n\geq 0} d\Gamma^{(n)}(\varepsilon), \tag{2.10}$$

using the same formula for $d\Gamma^{(n)}$ as in equation (2.1), so that

$$H_{\mathcal{P}}\Phi^{(n)}(f_1 \otimes \cdots \otimes f_n) = \Phi^{(n)} \circ d\Gamma^{(n)}(f_1 \otimes \cdots \otimes f_n)\,. \tag{2.11}$$

Evidently $H_{\mathcal{P}}\Omega = 0$.

[4] We could restrict attention to the first of these laws and dispense with the conditions as most writers do.

We have now introduced the qubit and Boson subsystems. As indicated previously, the system Hilbert space is $\mathcal{H} = \mathcal{Q} \otimes \mathcal{P}$ and the corresponding domain for the fields with qubit observables attached, is $\mathcal{Q} \otimes \mathcal{V}$. For the inner product of two vectors $\mathcal{X}$, $\mathcal{Y} \in \mathcal{H}$ we write $\langle \mathcal{X}, \mathcal{Y} \rangle$, anti-linear in the first variable.

Now we turn to the interaction Hamiltonian. This unbounded operator is formally defined on $\mathcal{Q} \otimes \mathcal{V}$ by the expression

$$H_{\mathrm{I}} = \mu \Big(d\Gamma^{(k)}(\sigma_-) \otimes a^+(\lambda) + d\Gamma^{(k)}(\sigma_+) \otimes a(\lambda) \Big) . \tag{2.12}$$

The parameter μ is a positive constant we use to control the strength of the interaction[5], leaving the function λ for other purposes. Since λ is an input of the model, we are free to impose certain conditions on it. We give the first of these here; the others will be given further on as needed.

λ_1: λ is a real-valued function in V satisfying

$$\int_{\mathbb{R}} \varepsilon(k)^{-1} \, | \, \lambda(k) \, |^2 \; dk \; < \; \infty \, . \tag{2.13}$$

This is a non-trivial constraint on the low momentum behaviour of the coupling function relative to the energy, controlling the creation of 'soft' Bosons. Davies [4] has shown that H_{I} has form bound zero with respect to $H_{\mathcal{S}} + H_{\mathcal{P}}$, so that the Hamiltonian[6]

$$H = H_{\mathcal{S}} + H_{\mathcal{P}} + H_{\mathrm{I}} \tag{2.14a}$$

is self-adjoint as a form sum, with

$$\mathrm{Quad}(H) = \mathrm{Quad}(H_{\mathcal{S}} + H_{\mathcal{P}}) = \mathrm{Quad}(H_{\mathcal{S}}) \cap \mathrm{Quad}(H_{\mathcal{P}}) \, . \tag{2.14b}$$

Consequently, H is the infinitesimal generator of the strongly continuous one-parameter unitary group U of time translations[7],

$$U(t) = e^{-itH} \, , \qquad t \in \mathbb{R} \, . \tag{2.15}$$

We write V_c for those $\lambda \in V$ for which (2.13) holds. Note that this is a proper subset of V. Henceforth we assume that the function λ in the interaction Hamiltonian satisfies the above condition.

[5] Of course, μ could simply be incorporated into the function λ. It is, however, useful to have μ as a separate variable so that, by altering its value, we can compare the effect of changing the strength of the coupling, without changing its shape.

[6] For brevity we sometimes write $H_{\mathcal{S}}$ in place of $H_{\mathcal{S}} \otimes I_{\mathcal{P}}$, $H_{\mathcal{P}}$ in place of $I_{\mathcal{S}} \otimes H_{\mathcal{P}}$, and similarly for other operators localized on $\mathcal{Q}$ or $\mathcal{S}$.

[7] We use units where $\hbar$ has the value 1.

We are using the Schrödinger representation of the CCR, so if $\mathcal{X} \in \mathcal{H}$, its time evolution is given by

$$\mathcal{X}_t = U_t \mathcal{X}, \qquad \text{where} \quad U_t = e^{-itH}, \tag{2.16a}$$

and if $\mathcal{X}$ is regular enough, say $\mathcal{X} \in \mathcal{V}$, the Schrödinger equation is satisfied:

$$\partial_t \mathcal{X}_t = -i\, H \mathcal{X}_t. \tag{2.16b}$$

Once we know $\mathcal{X}_t$, the function $t \mapsto \langle \mathcal{X}, \mathcal{X}_t \rangle$ is of particular interest, as it is the *decay law function* for $\mathcal{X}$ under the action of $U(t)$. Further, if $\mathcal{X}$ is normalized, we can evaluate the lifetime of the state it represents by means of the decay law:

$$\tau(\mathcal{X}) = \frac{1}{2} \int_0^\infty |\, \langle \mathcal{X}\,,\, \mathcal{X}_t \rangle\, |^2 \; dt\,, \tag{2.17}$$

according to the standard prescription, see Exner [5], Thirring [10].

3. The Classical Equations

Let us now establish the ansatz for $\mathcal{X}_t$ on the basis of the conservation law discussed in the introduction. For k qubits, an initial state of the class we consider is determined by a normalized vector of the form

$$\mathcal{X} = e[j] \otimes \Omega, \qquad j \in \Pi^k, \tag{3.1}$$

and we assume that η of the qubits are in their upper states. That is, exactly η of the j_i are equal to $+$. At a later time t, the state is of the following form (later on we shall lighten the notation considerably):

$$\mathcal{X}_t = \sum_{\substack{i \in \Pi^k \\ \langle j \rangle = 2\eta - k}} W_t^{(j;0;i)} e[i] \otimes \Omega \qquad \left[\binom{k}{\eta} \text{ no-Boson terms} \right] \tag{3.2a}$$

$$+ \sum_{\substack{i \in \Pi^k \\ \langle j \rangle = 2\eta - 2 - k}} e[i] \otimes \Phi^{(1)}[W_t^{(j;1;i)}] \qquad \left[\binom{k}{\eta - 1} \text{ one-Boson terms} \right] \tag{3.2b}$$

$$+ \sum_{\substack{i \in \Pi^k \\ \langle j \rangle = 2\eta - 6 - k}} e[i] \otimes \Phi^{(2)}[W_t^{(j;2;i)}] \qquad \left[\binom{k}{\eta - 2} \text{ two-Boson terms} \right] \tag{3.2c}$$

$$\vdots$$

$$+\, e[\underbrace{-,-,\ldots,-}_{k \text{ entries}}] \otimes \Phi^{(k)}[W_t^{(j;\eta)}] \qquad \left[\binom{k}{0} \;\; n\text{-Boson terms}\right]. \tag{3.2d}$$

The unknown functions in this expression are the $W_t^{(\sharp)}$, where the superscript identifies the term to which it belongs. Note that the functions in the argument of $\Phi^{(r)}$ are what one might call (time dependent) r-Boson wave functions; they depend on r momentum variables. If we apply the Schrödinger equation to this expression for $\mathcal{X}_t$, orthogonality in the system Hilbert space will give us equations of the form $\partial_t W_t^{(\sharp)} = \mathcal{L}(\ldots, W^{(\flat)}, \ldots)$ where $\mathcal{L}$ is a linear expression in certain $W_t^{(\flat)}$. The dependence will involve terms which are integrals with some $W^{(\flat)}$ as integrands (together with other factors). Hence we arrive at coupled integro-differential equations. Not only are these couplings liable to be quite intricate (depending on what state we are discussing), but for even a modest number of qubits the number of equations in a system, equal to the number of functions $W^{\sharp}$, is uncomfortably large.

3.1. The Lower States

By the lower states we mean the pure states with state vectors $\mathcal{X} = e[j] \otimes \Omega$ where $j \in \Pi^k$ is given by $j_m = -$ for $m = 1, 2, \ldots, k$. This case can be solved quite easily for all k, in particular, for the 1-qubit and 2-qubit lower states.

Proposition 1. *The k-qubit lower state $\mathcal{X} = e[j] \otimes \Omega$ is stationary, with*

$$U_t \mathcal{X} = e^{ik\xi t} \mathcal{X} \tag{3.3}$$

and decay law function

$$\langle \mathcal{X}, U_t \mathcal{X} \rangle = e^{ik\xi t}. \tag{3.4}$$

Hence the lifetime $\tau(\mathcal{X})$ is infinite.

Proof. It is enough to determine the action of H on $\mathcal{X}$ directly. As $\langle j \rangle = -k$, we have $H_{\mathcal{S}} \mathcal{X} = -k\xi \mathcal{X}$, and $H_{\mathcal{P}} \mathcal{X} = 0$ by virtue of the Fock condition. The action of the interaction Hamiltonian is $H_{\mathrm{I}} \mathcal{X} = 0$: on the one hand, $\sigma_- e[-] = 0$, and on the other, $a(\lambda)\Omega = 0$. Hence, for the total Hamiltonian we have $H\mathcal{X} = -k\xi\mathcal{X}$. The result for $U_t \mathcal{X}$, the decay law function, and the lifetime is now immediate. □

This result certainly accords with our intuition: there is no lower energy state available for $\mathcal{X}$ to decay to through spontaneous emission.

3.2. The Upper 1-Qubit State

We now derive the system of equations for the initial state vector $\mathcal{X} = e[+] \otimes \Omega$. We will discuss the existence and uniqueness of the solutions when we consider the problem of separating the integro-differential equations that result. Further on in the paper we will consider the solution in detail; here we do no more than determine the classical equations.

We will need to make use of the inner product *without a complex conjugation*; we introduce the notation $[\![f , g]\!] = \langle \overline{f} , g \rangle$ for all $f, g \in L^2(\mathbb{R})$.

Proposition 2. *The time evolute $\mathcal{X}_t$ of the initial state vector $\mathcal{X} = e[+] \otimes \Omega$ is*

$$\mathcal{X}_t = f(t)\mathcal{X} + e[-] \otimes \Phi^{(1)}(W_t)\,, \tag{3.5}$$

where the unknown functions $f\colon [0,\infty) \to \mathbb{C}$ and $W\colon [0,\infty) \to V$ are continuously differentiable in t and satisfy the pair of equations

$$i\frac{df(t)}{dt} = \xi f(t) + \mu [\![\lambda , W_t]\!] = \xi f(t) + \mu \int_{-\infty}^{\infty} \lambda(k) W_t(k)\, dk\ , \tag{3.6a}$$

$$i\frac{W_t(k)}{dt} = (\varepsilon(k) - \xi)\ W_t(k) + \mu f(t)\lambda(k)\ . \tag{3.6b}$$

The initial conditions are

$$f(0) = 1\ , \qquad W_0(k) = 0\ . \tag{3.7}$$

Equations (3.6*a*) *and* (3.6*b*) *together with the conditions* (3.7) *constitute the classical system associated with the state vector $e[+] \otimes \Omega$.*

When we separate this system, we will find that, for each set $(\xi, \varepsilon, \lambda)$ of Hamiltonian parameters (each set defining a different dynamics), this system has a unique solution with $f \in \mathcal{C}[0,\infty)$ and $W_t \in L^2(\mathbb{R})$ for each $t \geq 0$. Thus we can unambiguously refer to *the* classical solution (f, W) in what follows (the parameters being understood).

Proof. The conservation of probability for the state, $\|\, \mathcal{X}_t \,\|^2 = 1$ for all $t > 0$, follows from the equality

$$|\, f(t)\,|^2 + \|\, W_t \,\|^2 = 1\ . \tag{3.8}$$

To see this, suppose equation (3.8) holds. Then (in the third expression just below, the commutator is meant)

$$\left\|\, a^+(W_t)\Omega \,\right\|^2 = \langle \Omega\,,\, a(\overline{W_t}) a^+(W_t)\Omega\rangle = \langle \Omega\,,\, \left[a(\overline{W_t}), a^+(W_t)\right]_- \Omega\rangle$$

$$= \| W_t \|^2 \ ; \tag{3.9}$$

$\| \mathcal{X}_t \|^2 = 1$ is now immediate.

We now prove equation (3.8) assuming that the pair (f, W) satisfy the classical system. Calculating the time derivative of $| f(t) |^2 + \| W_t \|^2$, for the first term we have

$$\begin{aligned}\frac{d\,| f(t) |^2}{dt} &= 2\mathrm{Re}\Big\{ \overline{f(t)}\,\frac{df(t)}{dt} \Big\} = 2\mathrm{Re}\Big\{ -i\xi\,| f(t) |^2 \ - \ i\mu\overline{f(t)}\,[\![\,\lambda\,,\,W_t\,]\!] \Big\} \\ &= 2\mu\mathrm{Im}\Big\{ \overline{f(t)}\,[\![\,\lambda\,,\,W_t\,]\!] \Big\} \ ;\end{aligned}$$

and for the second we have

$$\begin{aligned}\frac{d\,\| W_t \|^2}{dt} &= 2\mathrm{Re}\Big\{ \int_{\mathbb{R}} \overline{W_t(k)}\,\frac{W_t(k)}{dt}\,dk \Big\} \\ &= 2\mathrm{Re}\Big\{ i\xi\,\| W_t \|^2 - i\int_{\mathbb{R}} \varepsilon(k)\,| W_t(k) |^2\ dk - i\mu\overline{f(t)}\,[\![\,\lambda\,,\,W_t\,]\!] \Big\} \\ &= 2\mu\mathrm{Im}\Big\{ f(t)[\![\,\lambda\,,\,\overline{W_t}\,]\!] \Big\} \ = \ -2\mu\mathrm{Im}\Big\{ \overline{f(t)}[\![\,\lambda\,,\,W_t\,]\!] \Big\} \ ,\end{aligned}$$

all $t > 0$. We see that the time derivative of $| f(t) |^2 + \| W_t \|^2$ vanishes. The initial conditions now give us equation (3.8).

It remains to prove that given a classical solution (f, W), the vector $\mathcal{X}_t$ given by equation (3.5) is equal to $U(t)\big(e[+] \otimes \Omega\big)$.

We proceed by means of the Schrödinger equation. In detail,

$$\begin{aligned} H_{\mathcal{S}}\,\mathcal{X}_t &= \xi f(t)\mathcal{X} - \xi e_- \otimes \Phi^{(1)}(W_t) \ , \\ H_{\mathcal{P}}\,\mathcal{X}_t &= e[-] \otimes \Phi^{(1)}(\varepsilon W_t) \ , \\ H_{\mathrm{I}}\,\mathcal{X}_t &= \mu[\![\,\lambda\,,\,W_t\,]\!]\mathcal{X} + \mu f(t) e[-] \otimes \Phi^{(1)}(\lambda) \ ,\end{aligned}$$

so that

$$\begin{aligned} H\,\mathcal{X}_t &= \Big(\xi f(t) + \mu[\![\,\lambda\,,\,W_t\,]\!] \Big)\mathcal{X} + e[-] \otimes \Phi^{(1)}\big(-\xi W_t + \varepsilon W_t + f(t)\mu\lambda \big) \\ &= i\frac{df(t)}{dt}\,\mathcal{X} + ie[-] \otimes \Phi^{(1)}\left(\frac{\partial W_t}{\partial t} \right) \ = \ i\frac{d\mathcal{X}_t}{dt} \ ,\end{aligned}$$

for all $t > 0$. Since $\mathcal{X}_0 = \mathcal{X}$, the proof is complete. □

Corollary 3. *The decay law function for* $e[+] \otimes \Omega$ *is given by* $t \mapsto f(t)$, *and so the lifetime of this state is given by the formula*

$$\tau\big(e[+] \otimes \Omega\big) \ = \ \frac{1}{2}\int_0^\infty \ | f(t) |^2\ dt \ , \tag{3.10}$$

and is finite or infinite as f is square integrable or not.

We will not be able to say anything about finiteness or not of τ until we analyze the solutions.

3.3. The Intermediate 2-Qubit States

By the intermediate 2-qubit states we mean the pair $e[-,+]\otimes\Omega$ and $e[+,-]\otimes\Omega$ together with their antisymmetric and symmetric linear combinations

$$\mathcal{X}_{\mathrm{A}} = e_{\mathrm{A}} \otimes \Omega \quad \text{and} \quad \mathcal{X}_{\mathrm{S}} = e_{\mathrm{S}} \otimes \Omega\,. \tag{3.11a}$$

Here we have introduced the notation

$$e_{\mathrm{A}} = \frac{1}{\sqrt{2}}\Big(e[-,+] - e[+,-] \Big) \text{ and } e_{\mathrm{S}} = \frac{1}{\sqrt{2}}\Big(e[-,+] + e[+,-] \Big)\,. \tag{3.11b}$$

The state vectors $\mathcal{X}_{\mathrm{A}}$ and $\mathcal{X}_{\mathrm{S}}$ are important in quantum computing theory, where they are known as Bell states.

The derivations of the classical equations for these states are entirely similar to that for $e[+]\otimes\Omega$, so we omit them. The results are as follows.

Proposition 4. *The state vector $\mathcal{X}_{\mathrm{A}}$ is annihilated by the Hamiltonian, $H\,\mathcal{X}_{\mathrm{A}} = 0$, and so $\mathcal{X}_{\mathrm{A}}$ is time invariant. Hence the lifetime, $\tau\,(\mathcal{X}_{\mathrm{A}})$, is infinite.*

For the symmetric state the results are as follows.

Proposition 5. *The time evolute $\mathcal{X}_{\mathrm{S}}(t)$ of the initial state vector $\mathcal{X}_{\mathrm{S}}$ is*

$$\mathcal{X}_{\mathrm{S}}(t) = e^{i\xi t} f(t)\mathcal{X}_{\mathrm{S}} + e[-,-] \otimes \Phi^{(1)}(e^{i\xi t} W_t)\,, \tag{3.12}$$

where (f,W) is the same classical pair we obtained for the one qubit upper state $e[+]\otimes\Omega$. Hence the decay law function for $\mathcal{X}_{\mathrm{S}}$ is $t \mapsto e^{i\xi t} f(t)$ and the lifetime $\tau(\mathcal{X}_{\mathrm{S}})$ is finite or infinite as $\tau(e[+]\otimes\Omega)$ is.

Corollary 6. *The time evolution of the other two intermediate 2-qubit states can be obtained from those of $\mathcal{X}_{\mathrm{A}}$ and $\mathcal{X}_{\mathrm{S}}$, since*

$$U_t\,(e[+,-]\otimes\Omega) = \frac{\mathcal{X}_{\mathrm{S}}(t) + \mathcal{X}_{\mathrm{A}}}{2}, \quad U_t\,(e[-,+]\otimes\Omega) = \frac{\mathcal{X}_{\mathrm{S}}(t) - \mathcal{X}_{\mathrm{A}}}{2}. \tag{3.13}$$

Hence the solutions to these evolution systems can also be obtained from that of the 1-qubit upper state.

3.4. The Upper 2-Qubit State

There remains one more 2-qubit vector state to consider, $e[+,+]\otimes\Omega$, the upper state. Using the same method as for the other states we obtain the following result.

Proposition 7. *The time evolution of the state vector* $e[+,+]\otimes\Omega$ *is given by*

$$U(t)\Big(e[+,+]\otimes\Omega\Big) = h(t)e[+,+]\otimes\Omega + e_{\mathrm{S}}\otimes\Phi^{(1)}(B_t) + e[-,-]\otimes\Phi^{(2)}(C_t), \quad (3.14)$$

where the associated classical system is as follows: $h\in\mathcal{C}^1[0,\infty)$, *and for each* $t\geq 0$, $B_t\in L^2(\mathbb{R})$, $C_t\in L^2_+(\mathbb{R}^2)$ *are continuously differentiable in* t *and satisfy the coupled integro-differential equations*

$$i\frac{dh(t)}{dt} = 2\xi h(t) + \mu[\![\,\lambda\,,\,B_t\,]\!] = 2\xi h(t) + \mu\int_{-\infty}^{\infty}\lambda(k)B_t(k)\,dk\ , \qquad (3.15a)$$

$$i\frac{dB_t}{dt} = \mu h(t)\lambda + \varepsilon B_t + \sqrt{2}\mu\int_{-\infty}^{\infty}C_t(\cdot,k)\lambda(k)\,dk\ , \qquad (3.15b)$$

$$i\frac{dC_t}{dt} = \big(d\Gamma^{(2)}(\varepsilon) - 2\xi\big)C_t\ +\ \mu\frac{\lambda\otimes B_t + B_t\otimes\lambda}{\sqrt{2}}\ , \qquad (3.15c)$$

with the initial conditions

$$h(0) = 1\ , \qquad B_0(k) = 0\ , \qquad C_0(k,q) = 0\ . \qquad (3.15d)$$

It follows that, for all $t\geq 0$,

$$|\,h(t)\,|^2 + \|\,B_t\,\|_2^2 + \|\,C_t\,\|_2^2 = 1\ , \qquad (3.16)$$

which is equivalent to the conservation of quantum probability.

We call attention to the fact that the one Boson contribution at time t involves the two qubits only in their symmetric combination. This symmetry reduces the possible number of equations in this system from 4 to 3. Hence the growth in the number of equations with qubit number in the upper states falls (just) short of exponential.

4. Separation, Upper 1-Qubit State

In this section we separate the classical system for the two unknown functions $f(t)$ and W_t associated with the vector $\mathcal{X}_t$. To do so, it is convenient to introduce a function $\varphi\colon \mathbb{R}^+ \to \mathbb{C}$ which differs from f by what is essentially a shift in the origin of the qubit energy scale. The function φ obeys a Volterra integral equation of the second kind from which f and W can be obtained. This will prove the existence and uniqueness of a solution (f, W) for each triple $(\xi, \varepsilon, \lambda)$. We do not solve the Volterra equation for φ in this section; that is done further on.

Parenthetically, if uniqueness did not hold there would be different orbits $(\mathcal{X}_t)_t$ passing through the initial point $\mathcal{X}$, which is physically absurd.

We begin be defining a certain integral transform[8] (in engineering terminology this is a filtering transform. For a connection to the spectrum of the operator ε, see Picard [9]).

Lemma 8. *For functions $g \in L^1(\mathbb{R})$ for which $\varepsilon g \in L^1(\mathbb{R})$, we introduce the* energy transform *by the formula*

$$\mathcal{E}[g](t) = \int_{\mathbb{R}} g(k) e^{-i\varepsilon(k)t}\, dk\ , \qquad t \geq 0\ . \tag{4.1}$$

Then $\mathcal{E}[g]$ is a uniformly continuous bounded function on $[0, \infty)$.

The proof is straightforward and we omit it. We now introduce two integral kernels that appear in the analysis.

Corollary 9. *The functions*

$$K(t) = \int_{\mathbb{R}} \lambda(k)^2 e^{-i\varepsilon(k)t}\, dk\ , \tag{4.2a}$$

$$\mathcal{K}(t) = 2i\xi - i\mu^2 \int_{\mathbb{R}} \lambda(k)^2\ \frac{1 - e^{-i\varepsilon(k)t}}{\varepsilon(k)}\, dk\ , \tag{4.2b}$$

are uniformly continuous bounded functions on $[0, \infty)$. Moreover $\mathcal{K}$ is continuously differentiable on $[0, \infty)$, with

$$\mathcal{K}' \;=\; \mu^2 K. \tag{4.2c}$$

[8] For uniformity of notation, we use square brackets around the argument for all integral transforms.

Proof. By definition $K = \mathcal{E}(\lambda^2)$ and

$$\mathcal{K} = 2i\xi \, - \, i\mu^2 \int_{\mathbb{R}} \frac{\lambda(k)^2}{\varepsilon(k)} \, dk + i\mu^2 \mathcal{E}[\lambda^2 \varepsilon^{-1}] \, .$$

Since $\lambda \in W$ and satisfies (2.13), the first results are immediate. Using Fubini's Theorem, it follows that, for all $t \geq 0$,

$$\mathcal{K}(t) = 2i\xi + \mu^2 \int_0^t K(u) \, du \, ,$$

from which the differentiability result follows. □

Now we introduce the function φ promised above, out of which the solution for f and W can be constructed. We shall reserve the symbol φ for this function throughout this paper.

Lemma 10. *Let φ: $[0, \infty) \to \mathbb{C}$ be defined as the unique solution to the Volterra integral equation*

$$\varphi(t) = 1 - \int_0^t \mathcal{K}(t-s)\varphi(s) \, ds \, , \qquad t \geq 0 \, . \tag{4.3}$$

Then $\varphi \in \mathcal{C}^1[0, \infty)$ and its derivative satisfies the integro-differential equation

$$i\frac{d\varphi}{dt} = 2\xi\varphi(t) - i\mu^2 \int_0^t K(t-s)\varphi(s) \, ds \, , \qquad t \geq 0 \, . \tag{4.4}$$

Proof. Since $\mathcal{K}$ is a bounded function on $[0, \infty)$, standard integral equation theory implies that there exists a unique solution $\varphi \in \mathcal{C}[0, \infty)$ to equation (4.3). Since $\mathcal{K}$ is differentiable, it follows that φ is continuously differentiable on $[0, \infty)$, and satisfies the integro-differential equation (4.4), completing the proof. □

The classical solution (f, W) can now be obtained in terms of φ.

Proposition 11. *The functions (f, W) constituting the unique classical solution for the state vector $\mathcal{X}$ are given by*

$$f(t) \;=\; e^{i\xi t}\varphi(t) \, , \tag{4.5a}$$

$$iW_t(k) \;=\; \mu \left(\int_0^t f(s) e^{i(\varepsilon(k)-\xi)(s-t)} \, ds \right) \lambda(k) \, , \tag{4.5b}$$

for $t \geq 0$.

Proof. It follows from what has gone before that f, so defined, is a continuously differentiable real-valued function on $[0, \infty)$, and that W is a continuously differentiable V-valued function on $[0, \infty)$. Hence (f, W) satisfy the required regularity conditions.

From Fubini's Theorem we obtain

$$
\begin{aligned}
i[\![\, \lambda \,,\, W_t \,]\!] &= \mu \int_0^t f(s) e^{i\xi(t-s)} \left(\int_{\mathbb{R}} \lambda(k)^2 e^{i\varepsilon(k)(s-t)} \, dk \right) ds \\
&= \mu \int_0^t f(s) e^{i\xi(t-s)} K(t-s) \, ds \,,
\end{aligned} \tag{4.6}
$$

and so

$$
\begin{aligned}
i\frac{df}{dt} &= \xi f(t) \,-\, i\mu^2 \int_0^t e^{i\xi(t-s)} K(t-s) f(s) \, ds \\
&= \xi f(t) \,+\, \mu[\![\, \lambda \,,\, W_t \,]\!] \,, \\
i\frac{\partial W_t(k)}{\partial t} &= \mu f(t)\lambda(k) \,+\, i\mu(\xi - \varepsilon(k)) \left(\int_0^t f(s) e^{i(\varepsilon(k)-\xi)(s-t)} \, ds \right) \lambda(k) \\
&= -\xi W_t(k) \,+\, \varepsilon(k) W_t(k) \,+\, \mu f(t) \lambda(k) \,,
\end{aligned}
$$

as required. The initial conditions for f and W are trivially satisfied. □

We recall that (f, W) also determines the evolution of $e_{\mathrm{S}} \otimes \Omega$.

5. Separation and Solution for the Upper 2-Qubit State

Earlier we introduced a condition for the equation coupling function λ which included equation (2.13), ensuring that the Hamiltonian is self-adjoint. We now impose further conditions on λ which will ensure a smooth solution to both this and the 1-qubit upper state classical system.

Definition 12. (Regulated Coupling) By a *coupling function* for this model we mean the nonnegative function ψ defined in terms of the interaction Hamiltonian function λ by[9]

$$
\psi(u) = \begin{cases} \dfrac{1}{\varepsilon'(\varepsilon^{-1}(u))} \left[\lambda(\varepsilon^{-1}(u))^2 + \lambda(-\varepsilon^{-1}(u))^2 \right] & u > 0 \,, \\ 0 \,, & u < 0 \,. \end{cases} \tag{5.1}
$$

From the condition λ_1 applied to ψ, we obtain the following, which while not new, we label λ_2:

[9] ψ is essentially the function that results when we transform the momentum variable in λ to the energy as independent variable. We include a denominator which will cancel the chain rule factor.

λ_2: $p\psi \in L^1(\mathbb{R})$ for any polynomial p, and

$$\int_{\mathbb{R}} \frac{\psi(u)}{u}\, du \;<\; \infty\,. \tag{5.2}$$

We say that the function ψ is *regulated* if it satisfies the following three conditions in addition to λ_1 and λ_2:

λ_3: $\psi \in \mathcal{C}^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

λ_4: $\psi' \in L^2(\mathbb{R})$,

λ_5: there exist constants $0 \leq \alpha < 2\xi < \beta \leq \infty$ such that $\psi(u) > 0$ if and only if $u \in (\alpha, \beta)$.

When the coupling function is regulated we say that *the model is regulated.*

Remark 13. Conditions λ_3 and λ_4 are basic smoothness properties; condition λ_5 is of a different character. It provides a smooth infrared cutoff if $\alpha > 0$ and adequate infrared control if $\alpha = 0$. Similarly, it provides an ultraviolet cutoff if $\beta < \infty$ and adequate control if $\beta = \infty$. The requirement for 2ξ to lie between α and β ensures that the energy transferred from field to atom is no greater than the qubit energy level difference. This prevents ionization.

Note that the differentiability of ψ at the origin implies that the function $u^{-1}\psi(u)$ is integrable, so that equation (5.2) is satisfied. Hence $\psi \in W_c$.

From now on we assume that ψ is regulated.

The next step in separation is to transform from the momentum k to the energy ε as independent variable in the unknown functions. At the same time we take the opportunity to scale out some factors of λ for convenience.

Lemma 14. *Let*

$$B_t(k) = \mathfrak{b}_t\big(\varepsilon(k)\big)\lambda(k)\;, \tag{5.3a}$$

$$C_t(k,\ell) = \mathfrak{c}_t\big(\varepsilon(k),\varepsilon(\ell)\big)\lambda(k)\lambda(\ell) \tag{5.3b}$$

define $\mathfrak{b}_t\colon [0,\infty) \to \mathbb{C}$ *and* $\mathfrak{c}_t\colon [0,\infty)\times[0,\infty) \to \mathbb{C}$, *with* $\mathfrak{c}_t$ *symmetric in its coordinates. For these functions, the integro-differential equations are (see equations* (3.15a) - (3.15c)*)*

$$ih'(t) = 2\xi h(t) \;+\; \mu \int_0^\infty \mathfrak{b}_t(x)\psi(x)\, dx\;, \tag{5.4a}$$

$$i\mathfrak{b}_t'(x) = \mu h(t) \;+\; x\mathfrak{b}_t(x) \;+\; \sqrt{2}\mu \int_0^\infty \mathfrak{c}_t(x,y)\psi(y)\, dy\;, \tag{5.4b}$$

$$i\mathfrak{c}_t'(x,y) = (x+y-2\xi)\mathfrak{c}_t(x,y) \;+\; \frac{\mu}{\sqrt{2}}\big(\mathfrak{b}_t(x)+\mathfrak{b}_t(y)\big)\;, \tag{5.4c}$$

with initial conditions (see equation (3.15d)*)*

$$\mathfrak{b}_0(x) = 0, \qquad \mathfrak{c}_0(x,y) = 0\;. \tag{5.5}$$

There is a unique triple (h, B_t, C_t) *satisfying equations* (3.15*a*)-(3.16) *if and only if there is a unique triple* $(h, \mathfrak{b}_t, \mathfrak{c}_t)$*, and this follows from the properties of* ε *and* ψ*.*

We omit the proof.

We have separated these equations by expressing the solution triple in terms of a function Z on the first octant, where Z itself is defined as the solution to a Volterra integral equation. We begin by introducing this function and its integral kernel $\mathfrak{X}$, then proving that the iterative sequence obtained from its kernel converges to Z uniformly on compact subsets of the first octant. Finally we connect Z to the classical solution triple.

Proposition 15. *Let* $\mathbf{\Delta} = \{(t,q) \,:\, t \geq q \geq 0\}$ *be the first octant in the* (t,q)*-plane and let* $\mathfrak{X} \in \mathcal{L}(\mathcal{C}(\mathbf{\Delta}))$ *be the operator defined by the formula*

$$\begin{aligned}(\mathfrak{X}F)(t,q) = & \int_q^t du \int_q^u ds F(s,q) e^{2i\xi(u-s)} K(u-s) \\ & + \int_q^t du \int_0^q ds F(q,s) e^{2i\xi(u-s)} K(u-s) \\ & + \int_0^q du \int_0^u ds F(u,s) e^{2i\xi(u-s)} K(u-s)\end{aligned} \tag{5.6}$$

for all $F \in \mathcal{C}(\mathbf{\Delta})$*.*

The integral equation

$$Z = 1 - \mathfrak{X}Z \tag{5.7}$$

has a unique solution $Z \in \mathcal{C}\,(\mathbf{\Delta})$*, which can be expressed as the limit of the series*

$$Z(t,q) = \sum_{n=0}^{\infty} (-1)^n Z_n(t,q)\,, \tag{5.8}$$

which converges uniformly on compact subsets of $\mathbf{\Delta}$*. Here the functions* $Z_n \in \mathcal{C}(\mathbf{\Delta})$ *(*$n \geq 0$*) are given by the recurrence formulae*

$$Z_0(t,q) \;=\; 1\;, \tag{5.9a}$$

$$Z_n(t,q) \;=\; (\mathfrak{X}Z_{n-1})(t,q)\;, \qquad n \in \mathbb{N}\;. \tag{5.9b}$$

Proof. It is convenient to define the function $\mathcal{G}\colon [0,\infty) \to \mathbb{C}$ by

$$\mathcal{G}(t) = \mu^2 \int_0^t e^{2i\xi u} K(u)\, du.$$

Just as for $\mathcal{K}$, we can show that the function $\mathcal{G}'$ is uniformly bounded on $[0,\infty)$ by $\mu^2 \,\|\, \lambda \,\|_2^2$.

Suppose that $F \in \mathcal{C}(\mathbf{\Delta})$ and $|\, F(t,q) \,| \le (t+q)^{2n}$ for all $(t,q) \in \mathbf{\Delta}$. Then

$$\begin{aligned}
&\left| \int_q^t du \int_q^u ds\, F(s,q)\mathcal{G}'(u-s) + \int_q^t du \int_0^q ds\, F(q,s)\mathcal{G}'(u-s) \right| \\
&\quad \le \; \big\|\mathcal{G}'\big\|_\infty \left\{ \int_q^t du \int_q^u ds\, (s+q)^{2n} + \int_q^t du \int_0^q ds\, (s+q)^{2n} \right\} \\
&\quad = \; \big\|\mathcal{G}'\big\|_\infty \int_q^t du \int_0^u ds\, (s+q)^{2n} \\
&\quad \le \; \frac{1}{2n+1} \big\|\mathcal{G}'\big\|_\infty \int_q^t du\, (u+q)^{2n+1} \\
&\quad \le \; \frac{1}{(2n+1)(2n+2)} \big\|\mathcal{G}'\big\|_\infty (t+q)^{2n+2} \,,
\end{aligned}$$

while

$$\begin{aligned}
&\left| \int_0^q du \int_0^u ds\, F(u,s)\mathcal{G}'(u-s) \right| \\
&\quad \le \big\|\mathcal{G}'\big\|_\infty \int_0^q du \int_0^u ds\, (u+s)^{2n} \quad \le \frac{1}{2n+1} \big\|\mathcal{G}'\big\|_\infty \int_0^q du\, (2u)^{2n+1} \\
&\le \frac{1}{2n+1} \big\|\mathcal{G}'\big\|_\infty \int_0^q du\, (t+u)^{2n+1} \le \frac{1}{(2n+1)(2n+2)} \big\|\mathcal{G}'\big\|_\infty (t+q)^{2n+2} \,,
\end{aligned}$$

and hence

$$|\, (\mathfrak{X}F)(t,q) \,| \;\le\; \frac{2}{(2n+1)(2n+2)} \big\|\mathcal{G}'\big\|_\infty (t+q)^{2n+2} \,, \qquad (t,q) \in \mathbf{\Delta} \,.$$

Thus if we define functions $Z_n \in \mathcal{C}(\mathbf{\Delta})$ as above, we deduce by induction that

$$|\, Z_n(t,q) \,| \;\le\; \frac{2^n}{(2n)!} \big\|\mathcal{G}'\big\|_\infty^n (t+q)^{2n} \,, \qquad (t,q) \in \mathbf{\Delta} \,,$$

from which it follows that the infinite series converges uniformly on compact subsets of $\mathbf{\Delta}$ to the solution of the integral equation for Z. $\square$

Corollary 16. *The triple* $(h, \mathfrak{b}_t, \mathfrak{c}_t)$ *may be expressed in terms of* Z *by*

$$\mathfrak{b}_t(x) = -i\mu \int_0^t Z(t,q) e^{-2i\xi q} e^{-i(t-q)x}\, dq \ , \tag{5.10a}$$

$$e^{2i\xi t} h(t) = 1 + \int_0^t du \int_0^u ds\, Z(u,s) e^{2i\xi(u-s)} K(u-s) \ , \tag{5.10b}$$

$$\mathfrak{c}_t(x,y) = 2^{-1/2} \int_0^t ds \int_0^s dq\, Z(s,q) e^{2i\xi(t-s-q)} \times \left\{ e^{-ix(t-q)} e^{-iy(t-s)} + e^{-ix(t-s)} e^{-iy(t-q)} \right\} \tag{5.10c}$$

Hence, from the properties of Z *just proved, the triple* $(h, \mathfrak{b}_t, \mathfrak{c}_t)$ *has a unique solution, and so, therefore, does the classical system* (h, B_t, C_t).

This completes our analysis of the upper 2-qubit state.

6. Technical Results Concerning Hardy Spaces

It remains to determine the function φ, from which the solutions to the one qubit and intermediate two qubit solutions can be inferred. An iterated kernel series solution is available as for the upper two qubit state but we can do better and obtain a closed form solution to equation (4.3). To do so, we resort to the Laplace transform and inversion. To manage the inversion rigorously requires detailed knowledge of the analytic properties of the Laplace transform of the kernel $\mathcal{K}$. Since we are working with an unspecified coupling function λ and its energy form ψ, analytic behaviour must be inferred from general principles. This has required an excursion into function theory on Hardy space (for the basic material, see Hoffman [6]).

Notation 17. We write $\mathbb{H}_\pm = \{ z \in \mathbb{C} : \mathrm{Re} z \gtrless 0 \}$ for the open right and left half-planes, and $H^p(\mathbb{H}_+)$ for the Hardy p-spaces of functions on the right open half-plane; only $p = 2$ and $p = \infty$ will be needed. Our convention for the Fourier transform is the following: if $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\mathcal{F}[g](y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x) e^{-ixy}\, dx \ . \tag{6.1}$$

We shall only apply the Laplace transform to functions $f\colon [0,\infty) \to \mathbb{C}$ which are bounded and continuously differentiable on their domain. Then if f is such a function,

$$\mathcal{L}[f](p) = \int_0^\infty e^{-px} f(x)\, dx \tag{6.2}$$

exists for all $p \in \mathbb{H}_+$.

As we are particularly interested in the case where $f \in L^2(0,\infty)$, we use the following application of the Paley-Wiener Theorem, see Hoffman [6].

Theorem 18. *A bounded and continuously differentiable function $f\colon [0,\infty) \to \mathbb{C}$ belongs to $L^2(0,\infty)$ if and only if $\mathcal{L}[f]$ belongs to $H^2(\mathbb{H}_+)$. In this case, let $\mathfrak{h} \in L^2(\mathbb{R})$ be the boundary function of $\mathcal{L}[f]$ in the L^2-sense:*

$$\lim_{\delta\to 0+} \int_{\mathbb{R}} |\, \mathcal{L}[f](\delta - iy) - \mathfrak{h}(y)\,|^2 \; dy \;=\; 0 \,, \tag{6.3}$$

and let $\mathcal{F}[\mathfrak{h}]$ be its Fourier transform. Further, let $\widehat{f} \in L^2(\mathbb{R})$ be the function f extended to non-negative times as

$$\widehat{f}(t) \;=\; \begin{cases} f(t)\,, & t \geq 0\,, \\ 0\,, & t < 0\,. \end{cases} \tag{6.4}$$

Then in the $L^2(\mathbb{R})$ sense,

$$\widehat{f} \;=\; \frac{1}{\sqrt{2\pi}}\, \mathcal{F}[\mathfrak{h}]\;. \tag{6.5}$$

We will use the following transform developed by Bremermann [1] in the context of distribution theory. When f is supported on a (finite) line, this transform is closely connected to potentials of simple and double layers distributed along the support, see Muskhelishvili [8].

Definition 19. For any $f \in L^2(\mathbb{R})$, the function

$$L[f](p) = \int_{-\infty}^{\infty} \frac{f(u)}{p + iu}\, du\;, \qquad p \in \mathbb{H}_+\,, \tag{6.6}$$

is analytic in $\mathbb{H}_+$, and we call it the Cauchy transform of f.

Remark 20. Bremermann's *Cauchy representation* is defined for distributions $f \in \mathcal{E}'$, with

$$p \mapsto \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u)}{u - p}\, du$$

analytic in the complement of the support of f. Relative to the class of functions we are considering, our Cauchy transform is seen to differ from this by a rotation of coordinates and a rescaling.

Definition 19 does not tell us to what class the Cauchy transforms belong; we consider that next, along with an expression for the boundary function.

Proposition 21. *The Cauchy transform is a linear map* $L^2(\mathbb{R}) \to H^2(\mathbb{H}_+)$.

For $f \in L^2(\mathbb{R})$*, the boundary function of* $L[f]$ *in the sense of equation* (6.3), *is* $2\pi\mathfrak{P}f$*, where* $\mathfrak{P}$ *is the orthogonal projection on* $L^2(\mathbb{R})$ *defined by*

$$\mathfrak{P} = \chi_{[0,\infty)}(P). \tag{6.7a}$$

Here P *is the momentum operator in the Schrödinger representation. Hence, if* Q *is the corresponding position operator, we have the intertwining relation*

$$\mathcal{F}\mathfrak{P} = \chi_{[0,\infty)}(Q)\mathcal{F}\,. \tag{6.7b}$$

The operator $\mathfrak{P}$ *is connected to the Hilbert transform* $\mathcal{H} = -i\,\mathrm{sign}(P)$ *by*

$$\mathfrak{P} = \frac{1}{2} + \frac{1}{2}i\mathcal{H}\,. \tag{6.7c}$$

Proof. For $f \in L^2(\mathbb{R})$, the function

$$f_\delta(y) \;=\; L[f](\delta - iy)\;, \qquad \delta > 0\,,$$

is the convolution $q_\delta \star f$, where $q_\delta \in L^2(\mathbb{R})$ is the function

$$q_\delta(u) \;=\; (\delta - iu)^{-1}\;.$$

Then $f_\delta \in L^2(\mathbb{R})$, with Fourier transform

$$\mathcal{F}[f_\delta] \;=\; \sqrt{2\pi}\mathcal{F}[q_\delta]\cdot\mathcal{F}[f] \;=\; 2\pi Q_\delta\,\mathcal{F}[f]\;,$$

$$Q_\delta(t) \;=\; \begin{cases} e^{-\delta t}\,, & t \ge 0\,,\\ 0\,, & t < 0\,. \end{cases}$$

This implies the bound $\|\, f_\delta \,\|_2 \;\le\; 2\pi \,\|\, f\,\|_2$ for all $\delta > 0$, so that $L[f] \in H^2(\mathbb{H}_+)$.

Moreover

$$\lim_{\delta\to 0+} \|\, f_\delta - 2\pi\mathfrak{P}f\,\|_2^2 \;=\; 4\pi^2 \lim_{\delta\to 0+}\int_0^\infty \left(1 - e^{-\delta t}\right)^2 |\, f(t)\,|^2\; dt \;=\; 0\;,$$

using the Monotone Convergence Theorem, which shows that $2\pi\mathfrak{P}f$ is the boundary function of $L[f]$. □

With additional conditions on f, its Cauchy transform converges to its limit function uniformly, and its limit function is continuous. To make these assertions precise, we first note a little lemma for Fourier transforms.

Lemma 22. *If $f \in \mathcal{C}^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$, then $\mathcal{F}[f] \in L^1(\mathbb{R})$.*

Proof. The given conditions are enough to imply that $\mathcal{F}[f'](t) = it\mathcal{F}[f](t)$ for almost all t, which implies that $\mathcal{F}[f]$ is integrable. □

Corollary 23. *Suppose f satisfies the conditions of Lemma 22. Then $L[f] \in H^\infty(\mathbb{H}_+)$, $\mathfrak{P}f \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and*

$$\lim_{\delta\to 0+} \| f_\delta - 2\pi\mathfrak{P}f \|_\infty \;=\; 0 \;. \tag{6.8}$$

Proof. We know that $\mathcal{F}[f] \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and so $\mathcal{F}[f_\delta] = 2\pi Q_\delta \cdot \mathcal{F}[f]$ and $\mathcal{F}[\mathfrak{P}f] = \chi_{[0,\infty)} \cdot \mathcal{F}[f]$ both belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as well. Thus

$$\| f_\delta \|_\infty \;\le\; \frac{1}{\sqrt{2\pi}} \| \mathcal{F}[f_\delta] \|_1 \;\le\; \sqrt{2\pi}\, \| \mathcal{F}[f] \|_1$$

for all $\delta > 0$, which implies that $L[f] \in H^\infty(\mathbb{H}_+)$. Moreover

$$\begin{aligned} \| f_\delta - 2\pi\mathfrak{P}f \|_\infty \;&\le\; \frac{1}{\sqrt{2\pi}} \| \mathcal{F}[f_\delta] - 2\pi\mathcal{F}[\mathfrak{P}f] \|_1 \\ &=\; \sqrt{2\pi} \int_0^\infty (1 - e^{-\delta t})\, |\, \mathcal{F}[f](t)\, |\; dt \end{aligned}$$

for all $\delta > 0$, and hence $\| f_\delta - 2\pi\mathfrak{P}f \|_\infty \to 0$ as $\delta \to 0+$, by the Monotone Convergence Theorem. It follows from these results that $\mathfrak{P}f \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, as required. □

7. Continuity of φ as a Function of $\boldsymbol{\lambda}$

It is important to know how φ behaves as a function of the coupling strength. This can be determined by utilizing the method[10] of successive approximations associated with Volterra integral equations.

For this next proposition only, it is typographically convenient to write $\ell = \mu\lambda \in V_c$.

[10] This method provides a means of calculating the function φ numerically, although we have not pursued this possibility.

Proposition 24. *If ℓ_1, $\ell_2 \in V_c$, defining Hamiltonians H_1, H_2 and hence solutions φ_1, φ_2 for the corresponding classical systems, then*

$$|\,\varphi_1(t) - \varphi_2(t)\,| \;\le\; t^2 \,\big\|\,\ell_1^2 - \ell_2^2\,\big\|_1 \,\exp\big[(2\xi + \|\,\ell_1\,\|_2^2\, t)t\big]\;, \qquad t \ge 0\;. \tag{7.1}$$

Proof. Denoting the integral kernels associated with the two coupling functions by $\mathcal{K}_j$, for $j = 1, 2$,

$$|\,\mathcal{K}_j(t)\,| \;\le\; 2\xi + \|\,\ell_j\,\|_2^2\, t\;, \qquad t \ge 0\;, \quad j = 1, 2\;,$$

and that

$$|\,\mathcal{K}_1(t) - \mathcal{K}_2(t)\,| \;\le\; \big\|\,\ell_1^2 - \ell_2^2\,\big\|_1\; t\;, \qquad t \ge 0\;.$$

Since $|\,\varphi_j(t)\,| \le 1$ for all $t \ge 0$, we see that

$$\begin{aligned}
&|\,\varphi_1(t) - \varphi_2(t)\,| = \left| \int_0^t \big(\mathcal{K}_2(t-s)\varphi_2(s) - \mathcal{K}_1(t-s)\varphi_1(s)\big)ds \right| \\
&\le \|\,\ell_1^2 - \ell_2^2\,\|_1 \int_0^t (t-s)\,|\,\varphi_2(s)\,|\,ds \\
&+ \int_0^t [2\xi + \|\,\ell_1\,\|_2^2\,(t-s)]\,|\,\varphi_1(s) - \varphi_2(s)\,|\,ds \\
&\le \frac{1}{2}\,\|\,\ell_1^2 - \ell_2^2\,\|_1\, t^2 + \int_0^t [2\xi + \|\,\ell_1\,\|_2^2\,(t-s)]\,|\,\varphi_1(s) - \varphi_2(s)\,|\,ds\;,
\end{aligned}$$

for $t \ge 0$. As $|\,\varphi_1(t) - \varphi_2(t)\,| \;\le\; 2$ for $t \ge 0$, we can show inductively that

$$\begin{aligned}
&|\,\varphi_1(t) - \varphi_2(t)\,| \\
&\qquad \le\; \|\,\ell_1^2 - \ell_2^2\,\|_1 \sum_{m=2}^{n+1} \frac{(2\xi + \|\,\ell_1\,\|_2^2\, t)^{m-2} t^m}{m!} \;+\; \frac{2(2\xi + \|\,\ell_1\,\|_2^2\, t)^n t^n}{n!}\;,
\end{aligned}$$

for all $t \ge 0$ and $n \ge 1$. Letting $n \to \infty$, we obtain the desired result. □

Corollary 25. *The function $V_c \to \mathcal{C}[0,\infty)$, $\ell \mapsto \varphi$, is continuous if V_c is given the metric topology it inherits as a subspace of $L^2(\mathbb{R})$, and $\mathcal{C}[0,\infty)$ the topology of uniform convergence on compact subsets of $[0,\infty)$.*

Proof. Note that $\big\|\,\ell_1^2 - \ell_2^2\,\big\|_1 \;\le\; \|\,\ell_1 - \ell_2\,\|_2\,(\|\,\ell_1\,\|_2 + \|\,\ell_2\,\|_2)$ for all $\ell_1, \ell_2 \in V_c$. Thus, for any $T > 0$ and $\ell_1 \in V_c$, we can find a constant $A > 0$ such that

$$\sup_{0 \le t \le T} |\,\varphi_1(t) - \varphi_2(t)\,| \;\le\; A(\|\,\ell_1\,\|_2 + \|\,\ell_2\,\|_2)\,\|\,\ell_1 - \ell_2\,\|_2$$

for all $\ell_2 \in V_c$. This establishes the required continuity at ℓ_1. □

We now revert to considering μ and λ separately. When the dependence of φ on μ needs emphasis we write φ_μ.

Corollary 26. *For any $t \geq 0$, $\varphi_\mu(t)$ is a continuous function of μ. Moreover, the function sending μ to $\varphi_\mu|_{[0,T]}$ is a continuous function from $[0,\infty)$ to $\mathcal{C}[0,T]$ for any $T > 0$.*

Writing $\mathcal{X}$ for the vector $e[+] \otimes \Omega$, if the coupling constant is μ, we write $\mathcal{X}_t^{(\mu)}$ for its time translates. Then this corollary means, in particular, that any solution $\mathcal{X}_t^{(\mu)}$ will converge to the no-coupling solution $\mathcal{X}_t^{(0)}$ as $\mu \to 0$. We note that $\mathcal{X}_t^{(0)} = \mathcal{X}$ is stationary and so has an infinite lifetime, but this does not mean that a solution for $\mu > 0$ cannot have a finite lifetime. Indeed, we will show that this occurs when μ is strictly positive but not too large. As μ gets larger, however, the lifetime again becomes infinite, although the solutions are perfectly well behaved.

8. The Laplace Transform of φ

We wish to take the Laplace transform of φ, but in order to do so we require the transform of K. In view of this, we begin by considering what class $\mathcal{L}[K]$ belongs to. As we are taking ψ to be a regulated coupling function, we can apply material from the previous section to determine the following.

Lemma 27. *Let ψ be a regulated coupling function. Then:*

1. *the function $\mathcal{L}[K] = L[\psi]$ belongs to $H^2(\mathbb{H}_+) \cap H^\infty(\mathbb{H}_+)$;*
2. *for $y \in \mathbb{R}$, the boundary function*

$$2\pi(\mathfrak{P}\psi)(y) = \lim_{\delta\to 0+} \psi_\delta(y) = \lim_{\delta\to 0+} L[\psi](\delta - iy) = \pi\psi(y) + i(\mathcal{H}[\psi])(y)\,, \quad (8.1a)$$

exists everywhere, belongs to $\mathcal{C}(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and satisfies the identities

$$\lim_{\delta\to 0+} \| \psi_\delta - 2\pi\mathfrak{P}\psi \|_2 = \lim_{\delta\to 0+} \| \psi_\delta - 2\pi\mathfrak{P}\psi \|_\infty = 0\,; \quad (8.1b)$$

3. *the function*

$$H(p) = p + 2i\xi + \mu^2 L[\psi](p)\,, \qquad p \in \mathbb{H}_+ \quad (8.2a)$$

equal to $\big(\mathcal{L}[\varphi](p)\big)^{-1}$ (see (8.3)), is analytic in $\mathbb{H}_+$, and the limit

$$\Theta(y) = \lim_{\delta\to 0+} H(\delta - iy) = \pi\mu^2\psi(y) + i[\pi\mu^2(\mathcal{H}[\psi])(y) + 2\xi - y]\,,$$

$$y \in \mathbb{R}\,, \quad (8.2b)$$

exists uniformly in y *over the whole of* $\mathbb{R}$*, defining* $\Theta \in \mathcal{C}(\mathbb{R})$.

We also note that, since ψ is real-valued, the formula $2\pi\mathfrak{P}\psi \;=\; \pi\psi + \pi i\mathcal{H}[\psi]$ is in fact the decomposition of $2\pi\mathfrak{P}\psi$ into its real and imaginary parts.

We can now take the Laplace transform of the integral equation for φ.

Proposition 28. *The Laplace transform of* φ *is well-defined on* $\mathbb{H}_+$ *and given by*

$$\mathcal{L}[\varphi](p) = \frac{1}{p + 2i\xi + \mu^2 \mathcal{L}[K](p)}\;, \qquad p \in \mathbb{H}_+\;, \tag{8.3}$$

where $\mathcal{L}[K]$ *is the Laplace transform of the kernel* K*:*

$$\mathcal{L}[K](p) = \int_{\mathbb{R}} \frac{\lambda(k)^2}{p + i\varepsilon(k)}\,dk \tag{8.4a}$$

$$= L[\psi](p)\;, \qquad p \in \mathbb{H}_+\;. \tag{8.4b}$$

Proof. Since φ is a continuously differentiable bounded function on $[0, \infty)$, we know that it possesses a Laplace transform on $\mathbb{H}_+$. From the Volterra integral equation (4.3) we deduce that (see below for $\mathcal{L}[K]$ and $\mathcal{L}[\mathcal{K}]$)

$$\mathcal{L}[\varphi](p) = \frac{1}{p} \;-\; \mathcal{L}[\mathcal{K}](p)\,\mathcal{L}[\varphi](p)\;, \qquad p \in \mathbb{H}_+\;,$$

and hence that

$$\mathcal{L}[\varphi](p) = \frac{1}{p\,[1 + \mathcal{L}[\mathcal{K}](p)]}\;, \qquad p \in \mathbb{H}_+\;.$$

The relationship between $\mathcal{K}$ and K implies that

$$p\,\mathcal{L}[\mathcal{K}](p) \;=\; 2i\xi \;+\; \mu^2 \mathcal{L}[K](p)\;, \qquad p \in \mathbb{H}_+\;,$$

and the result (8.4a) follows. Equation (8.4b) now follows directly from the definition of ψ. ☐

If we can invert $\mathcal{L}[\varphi]$, Proposition 11 enables us to determine f and W from φ. Equations (8.3) and (8.4a) then yield the solution to the dynamical equations (at this point μ is strictly greater than zero but otherwise arbitrary).

The model parameters ε, μ, and ψ will determine the properties of φ, for example the limit $\lim_{t\to\infty} f(t)$, and in particular whether or not the states $e[+] \otimes \Omega$ and $e_{\mathrm{S}} \otimes \Omega$ have finite lifetimes. Recall that this occurs if and only if $\varphi \in L^2(0, \infty)$. The Paley-Wiener Theorem now tells us that this is so if and only if $\mathcal{L}[\varphi] \in H^2(\mathbb{H}_+)$, in which case φ is proportional to the Fourier transform of the boundary function $\mathfrak{h}$ of $\mathcal{L}[\varphi]$.

9. Analytic Extensions of H and $L[\varphi]$

It is clear from inspection of (6.6) that $\mathcal{L}[K] = L[\psi]$ can be defined analytically on a larger domain than $\mathbb{H}_+$. Indeed, this integral formula also defines an analytic function on the domains $\mathbb{H}_-$,

$$\mathbb{K}_+(-\alpha) = \{ p \in \mathbb{C} \,:\, \operatorname{Im} p > -\alpha \} ,$$

and, if β is finite, on

$$\mathbb{K}_-(-\beta) = \{ p \in \mathbb{C} \,:\, \operatorname{Im} p < -\beta \} .$$

Consequently we deduce that $\mathcal{L}[K]$, and so also H, can be defined analytically on the domain

$$\mathbb{V} \;=\; \begin{cases} \mathbb{C} \setminus [-i\beta, -i\alpha] , & \beta < \infty , \\ \mathbb{C} \setminus (-i\infty, -i\alpha] , & \beta = \infty . \end{cases} \tag{9.1}$$

Additionally, $\mathcal{L}[K]$ and H satisfy the symmetry properties

$$\mathcal{L}[K](-\overline{p}) \;=\; -\overline{\mathcal{L}[K](p)} , \qquad H(-\overline{p}) \;=\; -\overline{H(p)} , \qquad p \in \mathbb{V} . \tag{9.2}$$

We can then use equation (8.3) to define an analytic extension of $\mathcal{L}[\varphi]$ to the whole of $\mathbb{V}$, excluding those points of $\mathbb{V}$ where H vanishes. Our first aim, therefore, is to identify the zeros of H.

Since $\mathcal{L}[\varphi]\,(p)H(p) = 1$ for all $p \in \mathbb{H}_+$, it follows that H has no zeros in $\mathbb{H}_+$. The above symmetry property for H shows that H has no zeros in $\mathbb{H}_-$ either. Thus the zeros of H, if any, lie in $\mathbb{V} \cap i\mathbb{R}$. We must therefore determine whether the function Θ possesses any zeros outside the interval $[\alpha, \beta]$; the properties of the regulated coupling function ψ ensure that Θ can have no zeros in (α, β).

The number and location of the zeros of Θ depends crucially on the size of the coupling constant[11] μ. To help distinguish between the various cases, we introduce the following constants:

$$M(\alpha)^2 = (2\xi - \alpha) \left(\int_\alpha^\beta \frac{\psi(u)}{u - \alpha}\, du \right)^{-1} , \tag{9.3a}$$

[11] Together with the values of α and β, this is a point where the nature of the mathematical solution is determined by physical requirements.

$$M(\beta)^2 = \begin{cases} (\beta - 2\xi)\left(\int_\alpha^\beta \frac{\psi(u)}{\beta - u}\, du\right)^{-1}, & \beta < \infty\,, \\ \infty\,, & \beta = \infty\,. \end{cases} \tag{9.3b}$$

The differentiability of ψ and the fact that ψ vanishes at α (and at β when β is finite) ensures the existence of the integrals necessary for these definitions.

We can now locate the zeros of Θ precisely, in terms of ψ.

Proposition 29. *Let ψ be a regulated coupling function and μ the coupling constant.*

1. *If $\mu < M(\alpha)$, then Θ has no zeros in $(-\infty, \alpha]$.*
2. *If $\mu > M(\alpha)$, then Θ has exactly one zero α_μ in $(-\infty, \alpha]$, with $\alpha_\mu < \alpha$.*
3. *If $\mu < M(\beta)$, then Θ has no zeros in $[\beta, \infty)$.*
4. *If β is finite and $\mu > M(\beta)$, then Θ has exactly one zero β_μ in $[\beta, \infty)$, with $\beta_\mu > \beta$.*

Proof. Θ is purely imaginary on $(-\infty, \alpha]$, with

$$(\operatorname{Im}\Theta)(y) \;=\; -y + 2\xi - \mu^2 \int_\alpha^\beta \frac{\psi(u)}{u - y}\, du\,, \qquad y \le \alpha\,. \tag{9.4a}$$

We deduce from this that $\operatorname{Im}\Theta$ is differentiable in $(-\infty, \alpha)$, with

$$(\operatorname{Im}\Theta)'(y) \;=\; -1 - \mu^2 \int_\alpha^\beta \frac{\psi(u)}{(u - y)^2}\, du \;\le\; -1\,, \qquad y < \alpha\,. \tag{9.4b}$$

Thus $\operatorname{Im}\Theta$ is strictly monotonically decreasing on $(-\infty, \alpha)$, and $\Theta(y) \to \infty$ as $y \to -\infty$. Since

$$(\operatorname{Im}\Theta)(\alpha) \;=\; 2\xi - \alpha - \mu^2 \int_\alpha^\beta \frac{\psi(u)}{u - \alpha}\, du \;=\; (M(\alpha)^2 - \mu^2) \int_\alpha^\beta \frac{\psi(u)}{u - \alpha}\, du\,,$$

we see that $(\operatorname{Im}\Theta)(\alpha) > 0$ if $\mu < M(\alpha)$, and $(\operatorname{Im}\Theta)(\alpha) < 0$ if $\mu > M(\alpha)$. This establishes the existence, or otherwise, of the zero α_μ of Θ. The calculations for $y \ge \beta$ are performed similarly. ☐

Corollary 30. *The Laplace transform $\mathcal{L}[\varphi]$ can be extended to a function which is meromorphic on $\mathbb{V}$.*

1. *If $\mu < M(\alpha)$, then $\mathcal{L}[\varphi]$ has a simple pole of residue* $\operatorname{Res}_{-i\alpha_\mu} \mathcal{L}[\varphi] = -(\operatorname{Im}\Theta)'(\alpha_\mu)^{-1}$ *at $-i\alpha_\mu$.*

2. *If* $\beta < \infty$ *and* $\mu < M(\beta)$*, then* $\mathcal{L}[\varphi]$ *has a simple pole of residue* $\operatorname{Res}_{-i\beta_\mu} \mathcal{L}[\varphi] = -(\operatorname{Im}\Theta)'(\beta_\mu)^{-1}$ *at* $-i\beta_\mu$.

The Laplace transform $\mathcal{L}[\varphi]$ *has no other singularities in* $\mathbb{V}$.

Proof. It is clear that $\mathcal{L}[\varphi](p)$ can be defined as $H(p)^{-1}$ for all $p \in \mathbb{V}$ for which $H(p) \neq 0$. If either α_μ or β_μ exist, then the fact that H has nonzero derivative at these point(s) implies that H has a simple zero, and hence that $\mathcal{L}[\varphi]$ has a simple pole, at these point(s), and the residue of $\mathcal{L}[\varphi]$ at these point(s) can be calculated simply in terms of the derivative of H there.

If $\mu = M(\alpha)$, then Θ has a zero at α. However, $-i\alpha \notin \mathbb{V}$, so no unexpected singularities are created in this case. A similar observation is the case when $\beta = \infty$ and $\mu = M(\beta)$. □

10. Inverting the Laplace Transform

The fact that the existence of singularities for $\mathcal{L}[\varphi]$ depends crucially on the size of μ makes subsequent analysis complicated. The simplest way to proceed is to *assume* that μ does not take either of the critical values $M(\alpha)$, $M(\beta)$. If we invert the Laplace transform to obtain φ as a function of μ in this favourable case, we can obtain φ for μ equal to the critical values by continuity, using Corollary 26.

The assumption concerning μ enables us to subtract the singularities of $\mathcal{L}[\varphi]$, allowing us to work with the function G defined just below which is analytic on the whole of $\mathbb{V}$.

First we define the terms that must be subtracted from φ, following from the residue results given in Corollary 30.

For $\mu \in (0,\infty)\backslash\{M(\alpha), M(\beta)\}$, define the meromorphic functions $F_{\alpha,\mu}$, $F_{\beta,\mu}$ on $\mathbb{V}$ by the formulae

$$F_{\alpha,\mu}(p) = \begin{cases} 0\,, & \mu < M(\alpha)\,, \\ -\dfrac{1}{(\operatorname{Im}\Theta)'(\alpha_\mu)}(p + i\alpha_\mu)^{-1}\,, & \mu > M(\alpha)\,, \end{cases} \tag{10.1a}$$

$$F_{\beta,\mu}(p) = \begin{cases} 0\,, & \mu < M(\beta)\,, \\ -\dfrac{1}{(\operatorname{Im}\Theta)'(\beta_\mu)}(p + i\beta_\mu)^{-1}\,, & \mu > M(\beta)\,, \end{cases} \tag{10.1b}$$

and the function G:

$$G(p) \;=\; \mathcal{L}[\varphi](p) - F_{\alpha,\mu}(p) - F_{\beta,\mu}(p)\,, \qquad p \in \mathbb{V}\,. \tag{10.2}$$

Lemma 31. *The function G is analytic in $\mathbb{V}$, and*

$$G(-\overline{p}) \;=\; -\overline{G(p)}\,, \qquad p \in \mathbb{V}\,. \tag{10.3}$$

Proof. If either α_μ or β_μ exist, then G has a removable singularity at these point(s). The symmetry property for G follows from the analogous symmetry properties for $\mathcal{L}[K]$, H and $\mathcal{L}[\varphi]$. ☐

Regarded as a function on $\mathbb{H}_+$, G has many desirable properties.

Theorem 32. *For any $\mu \in (0,\infty)\backslash\{M(\alpha), M(\beta)\}$, G belongs to $H^2(\mathbb{H}_+)\cap H^\infty(\mathbb{H}_+)$. If $\mathfrak{h}$ is its boundary function, then $\mathfrak{h} \in \mathcal{C}(\mathbb{R})\cap L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$, and the limit*

$$\mathfrak{h}(y) \;=\; \lim_{\delta\to 0+} G(\delta - iy) \tag{10.4}$$

is uniform in y over compact subsets of $\mathbb{R}$.

Proof. Since $\operatorname{Re} L[\psi](p) \ge 0$ for all $p \in \mathbb{H}_+$, it follows from equation (8.2a) that $|\,H(p)\,| \ge \operatorname{Re} p$ for all $p \in \mathbb{H}_+$, and hence that

$$|\,\mathcal{L}[\varphi](p)\,| \;\le\; (\operatorname{Re} p)^{-1}\,, \qquad p \in \mathbb{H}_+\,.$$

It is elementary that both $(\operatorname{Re} p)F_{\alpha,\mu}(p)$ and $(\operatorname{Re} p)F_{\beta,\mu}(p)$ are bounded functions on $\mathbb{H}_+$, so we deduce that there exists some constant $Q > 0$ such that

$$|\,G(p)\,| \;\le\; Q(\operatorname{Re} p)^{-1}\,, \qquad p \in \mathbb{H}_+\,.$$

Since $|\,p + i\zeta\,| \ge |\operatorname{Im} p\,| - |\,\zeta\,|$ for any $p \in \mathbb{H}_+$ and $\zeta \in \mathbb{R}$, and since

$$|\,H(p)\,| \;\ge\; |\,p\,| - 2\xi - \mu^2\,|\,L[\psi](p)\,| \;\ge\; |\operatorname{Im} p\,| - 2\xi - \mu^2\,\|\,L[\psi]\,\|_\infty\,, \quad p \in \mathbb{H}_+\,,$$

we can find constants $R, X > 0$ such that

$$|\,G(p)\,| \;\le\; R\,|\operatorname{Im} p\,|^{-1}\,, \qquad p \in \mathbb{H}_+\,, \quad |\operatorname{Im} p\,| \ge X\,.$$

We already know that the limit as $\delta \to 0+$ of $\mathcal{L}[\varphi](\delta - iy)$, and hence of $G(\delta - iy)$, exists uniformly in y over any compact subset C of $\mathbb{R}$ such that $\Theta(y) \neq 0$ for all $y \in C$. If $\mu > M(\alpha)$ then the analyticity of G on $\mathbb{V}$ ensures that the limit of $G(\delta - iy)$ as $\delta \to 0+$ exists uniformly in y over the interval $[\alpha_\mu - \eta, \alpha_\mu + \eta]$ for any $0 < \eta < \alpha - \alpha_\mu$. If β is finite and $\mu > M(\beta)$ then (similarly) the limit of $G(\delta - iy)$ as $\delta \to 0+$ exists uniformly in y over the interval $[\beta_\mu - \eta, \beta_\mu + \eta]$ for any $0 < \eta < \beta_\mu - \beta$. Putting these results together, it follows that the limit

$$\mathfrak{h}(y) \;=\; \lim_{\delta\to 0+} G(\delta - iy)\;, \qquad y\in\mathbb{R}\,,$$

is uniform in y over compact subsets of $\mathbb{R}$. Clearly, then, $\mathfrak{h}\in\mathcal{C}(\mathbb{R})$. This implies that $G|_{\mathbb{H}_+}$ can be extended[12] to a function continuous on $\overline{\mathbb{H}_+} = \{p\in\mathbb{C} : \operatorname{Re} p \geq 0\}$, and hence that there exists a constant $S>0$ such that

$$|\,G(p)\,| \;\leq\; S\;, \qquad 0<\operatorname{Re} p\leq 1\;, \quad |\operatorname{Im} p\,|\leq X\,.$$

This implies that G is bounded over the whole of $\mathbb{H}_+$, and so $G\in H^\infty(\mathbb{H}_+)$, as required.

Finally, consider the function $Z\in L^2(\mathbb{R})$ defined by the formula

$$Z(y) \;=\; \begin{cases} \max(Q,S)\,, & |\,y\,|\leq X\,,\\ R\,|\,y\,|^{-1}\,, & |\,y\,|>X\,.\end{cases}$$

Clearly $|\,G(\delta-iy)\,| \;\leq\; Z(y)$ for all $y\in\mathbb{R}$ and $\delta>0$, so that $G\in H^2(\mathbb{H}_+)$, and hence $\mathfrak{h}\in L^2(\mathbb{R})$. □

We can now invert the Laplace transform to obtain the function φ.

Corollary 33. *Let the coupling function be regulated and $\mu>0$, but $\mu\neq M(\alpha)$, $M(\beta)$. Define the functions $\varphi_{\alpha,\mu}$ and $\varphi_{\beta,\mu}$ on $[0,\infty)$ by the formulae*

$$\varphi_{\alpha,\mu}(t) = \begin{cases} 0\,, & \mu<M(\alpha)\,,\\ -\dfrac{1}{(\operatorname{Im}\Theta)'(\alpha_\mu)}\,e^{-i\alpha_\mu t}\,, & \mu>M(\alpha)\,,\end{cases} \tag{10.5a}$$

$$\varphi_{\beta,\mu}(t) = \begin{cases} 0\,, & \mu<M(\beta)\,,\\ -\dfrac{1}{(\operatorname{Im}\Theta)'(\beta_\mu)}\,e^{-i\beta_\mu t}\,, & \mu>M(\beta)\,,\end{cases} \tag{10.5b}$$

Then G is the Laplace transform of $\varphi-\varphi_{\alpha,\mu}-\varphi_{\beta,\mu}$, and the Paley-Wiener Theorem allows us to identify the restricted Fourier transform (6.4) *of φ as*

$$\widehat{\varphi} \;=\; \widehat{\varphi}_{\alpha,\mu}+\widehat{\varphi}_{\beta,\mu}+\frac{1}{\sqrt{2\pi}}\,\mathcal{F}[\mathfrak{h}]\,. \tag{10.5c}$$

So far we have obtained an expression for φ when $\mu\in(0,\infty)\backslash\{M(\alpha), M(\beta)\}$. It is important for the question of the lifetime of the quantum states constructed from it to note that it is the sum of a function belonging to $L^2(0,\infty)$ possibly plus one or two periodic terms. We shall return to discussing the implications of this further on.

[12] Note that this is a different extension to the extension of $G|_{\mathbb{H}_+}$ to G on $\mathbb{V}$.

As we remarked above, in principle φ can be calculated when μ takes either of the critical values $M(\alpha)$, $M(\beta)$ by continuity arguments. However, the expression that we have obtained above is not well suited to this and various other purposes, since all the terms are functions of α_μ and β_μ, whose values are only known implicitly. It is desirable, therefore, to reformulate (10.5c) in a more direct and convenient manner. The result is described in this next theorem.

Theorem 34. *When $\mu \in (0,\infty)\backslash\{M(\alpha), M(\beta)\}$, the dynamical function φ is given by the formula*

$$\varphi(t) = \varphi_{\alpha,\mu}(t) + \varphi_{\beta,\mu}(t) + \sqrt{\frac{2}{\pi}}\,\mathcal{F}[\Xi](t)\,, \quad t > 0\,, \tag{10.6}$$

where $\Xi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by the formula

$$\Xi(y) = \begin{cases} \dfrac{\pi\mu^2\psi(y)}{\pi^2\mu^4\psi(y)^2 + [y - 2\xi - \pi\mu^2(\mathcal{H}[\psi])(y)]^2}\,, & \alpha \le y \le \beta\,, \\ 0\,, & \textit{otherwise}\,. \end{cases} \tag{10.7}$$

Proof. We begin by applying the Bromwich integral formula to the function G: for any $\kappa > 0$ and $t > 0$,

$$\varphi(t) - \varphi_{\alpha,\mu}(t) - \varphi_{\beta,\mu}(t) \;=\; \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\kappa-iT}^{\kappa+iT} G(p)e^{pt}\,dp\,. \tag{10.8}$$

We shall be using the technical results found in the proof of Theorem 32 in what follows. Consider the contour above, where $t > \alpha$, $t > |\,\alpha_\mu\,|$, (and $T > \beta_\mu$ if β_μ exists) $T > X\sqrt{2}$ and $0 < \delta < \min(\kappa, \alpha - \alpha_\mu)$. Observing that G is analytic both inside and on this contour, the integral of $G(p)e^{pt}$ around it vanishes. If $\Gamma_{\delta,T}$ is the positively oriented semicircular contour, centre $-\delta$ and radius T, running from $-\delta + iT$ to $-\delta - iT$, then $|\,G(p)\,| \le \sqrt{2}\max(Q,R)T^{-1}$ for all $p \in \Gamma_{\delta,T}$.

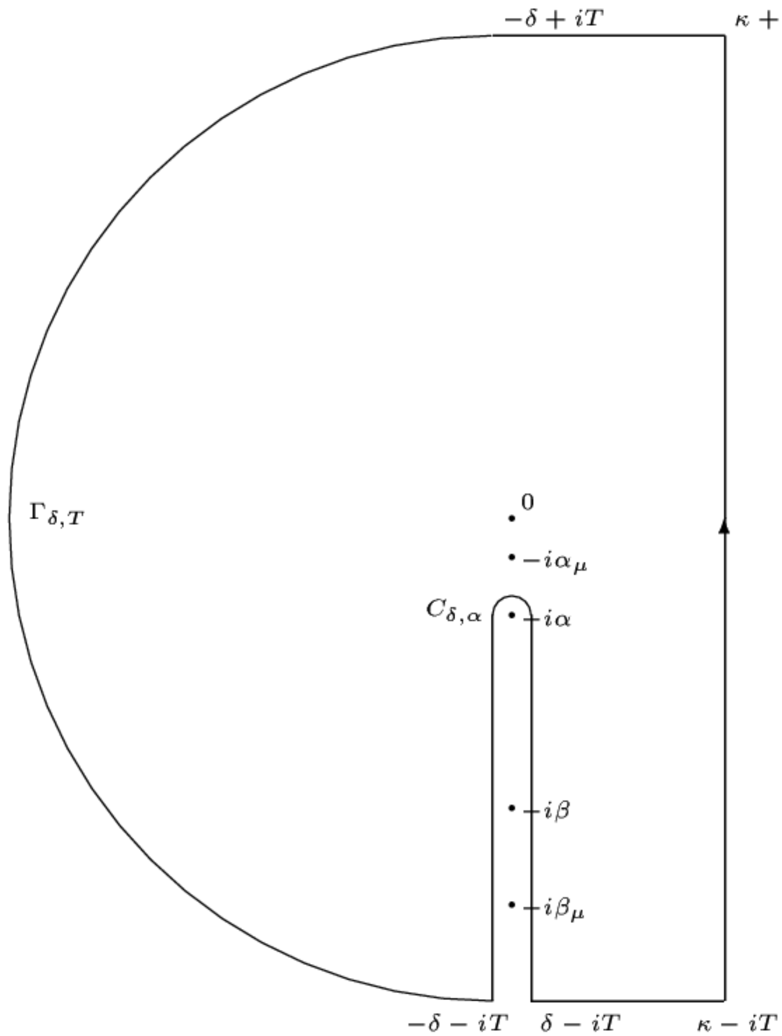
Direct calculations show that

$$\left|\int_{-\delta+iT}^{\kappa+iT} G(p)e^{pt}\,dp\right| \le \frac{2\kappa R}{T}e^{\kappa t}\,, \qquad \left|\int_{\delta-iT}^{\kappa-iT} G(p)e^{pt}\,dp\right| \;\le\; \frac{\kappa R}{T}e^{\kappa t}\,,$$

and

$$\left|\int_{\Gamma_{\delta,T}} G(p)e^{pt}\,dp\right| \le \sqrt{2}\max(Q,R)e^{-\delta t}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} e^{Tt\cos\theta}\,d\theta \;\le\; \frac{\pi\sqrt{2}}{tT}\max(Q,R)\,,$$

where we have used Jensen's inequality. Hence we can find a constant $A > 0$,



independent of δ, such that

$$\left| \left(\int_{\kappa-iT}^{\kappa+iT} - \int_{-\delta-i\alpha}^{-\delta-iT} + \int_{\delta-i\alpha}^{\delta-iT} - \int_{C_{\delta,\alpha}} \right) G(p)e^{pt}\, dp \right| \leq \frac{A}{T} \,,$$

where $C_{\delta,\alpha}$ is the positively oriented semicircular contour, centre $-i\alpha$ and radius δ, running from $\delta - i\alpha$ to $-\delta - i\alpha$. Since $G|_{\mathbb{H}_+}$ can be extended continuously to $\overline{\mathbb{H}_+}$ and, by symmetry, $G|_{\mathbb{H}_-}$ can be extended continuously to $\overline{\mathbb{H}_-}$, we see that G is bounded in $\mathbb{V} \cap \{p \in \mathbb{C} \,:\, |\,p + i\alpha\,| \leq \eta\}$ for suitably small values of η. It follows that

$$\int_{C_{\delta,\alpha}} G(p)e^{pt}\,dp \;=\; \mathrm{O}(\delta)\;, \qquad \delta \to 0+\;.$$

Then

$$\varphi(t) - \varphi_{\alpha,\mu}(t) - \varphi_{\beta,\mu}(t) \;=\; \frac{1}{2\pi i}\lim_{T\to\infty}\lim_{\delta\to 0+}\left(\int_{-\delta-i\alpha}^{-\delta-iT} - \int_{\delta-i\alpha}^{\delta-iT}\right) G(p)e^{pt}\,dp\;,$$

noting carefully the order in which these two limits are to be taken. Using the symmetry property of G, we deduce that

$$\begin{aligned}
&\lim_{\delta\to 0+}\left(\int_{-\delta-i\alpha}^{-\delta-iT} - \int_{\delta-i\alpha}^{\delta-iT}\right) G(p)e^{pt}\,dp \\
&\quad = i\lim_{\delta\to 0+}\int_{\alpha}^{T}\big[G(\delta-iy)e^{\delta t} + \overline{G(\delta-iy)}e^{-\delta t}\big]e^{-iyt}\,dy \\
&\quad = 2i\int_{\alpha}^{T}(\mathrm{Re}\,\mathfrak{h})(y)e^{-iyt}\,dy\;.
\end{aligned}$$

We note that, if $\beta < \infty$, $\mathfrak{h}(y) = G(-iy)$ is purely imaginary for all $y > \beta$, and so $(\mathrm{Re}\,\mathfrak{h})(y) = 0$ for all $y > \beta$. Additionally the limits

$$\lim_{\delta\to 0+} F_{\alpha,\mu}(\delta - iy)\;, \qquad \lim_{\delta\to 0+} F_{\beta,\mu}(\delta - iy)$$

exist, and are purely imaginary for all $\alpha \le y \le \beta$.

As $\Theta \in \mathcal{C}(\mathbb{R})$ does not vanish in $[\alpha,\beta]$, the denominator in the expression for Ξ given above is bounded away from 0 throughout $[\alpha,\gamma]$ for any finite $\beta \ge \gamma > \alpha$. Additionally, as $\mathcal{H}[\psi]$ is the imaginary part of the boundary function of $L[\psi]$, it belongs to $L^\infty(\mathbb{R})$, and so $\Xi(y) \;=\; \mathrm{O}\big(|\,y\,|^{-2}\,\psi(y)\big)$ as $|\,y\,| \to \infty$. Thus Ξ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ whether β is infinite or not. It follows that

$$\varphi(t) - \varphi_{\alpha,\mu}(t) - \varphi_{\beta,\mu}(t) \;=\; \frac{1}{\pi}\lim_{T\to\infty}\int_{\alpha}^{T}\Xi(y)e^{-iyt}\,dy\;.$$

Since $\Xi \in L^1(\mathbb{R})$, the Dominated Convergence Theorem now completes the proof. □

11. Weak Coupling and Lifetimes

We can now say when $\mathcal{X}$ has a finite lifetime:

Corollary 35. *Let ψ be a regulated coupling function. By* weak coupling *we mean that* $0 < \mu < \min(M(\alpha), M(\beta))$.

In this range of μ, φ belongs to $L^2[0,\infty)$ and so $\mathcal{X}$ has a finite lifetime.

Proof. For weak coupling, both $\varphi_{\alpha,\mu}$ and $\varphi_{\beta,\mu}$ vanish. □

This result is the peculiar advantage of weak coupling, since once μ exceeds $\min(M(\alpha), M(\beta))$, φ ceases to be square-integrable as it contains (one or two) periodic terms, whence the lifetime of $\mathcal{X}$ become infinite. In contrast, for strong coupling the energy interchange between field and atom sustains $\mathcal{X}$ sufficiently for its lifetime to be infinite.

Our model predicts that for weak coupling (finite lifetime), $\mathcal{X}$ will decay with probability 1 to a state containing one Boson. Granting this, if one describes the process entirely in terms of qubits, by projecting out the Boson field, the qubit states $e[+]$ and e_{S} will appear to have decayed to the ground states $e[-]$ and $e[-,-]$ respectively.

For moderately strong coupling, that is, for a certain range of μ greater than $\min(M(\alpha), M(\beta))$, the magnitude of the periodic terms $\varphi_{\alpha,\mu}$ and $\varphi_{\beta,\mu}$ are small. Consequently, $|\,\varphi(t)\,|^2$ will become small as $t \to \infty$. This implies that the probability that $\mathcal{X}$ has not decayed to a one Boson state becomes small, although nonzero. In this case, too, the qubit states $e[+]$ and e_{S} will appear to have decayed to the ground states $e[-]$ and $e[-,-]$ respectively, with a probability nearly equal to 1.

To obtain the most reliable physical results, *until further notice we restrict μ to the weak coupling range* $0 < \mu < \min(M(\alpha), M(\beta))$.

Proposition 36. *In the weak coupling case $0 < \mu < \min(M(\alpha), M(\beta))$, the (finite) lifetime of the state $\mathcal{X}$ is*

$$\tau(\mathcal{X}) \;=\; \frac{1}{2\pi}\,\|\,\Xi\,\|_2^2 \;. \tag{11.1}$$

Proof. In this weak coupling case, $\varphi_{\alpha,\mu}$ and $\varphi_{\beta,\mu}$ both vanish, and hence we simply have

$$\varphi(t) \;=\; \sqrt{\frac{2}{\pi}}\mathcal{F}[\Xi](t)\;, \qquad t > 0\;,$$

which certainly implies that $\varphi \in L^2(0,\infty)$, and that $\mathcal{X}$ has finite lifetime. However, since Ξ is real-valued, $\mathcal{F}[\Xi](-t) \;=\; \overline{\mathcal{F}[\Xi](t)}$ for all $t \in \mathbb{R}$, and so

$$\overline{\varphi(-t)} \;=\; \sqrt{\frac{2}{\pi}}\mathcal{F}[\Xi](t)\;, \qquad t < 0\;.$$

Consequently

$$\tau(\mathcal{X}) \;=\; \frac{1}{2}\int_0^\infty |\,\varphi(t)\,|^2 \; dt \;=\; \frac{1}{2\pi}\int_{\mathbb{R}} |\,\mathcal{F}[\Xi](t)\,|^2 \; dt \;=\; \frac{1}{2\pi}\,\|\,\Xi\,\|_2^2 \;,$$

as required. ☐

As noted above, when $\mu = 0$ there is no interaction between the atoms and the Boson field, and $\varphi^{(0)}(t) = e^{-2i\xi t}$. Therefore the state $\mathcal{X}^{(0)}$ is stationary and so has an infinite lifetime. We have seen that the lifetime of $\mathcal{X}^{(\mu)}$ is finite for small values of μ, and as we have shown in Corollary 25, φ_μ is a continuous function of μ in a certain sense. Let us now show that the lifetime of $\mathcal{X}^{(\mu)} \to \infty$ as $\mu \to 0+$.

Proposition 37. $\lim_{\mu\to 0+} \tau(\mathcal{X}^{(\mu)}) \;=\; \infty$.

Proof. For any $M > 0$, we can find $0 < \widehat{\mu} < \min(M(\alpha), M(\beta))$ such that for all $\mu \in (0, \widehat{\mu})$ and $0 \le t \le 8M$,

$$|\,|\,\varphi_\mu(t)\,| - 1\,| \;=\; |\,|\,\varphi_\mu(t)\,| - |\,\varphi_0(t)\,|\,| \;\le\; |\,\varphi_\mu(t) - \varphi_0(t)\,| \;\le\; \frac{1}{2}\,.$$

Thus

$$|\,\varphi_\mu(t)\,| \;\ge\; \frac{1}{2}$$

and hence

$$\tau(\mathcal{X}^{(\mu)}) \;=\; \frac{1}{2}\int_0^\infty |\,\varphi_\mu(t)\,|^2 \; dt \;\ge\; M \;,$$

which proves the result. ☐

At the beginning of this work we suggested that there are other times characteristic of the stability of the initial state[13]. To this end, we make the following definition.

Definition 38. An initial state Ψ is said to possess a *threshold time* if a finite value $T(\Psi) > 0$ can be found so that $|\,\langle \Psi\,,\, U(t)\Psi\rangle\,|^2 \;<\; e^{-1}$ whenever $t > T(\Psi)$.

[13] Recently, van Wezel, van den Brink and Zaanen [11] have proposed that the characteristic time that a system can stay coherent is $2\pi N\hbar/(k_B T)$, where N is the number of degrees of freedom and T is the temperature at which the system is held. Their analysis is based on the Lieb-Mattis Heisenberg anti-ferromagnet model, which is rather different from the model we are using. Nonetheless, their results can be said to complement what we have found.

In other words, if an initial state Ψ possesses the threshold time $T(\Psi)$, then at times $t > T(\Psi)$ the probability is greater than $1 - e^{-1}$ that the state will be observed to have decayed to another state. The existence and value of a threshold time for an initial state will certainly be more stable under uniform perturbations than is the standard lifetime.

To determine the threshold time for $\mathcal{X}$, we need to have information concerning the behaviour of φ as $t \to \infty$ (the restriction to weak coupling is still in force). That $T(\mathcal{X})$ exists is clear, since the conditions we have imposed upon ψ ensure that $\varphi(t) \to 0$ as $t \to \infty$. However, they are not sufficient to provide a value for $T(\mathcal{X})$. We now further restrict ψ as a result of which we will be able to find an upper bound for $T(\mathcal{X})$.

We defined a regulated model as one where the coupling function is regulated, Definition 12. We now require additional smoothness for ψ.

Definition 39. For any integer $n \geq 1$, an n-regulated model is a regulated model for which the function ψ satisfies the following additional conditions:

- $\psi \in \mathcal{C}^n(\mathbb{R})$;
- $\psi^{(m)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for $0 \leq m < n$;
- $\psi^{(n)} \in L^2(\mathbb{R})$.

Thus a 1-regulated model is simply a regulated one.

Lemma 40. *Suppose that $n \geq 2$, and that we have an n-regulated model. Then the function h belongs to $\mathcal{C}^{n-1}(\mathbb{R})$, and $h^{(m)} \in L^\infty(\mathbb{R})$ for all $1 \leq m \leq n-1$.*

Proof. For any $0 \leq m \leq n-1$ it is clear that the derivative $\psi^{(m)}$ satisfies the conditions of Lemma 22, and so it follows that $\mathfrak{P}\big(\psi^{(m)}\big) \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and that $\big(\psi^{(m)}\big)_\delta \to 2\pi\mathfrak{P}\big(\psi^{(m)}\big)$ uniformly over $\mathbb{R}$ as $\delta \to 0+$. However, inspection of the detail of Proposition 21 shows that $\psi_\delta \in \mathcal{C}^{n-1}(\mathbb{R})$, with $(\psi_\delta)^{(m)} = \big(\psi^{(m)}\big)_\delta$ for all $0 \leq m \leq n-1$. From this we deduce that $\mathfrak{P}(\psi) \in \mathcal{C}^{n-1}(\mathbb{R})$, and that $\mathfrak{P}(\psi) \in L^\infty(\mathbb{R})$ for all $0 \leq m \leq n-1$. The results of this lemma are now immediate. ☐

Lemma 41. *Suppose that $n \geq 2$, and that we have an n-regulated model in the weak coupling case $0 < \mu < \min(M(\alpha), M(\beta))$. Then $\Xi \in \mathcal{C}^{n-1}(\mathbb{R})$, and the function $y \mapsto (1+y^2)\Xi^{(m)}(y)$ belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for all $0 \leq m \leq n-1$.*

Proof. With weak coupling, Θ has no zeros. Since $\Theta(y) \sim -iy$ as $|\,y\,| \to \infty$, it follows that there exists a constant $Q > 0$ such that $|\,\Theta(y)\,| \geq Q$ for all $y \in \mathbb{R}$. From this, it is clear that $|\,\Theta\,|^2 \in \mathcal{C}^{n-1}(\mathbb{R})$, and that $(|\,\Theta\,|^2)^{(m)}(y) = \mathrm{O}(y)$ as $|\,y\,| \to \infty$ for any $0 \leq m \leq n-1$. Writing $\tilde{\Theta} = |\,\Theta\,|^{-2}$, we deduce that $\tilde{\Theta} \in \mathcal{C}^{n-1}(\mathbb{R})$, and that $y^2\,\tilde{\Theta}^{(m)}(y) = \mathrm{O}(1)$ as $|\,y\,| \to \infty$ for any $0 \leq m \leq n-1$. Since $\Xi = \pi\mu^2\varphi\tilde{\Theta}$, the lemma follows. ☐

Proposition 42. *In an n-regulated model, with ($n \geq 2$ and) weak coupling, we have the asymptotic identity*

$$\varphi(t) \;=\; \mathrm{o}\big(t^{-(n-1)}\big)\,, \qquad t \to \infty\,, \tag{11.2a}$$

and so the threshold time $T(\mathcal{X})$ has upper bound

$$T(\mathcal{X}) \;\leq\; \left(\frac{\sqrt{e}}{\pi}\big\|\Xi^{(n-1)}\big\|_1\right)^{\frac{1}{n-1}}. \tag{11.2b}$$

Proof. From the results of Lemma 41 it is clear that

$$i^m t^m \mathcal{F}[\Xi](t) \;=\; \mathcal{F}[\Xi^{(m)}](t)\,, \qquad t \in \mathbb{R}\,, \quad 0 \leq m \leq n-1\,,$$

and hence

$$|\,\varphi(t)\,| \;\leq\; \sqrt{\frac{2}{\pi}}\,|\,\mathcal{F}[\Xi](t)\,| \;\leq\; \frac{1}{\pi t^{n-1}}\big\|\Xi^{(n-1)}\big\|_1\,.$$

Thus $|\,\varphi(t)\,|^2 \leq e^{-1}$ whenever $e\,\|\,\Xi'\,\|_1^2 \;\leq\; \pi^2 t^{2(n-1)}$, yielding the desired upper bound for the threshold time. ☐

We can do even better than this by not restricting n.

Definition 43. A hyperregulated model is a regulated model which is n-regulated for all integers $n \geq 1$, so that $\psi \in \mathcal{C}^\infty(\mathbb{R})$, and $\psi^{(m)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Corollary 44. *In a hyperregulated model with weak coupling, $\varphi(t) = \mathrm{O}(t^{-n})$ as $t \to \infty$ for all integers $n \geq 1$ and so the threshold time $T(\mathcal{X})$ has upper bound*

$$T(\mathcal{X}) \;\leq\; \inf_{n \geq 2}\left(\frac{\sqrt{e}}{\pi}\big\|\Xi^{(n-1)}\big\|_1\right)^{\frac{1}{n-1}}. \tag{11.3}$$

Remark 45. We note in passing that any test function $\psi \in \mathcal{D}(\mathbb{R})$ with support $[\alpha, \beta]$ describes a hyperregulated model. A simple extension of the above proofs shows that, in the weak coupling case, Ξ is then a Schwartz function, and hence so is φ. In this case, then, we not only have the rapid decrease of φ, but we also have the fact that φ is smooth.

Once the time evolution of the states $e[\pm]\otimes\Omega$ are known, the time evolution of $u \otimes \Omega$ is known for any qubit state vector u. Consider the state vector $\Psi = \eta e[+]\otimes\Omega + \zeta e[-]\otimes\Omega$, with $|\,\eta\,|^2 + |\,\zeta\,|^2 = 1$. After a time t, it has evolved to the state vector

$$U(t)\Psi \;=\; \big(\eta f(t) e[+] + \zeta e^{i\xi t} e[-]\big)\otimes\Omega \;+\; \eta e[-]\otimes a^{+}(W_t)\Omega \,. \tag{11.4}$$

Its decay law function is thus

$$\langle \Psi\,,\, U(t)\Psi\rangle \;=\; |\,\eta\,|^2 f(t) \;+\; |\,\zeta\,|^2 e^{i\xi t} \;=\; e^{i\xi t}\big(\,|\,\eta\,|^2\varphi(t) \;+\; |\,\zeta\,|^2\big)\,. \tag{11.5}$$

Thus Ψ has infinite lifetime except when $\zeta = 0$ in the weak coupling case. This is very understandable: the contribution of $e[-]\otimes\Omega$ to the decay law function reflects its stationarity. Including the upper state does not change this. We interpret this as a limitation on the usefulness of the lifetime as a measure of stability in these circumstances.

Furthermore, this class of state reinforces the view that the threshold time provides useful information about stability. For the expression

$$\lim_{t\to\infty} |\,\langle \Psi\,,\, U(t)\Psi\rangle\,|^2 \;=\; |\,\zeta\,|^4$$

shows that a threshold time $T(\Psi)$ exists for all Ψ with $|\,\zeta\,| < e^{-1/4} \approx 0.7788$.

Suppose now that $\zeta \neq 0$, since that case is just the upper state stability problem already discussed. When $0 < |\,\zeta\,| < e^{-1/4}$ and the model is n-regulated,

$$|\,\langle \Psi\,,\, U(t)\Psi\rangle\,|^2 \;=\; |\,\zeta\,|^4 \;+\; \mathrm{O}(t^{-(n-1)})\,, \qquad t\to\infty\,,$$

the equality provides an upper bound for $T(\Psi)$.

Having obtained the evolution laws when taking the field into account, consider what happens when we project out the explicit appearance of the field, resulting in the reduced qubit dynamics. We obtain a mixed qubit state $\omega_{\Psi,t}$ corresponding to the full state Ψ_t,

$$\omega_{\Psi,t}(B) \;=\; \langle U(t)\Psi\,,\,(B\otimes I)U(t)\Psi\rangle\,, \qquad B\in\mathbb{B}(\mathcal{Q})\,, \tag{11.6a}$$

determined by the reduced density matrix $\rho_{\Psi,t}$,

$$\omega_{\Psi,t}(B) \;=\; \mathrm{Tr}(\rho_{\Psi,t}B)\,, \qquad B\in\mathbb{B}(\mathcal{Q})\,. \tag{11.6b}$$

given by

$$\rho_{\Psi,t} \;=\; \begin{pmatrix} |\,\eta\,|^2\,|\,\varphi(t)\,|^2 & \eta\overline{\zeta}\varphi(t) \\ \overline{\eta}\zeta\overline{\varphi(t)} & |\,\eta\,|^2\,(1-|\,\varphi(t)\,|^2) + |\,\zeta\,|^2 \end{pmatrix}\,. \tag{11.7}$$

It is interesting to note that some authors find approximate solutions to this model with $\varphi(t)$ replaced by $\exp\,(-\gamma t/2)$ for some decay constant γ, cf. Bouwmeester et al [2]. We have not obtained any such result. The nearest connection we can be make to exponential decay is that hyperregulated models show a fast, but not generally exponential, rate of decay. It may be that the many discussions of spontaneous decay assuming exponential behaviour have sacrificed accuracy for manageability. The error introduced may be small, however. It might be worth attempting an estimate of this, but we have not done so.

12. Sharp Cut-Off Solutions

Although regulated models are the most realistic, it is common in quantum mechanics to cut off interactions sharply in order to simplify calculations. We shall now reconsider our model with this class of interaction. As the smoothness conditions we imposed previously were central to the inversion of the Laplace transforms, we must redo that work under the new conditions.

It turns out that the discontinuities introduce periodic terms whatever the value of μ. Hence there is no range of μ for which the lifetimes of $\mathcal{X}^{(\mu)}$ are finite. However, for small μ the amplitudes of the periodic terms are small, though nonzero. This allows a non-trivial probability of decay, as we shall see.

13. Cut-Off Couplings

In the class of models we now consider, we require that the support of ψ be a compact interval $[\alpha, \beta]$ where $0 < \alpha < 2\xi < \beta < \infty$, and that ψ be continuous and strictly positive on $[\alpha, \beta]$. This guarantees a jump discontinuity at the end points. Nonetheless, enough of the regulated model constraints remain for the general theory to apply. To be more precise about what remains, (1) the conditions on the dispersion law ε still apply; (2) since ψ has compact support, it follows that $\psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and that $p\psi \in L^1(\mathbb{R})$ for all polynomials p; (3) since $\alpha > 0$, the function $u \mapsto u^{-1}\psi(u)$ is also integrable. Thus ψ satisfies a modified form of Definition 12.

One important implication of our new assumptions for ψ is that there exists a constant $\gamma > 0$ such that

$$\psi(u) \;\geq\; \gamma \;>\; 0 \,, \qquad \alpha \leq u \leq \beta \,. \tag{13.1}$$

Remark 46. As we will be proving results which are parallel to those for regulated models, we will occasionally distinguish the two cases by referring to the cut-off or the regulated case as appropriate.

We now specify the basic properties of the Cauchy transform of ψ in the cut-off case.

Proposition 47. *The function $\mathcal{L}[K] = L[\psi]$ can be extended to a function analytic on the domain $\mathbb{V} = \mathbb{C}\backslash[-i\beta, -i\alpha]$, on which domain it satisfies the following symmetry condition:*

$$L[\psi](-\overline{p}) \;=\; -\overline{L[\psi](p)}\,, \qquad p \in \mathbb{V}\,. \tag{13.2}$$

Additionally, $L[\psi]$ belongs to $H^2(\mathbb{H}_+)$, with boundary function $2\pi\mathfrak{P}\psi = \pi\psi + \pi i\mathcal{H}[\psi]$ in $L^2(\mathbb{R})$, and $(\psi_\delta)_{\delta>0}$ is a collection of functions in $L^2(\mathbb{R})$ which converges to $2\pi\mathfrak{P}\psi$ in $L^2(\mathbb{R})$-norm as $\delta \to 0+$. As well, there exists a sequence $(r_n)_{n\geq 0}$, with $0 < r_n < \frac{1}{2}(\beta-\alpha)$ for all $n \geq 0$, such that $r_n \to 0$ as $n \to \infty$ and such that $\psi_{r_n} \to 2\pi\mathfrak{P}\psi$ almost everywhere as $n \to \infty$.

Proof. All but the last of these results may be proved using similar methods to those employed in the section on smooth couplings. Since $\psi_\delta \to 2\pi\mathfrak{P}\psi$ in mean as $\delta \to 0+$, standard measure theory implies that this convergence also takes place in measure, and hence that convergence takes place almost everywhere for an appropriately chosen sequence $(r_n)_{n\geq 0}$. □

However $L[\psi]$ no longer necessarily belongs to $H^\infty(\mathbb{H}_+)$, and so other aspects of our analysis need to be handled differently now.

Lemma 48. *In the cut-off case, the function $L[\psi](p) \to 0$ as $|\,p\,| \to \infty$, and*

$$\operatorname{Re} L[\psi](\delta - iy) \;>\; \frac{1}{4}\pi\gamma\,, \qquad \alpha < y < \beta\,, \quad 0 < \delta < \frac{1}{2}(\beta-\alpha)\,. \tag{13.3}$$

Proof. Since

$$|\,L[\psi](x+iy)\,| \;\leq\; (|\,p\,| - \beta)^{-1}\,\|\,\psi\,\|_1\,, \qquad |\,p\,| > \beta\,,$$

the first result is straightforward. We see that

$$\begin{aligned}
\operatorname{Re} L[\psi](\delta - iy) \;&=\; \delta\int_\alpha^\beta \frac{\psi(u)}{\delta^2 + (u-y)^2}\,du \\
&\geq\; \gamma\delta\int_\alpha^\beta \frac{du}{\delta^2+(u-y)^2} \\
&=\; \gamma\Big[\tan^{-1}\big(\frac{\beta-y}{\delta}\big) + \tan^{-1}\big(\frac{y-\alpha}{\delta}\big)\Big]
\end{aligned}$$

for any $\alpha < y < \beta$ and $\delta > 0$. If $0 < \delta < (\beta - \alpha)/2$, at least one of $\delta^{-1}(\beta - y)$ and $\delta^{-1}(y - \alpha)$ must be greater than 1, so at least one of the two arctangents must be greater than $\pi/4$, and both are positive. This establishes the final result. □

14. Analytic Extensions of H and $L[\varphi]$

Now consider the denominator for the Laplace transform of φ on the domain $\mathbb{V}$. It has properties following directly from those of $L[\psi]$ given in Lemma 48.

Lemma 49. *In the cut-off case, the function*

$$H(p) \;=\; p + 2i\xi + \mu^2 L[\psi](p)\,, \qquad p \in \mathbb{V}\,, \tag{14.1}$$

is analytic in $\mathbb{V}$ *and satisfies the symmetry condition*

$$H(-\overline{p}) \;=\; -\overline{H(p)}\,, \qquad p \in \mathbb{V}\,. \tag{14.2}$$

Moreover, $|\,H(p)\,| \to \infty$ *as* $|\,p\,| \to \infty$*, and*

$$\operatorname{Re} H(\delta - iy) \;>\; \frac{1}{4}\pi\gamma\mu^2\,, \qquad \alpha < y < \beta\,, \quad 0 < \delta < \frac{1}{2}(\beta - \alpha)\,. \tag{14.3}$$

The next result begins to indicate how this discontinuous solution differs from the smooth solution of the previous section.

Proposition 50. *For any* $\mu > 0$*,* H *has exactly two zeros in* $\mathbb{V}$*, and these occur at points* $-i\alpha_\mu$ *and* $-i\beta_\mu$*, where* $\alpha_\mu, \beta_\mu \in \mathbb{R}$*, with* $\alpha_\mu < \alpha$ *and* $\beta < \beta_\mu$*. Both of these zeros of* H *are simple, with*

$$\begin{aligned} H'(-i\alpha_\mu) \;&=\; 1 + \mu^2 \int_\alpha^\beta \frac{\psi(u)\,du}{(u - \alpha_\mu)^2}\,, \\ H'(-i\beta_\mu) \;&=\; 1 + \mu^2 \int_\alpha^\beta \frac{\psi(u)\,du}{(\beta_\mu - u)^2}\,. \end{aligned} \tag{14.4}$$

Proof. Since $\mathcal{L}[\varphi](p)H(p) = 1$ for all $p \in \mathbb{H}_+$, H has no zeros in $\mathbb{H}_+$. In view of the symmetry of H given by (14.2), H has no zeros in $\mathbb{H}_-$ either, so the only possible zeros of H are purely imaginary. Moreover, H is purely imaginary on $\mathbb{V} \cap i\mathbb{R}$, and the real-valued function $k(y) = -iH(iy)$ is defined on $(-\infty, -\beta) \cup (-\alpha, \infty)$ by the formula

$$k(y) \;=\; y + 2\xi - \mu^2 \int_\alpha^\beta \frac{\psi(u)}{u + y}\,du\,, \qquad y \in (-\infty, -\beta) \cup (-\alpha, \infty)\,.$$

Evidently k is differentiable on this disconnected domain, with

$$k'(y) \;=\; 1+\mu^2\int_\alpha^\beta \frac{\psi(u)}{(y+u)^2}\,du \;>\; 1\,, \qquad y\in(-\infty,-\beta)\cup(-\alpha,\infty)\,.$$

Since

$$\left|\int_\alpha^\beta \frac{\psi(u)}{u+y}\,du\right| \;\geq\; \gamma\int_\alpha^\beta \frac{du}{|\,u+y\,|} \;=\; \gamma\log\big(\frac{y+\beta}{y+\alpha}\big)$$

for all $y\in(-\infty,-\beta)\cup(-\alpha,\infty)$,

$$\lim_{y\to-\alpha+} k(y) \;=\; -\infty\,, \qquad \lim_{y\to-\beta-} k(y) \;=\; \infty\,,$$

while

$$\lim_{y\to\infty} k(y) \;=\; \infty\,, \qquad \lim_{y\to-\infty} k(y) \;=\; -\infty\,,$$

from which it follows that k has exactly one zero $-\beta_\mu$ in $(-\infty,-\beta)$, and exactly one zero $-\alpha_\mu$ in $(-\alpha,\infty)$. Since $H'(iy) \;=\; k'(y)$ for all $y\in(-\infty,-\beta)\cup(-\alpha,\infty)$, the proposition follows. □

Corollary 51. *In the cut-off case, $\mathcal{L}[\varphi]$ can be extended analytically to the domain $\mathbb{V}\backslash\{-i\alpha_\mu,-i\beta_\mu\}$ by the formula*

$$\mathcal{L}[\varphi](p) \;=\; \frac{1}{H(p)}\,, \qquad p\in\mathbb{V}\backslash\{-i\alpha_\mu,-i\beta_\mu\}\,, \tag{14.5}$$

with symmetry

$$\mathcal{L}[\varphi](-\overline{p}) \;=\; -\overline{\mathcal{L}[\varphi](p)}\,, \qquad p\in\mathbb{V}\backslash\{-i\alpha_\mu,-i\beta_\mu\}\,. \tag{14.6}$$

The function $\mathcal{L}[\varphi]$ has simple poles at $-i\alpha_\mu$ and $-i\beta_\mu$, with residues $H'(-i\alpha_\mu)^{-1}$ and $H'(-i\beta_\mu)^{-1}$ respectively. In addition, $|\,\mathcal{L}[\varphi](p)\,| \to 0$ as $|\,p\,|\to\infty$, and $\mathcal{L}[\varphi](\delta-iy)$ is bounded for $\alpha<y<\beta$ and $0<\delta<(\beta-\alpha)/2$.

Proof. These results are simple consequences of Lemma 49 and Proposition 50. In particular, since

$$|\,H(\delta-iy)\,| \;\geq\; |\,\mathrm{Re}\,H(\delta-iy)\,| \;\geq\; \frac{1}{4}\pi\gamma\mu^2$$

for all $\alpha<y<\beta$ and $0<\delta<\frac{1}{2}(\beta-\alpha)$, it follows that $|\,\mathcal{L}[\varphi](\delta-iy)\,|$ is bounded above by $4/(\pi\gamma\mu^2)$ in the same range. □

Although we no longer have $\mathcal{L}[\varphi]$ belonging to any particular Hardy space, we can still prove the existence of a boundary function for $\mathcal{L}[\varphi]$, at least on the critical part of the imaginary axis.

Remark 52. To emphasize the similarities with the regulated cases, we use the same symbols as nearly as possible: φ, H, $\mathfrak{h}$, Ξ, and so on.

Proposition 53. *The function* $\mathfrak{h} : [\alpha, \beta] \to \mathbb{C}$ *be given by*

$$\mathfrak{h}(y) \;=\; \frac{1}{\pi\mu^2\psi(y) - i[y - 2\xi - \pi\mu^2(\mathcal{H}[\psi])(y)]}\,, \qquad \alpha \le y \le \beta\,. \tag{14.7}$$

belongs to $L^1[\alpha,\beta] \cap L^2[\alpha,\beta]$*, and is the boundary function for* $\mathcal{L}[\varphi]$ *in the cut-off case, in the sense that* $\mathcal{L}[\varphi](r_n - iy) \to \mathfrak{h}(y)$ *as* $n \to \infty$ *for almost all* $y \in [\alpha,\beta]$*, and also in both* $L^1[\alpha,\beta]$*-norm and* $L^2[\alpha,\beta]$*-norm.*

Proof. The integrability properties of $\mathfrak{h}$, and the convergence results, come from the estimate

$$0 \;\le\; |\,\mathcal{L}[\varphi](r_n - iy)\,| \;\le\; \frac{4}{\pi\gamma\mu^2}\,, \qquad n \ge 0\,, \quad \alpha < y < \beta$$

and the Dominated Convergence Theorem. □

We can therefore define the function $\Xi \in L^1[\alpha,\beta] \cap L^2[\alpha,\beta]$ by the formula

$$\Xi(y) \;=\; (\operatorname{Re}\mathfrak{h})(y) \;=\; \frac{\pi\mu^2\psi(y)}{\pi^2\mu^4\psi(y)^2 + [y - 2\xi - \pi\mu^2(\mathcal{H}[\psi])(y)]^2}\,, \qquad \alpha \le y \le \beta. \tag{14.8}$$

15. Inverting the Laplace Transform

We now have enough information to invert the Laplace transform to obtain the function φ. The Bromwich integral formula implies that, for any $\kappa > 0$,

$$\varphi(t) \;=\; \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\kappa - iR}^{\kappa + iR} \mathcal{L}[\varphi](p)e^{pt}\,dp\,, \tag{15.1}$$

for any $t > 0$. Consider the contour shown below, where $T > \max\{\,|\,\alpha_\mu\,|\,, \beta_\mu\,\}$. Since $\mathcal{L}[\varphi](p)$ tends to 0 as $|\,p\,| \to \infty$, it is easy to show that

$$\lim_{T\to\infty}\int_{[-iT,\kappa - iT]} \mathcal{L}[\varphi](p)e^{pt}\,dp \;=\; 0\,,$$

$$\lim_{T\to\infty}\int_{[iT,\kappa + iT]} \mathcal{L}[\varphi](p)e^{pt}\,dp \;=\; 0\,.$$

If Γ_T is the positively oriented semicircular contour, centre 0 and radius T, running from iT to $-iT$, then

$$\lim_{T\to\infty}\int_{\Gamma_T}\mathcal{L}[\varphi](p)e^{pt}\,dp \;=\; 0\,,$$

using Jensen's inequality, and so

$$\varphi(t) \;=\; \frac{1}{H'(-i\alpha_\mu)}e^{-i\alpha_\mu t}+\frac{1}{H'(-i\beta_\mu)}e^{-i\beta_\mu t}+\frac{1}{2\pi i}\int_C\mathcal{L}[\varphi](p)e^{pt}\,dp\,,\; t>0\,,$$

where C is any positively oriented simple contour in $\mathbb{V}\backslash\{-i\alpha_\mu,-i\beta_\mu\}$ which encloses the line segment $[-i\beta,-i\alpha]$, but which encloses neither $-i\alpha_\mu$ nor $-i\beta_\mu$.

Choose $0<\delta<\min\big(\frac{1}{2}(\beta-\alpha),b-\beta,\alpha-a\big)$, and consider the contour $C(\delta)$, made up of the following four components:

- a positively oriented semicircular contour $S(\alpha,\delta)$ with centre $-i\alpha$ and radius δ, running from $\delta-i\alpha$ to $-\delta-i\alpha$,
- the straight line segment from $-\delta-i\alpha$ to $-\delta-i\beta$,
- a positively oriented semicircular contour $S(\beta,\delta)$ with centre $-i\beta$ and radius δ, running from $-\delta-i\beta$ to $\delta-i\beta$,
- the straight line segment from $\delta-i\beta$ to $\delta-i\alpha$,

noting that $C(\delta)$ satisfies the conditions outlined above. If $0\le\theta\le\pi$ then

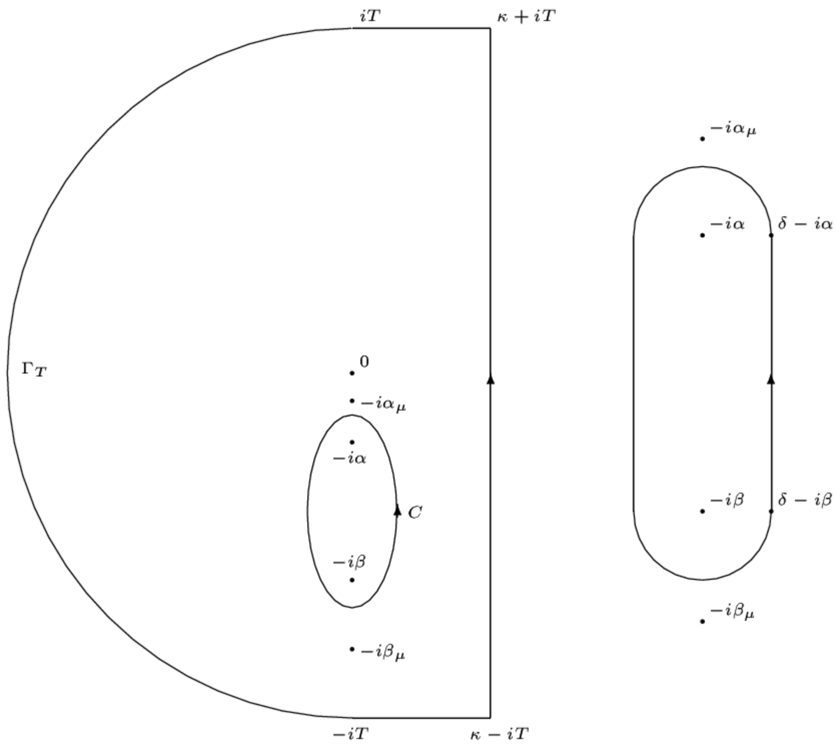
$$\begin{aligned}&\Big|L[\psi](-i\alpha+\delta e^{-\theta})\Big|\ge\Big|\operatorname{Im}L[\psi](-i\alpha+\delta e^{i\theta})\Big|\\&=\int_\alpha^\beta\frac{\psi(u)(u-\alpha+\delta\sin\theta)}{\delta^2\cos^2\theta+(u-\alpha+\delta\sin\theta)^2}\,du\\&\ge\gamma\int_\alpha^\beta\frac{u-\alpha+\delta\sin\theta}{\delta^2\cos^2\theta+(u-\alpha+\delta\sin\theta)^2}\,du\;\ge\;\frac{1}{2}\gamma\log\Big[1+(\frac{\beta-\alpha}{\delta})^2\Big]\,,\end{aligned}$$

so $L[\psi](p)\to\infty$ as $p\to-i\alpha$ through values in $\mathbb{K}_+(-\alpha)$. Then $\mathcal{L}[\varphi](p)\to 0$ as $p\to-i\alpha$ through values in $\mathbb{K}_+(-\alpha)$. Consequently

$$\int_{S(\alpha,\delta)}\mathcal{L}[\varphi](p)e^{pt}\,dp \;=\; \mathrm{O}(\delta)\,,\qquad \delta\to 0+\,,$$

and hence

$$\lim_{\delta\to0+}\int_{S(\alpha,\delta)}\mathcal{L}[\varphi](p)e^{pt}\,dp \;=\; 0\,.$$



Similarly

$$\lim_{\delta\to 0+} \int_{S(\beta,\delta)} \mathcal{L}[\varphi](p) e^{pt}\, dp \;=\; 0\ .$$

Moreover

$$\int_{\delta-i\alpha}^{\delta-i\beta} \mathcal{L}[\varphi](p) e^{pt}\, dp = -i \int_\alpha^\beta \mathcal{L}[\varphi](\delta - iy) e^{\delta t} e^{-iyt}\, dy\ ,$$

$$\begin{aligned}\int_{-\delta-i\alpha}^{-\delta-i\beta} \mathcal{L}[\varphi](p) e^{pt}\, dp &= -i \int_\alpha^\beta \mathcal{L}[\varphi](-\delta - iy) e^{-\delta t} e^{-iyt}\, dy \\ &= i \int_\alpha^\beta \overline{\mathcal{L}[\varphi](\delta - iy)} e^{-\delta t} e^{-iyt}\, dy\ ,\end{aligned}$$

so that

$$\begin{aligned}&\left(\int_{-\delta-i\alpha}^{-\delta-i\beta} - \int_{\delta-i\alpha}^{\delta-i\beta}\right) \mathcal{L}[\varphi](p) e^{pt}\, dp \\ &\quad = 2i \int_\alpha^\beta \Big[(\operatorname{Re}\mathcal{L}[\varphi])(\delta - iy)\cosh\delta t + i(\operatorname{Im}\mathcal{L}[\varphi])(\delta - iy)\sinh\delta t\Big] e^{-iyt}\, dy\ .\end{aligned}$$

Using Lemma 51, these results imply that as the inner contour shrinks,

$$\lim_{\delta\to 0+}\int_{C(\delta)} \mathcal{L}[\varphi](p)e^{pt}\,dp \;=\; 2i\lim_{\delta\to 0+}\int_{\alpha}^{\beta}(\operatorname{Re}\mathcal{L}[\varphi])(\delta - iy)e^{-iyt}\,dy\;.$$

Now, restricting our attention to the particular sequence $(r_n)_{n\geq 0}$ introduced previously, we can use the Dominated Convergence Theorem to obtain

$$\lim_{n\to\infty}\int_{C(r_n)} \mathcal{L}[\varphi](p)e^{pt}\,dp \;=\; 2i\int_{\alpha}^{\beta}\Xi(y)e^{-iyt}\,dy\;.$$

In summary, then, we obtain the following proposition.

Proposition 54. *In the cut-off case, the function φ has the formula*

$$\varphi(t) \;=\; \frac{1}{H'(-i\alpha_\mu)}e^{-i\alpha_\mu t} + \frac{1}{H'(-i\beta_\mu)}e^{-i\beta_\mu t} + \frac{1}{\pi}\int_{\alpha}^{\beta}\Xi(y)e^{-iyt}\,dy\;, \qquad t > 0\,. \quad (15.2)$$

Since φ always possesses periodic terms, the state $\mathcal{X}$ always has infinite lifetime.

16. Asymptotic Behaviour of the Periodic Terms

We have now shown that the cut-off coupling function always produces a solution φ containing two periodic terms, no matter how small (but nonzero) the coupling constant μ. A consequence is that the standard notion of lifetime is not well-suited to this model, since it gives no information about the rate and probability of a transition from $\mathcal{X}$. Moreover, the square-integrability property of φ, necessary for a finite lifetime, is not a property which is stable under small perturbations of μ even in the regulated model. For although φ is, in a certain sense, a continuous function of μ (see Corollary 26), the lifetime of the state $\mathcal{X}$ switches abruptly from being infinite to being finite at two points.

Another aspect of the notion of lifetime that is less than satisfactory arises from the fact that the sharp cut-off model can be approximated by a regulated model through replacing the jump discontinuities of ψ at α and β by smooth functions. Corollary 25 then tells us that the function φ must be locally uniformly close to some φ for a regulated model. One could then hope that an appropriate notion of stability would show similar behaviour in the two cases. The lifetime certainly does not: for small nonzero μ, the lifetime for $\mathcal{X}$ is always infinite for the cut-off model and always finite for the associated regulated model.

In view of this, it seems worthwhile trying to define a notion of "effective lifetime" which is consonant with the properties of the sharp cut-off model, and complements the lifetime and the threshold time.

In order to accomplish this, we need an understanding of the manner in which the periodic components of φ, vary with the coupling constant μ in the cut-off case.

Proposition 55. *The quantity α_μ is a differentiable, decreasing function of $\mu > 0$, while β_μ is a differentiable, increasing function of $\mu > 0$. Consequently the limits of both α_μ and β_μ exist as $\mu \to 0+$, and*

$$\lim_{\mu\to 0+} \alpha_\mu \;=\; \alpha\,, \qquad \lim_{\mu\to 0+} \beta_\mu \;=\; \beta\,. \tag{16.1a}$$

Moreover

$$\lim_{\mu\to 0+} H'(-i\alpha_\mu) \;=\; \lim_{\mu\to 0+} H'(-i\beta_\mu) \;=\; \infty\,. \tag{16.1b}$$

Proof. Note that β_μ satisfies the integral equation

$$-\beta_\mu + 2\xi + \mu^2 \int_\alpha^\beta \frac{\psi(u)\,du}{\beta_\mu - u} \;=\; 0\,, \qquad \mu > 0\,.$$

Thus β_μ is a differentiable function of μ, with

$$\Big[1 + \mu^2 \int_\alpha^\beta \frac{\psi(u)\,du}{(\beta_\mu - u)^2}\Big]\beta'_\mu \;=\; 2\mu \int_\alpha^\beta \frac{\psi(u)\,du}{\beta_\mu - u}\,, \qquad \mu > 0\,.$$

This implies that $\beta'_\mu > 0$ for all $\mu > 0$, and hence that β_μ is an increasing function of μ. Since $\beta_\mu > \beta$ for all $\mu > 0$, this implies that $\widehat{\beta} = \lim_{\mu\to 0+}\beta_\mu$ exists, with $\widehat{\beta} \geq \beta$. If $\widehat{\beta} > \beta$, then

$$\lim_{\mu\to 0+} \int_\alpha^\beta \frac{\psi(u)\,du}{\beta_\mu - u} \;=\; \int_\alpha^\beta \frac{\psi(u)\,du}{\widehat{\beta} - u}\,,$$

which is finite, so that

$$0 = \lim_{\mu\to 0+} \mu^2 \int_\alpha^\beta \frac{\psi(u)\,du}{\beta_\mu - u} \;=\; \lim_{\mu\to 0+} \beta_\mu - 2\xi \;=\; \widehat{\beta} - 2\xi \;\geq\; \beta - 2\xi \;>\; 0\,,$$

providing a contradiction. Hence $\widehat{\beta} = \beta$, as required. Note that

$$\begin{aligned} 0 \;<\; 2\mu \int_\alpha^\beta \frac{\psi(u)\,du}{\beta_\mu - u} \;&\leq\; 2\mu \left(\int_\alpha^\beta \psi(u)\,du\right)^{\frac12} \left(\int_\alpha^\beta \frac{\psi(u)\,du}{(\beta_\mu - u)^2}\right)^{\frac12} \\ &\leq\; \|\,\psi\,\|_1^{\frac12} \Big[1 + \mu^2 \int_\alpha^\beta \frac{\psi(u)\,du}{(\beta_\mu - u)^2}\Big]\,, \end{aligned}$$

and so $0 < \beta'_\mu < \| \psi \|_1^{\frac{1}{2}}$ for all $\mu > 0$. Since

$$\mu H'(-i\beta_\mu)\beta'_\mu \;=\; 2\mu^2 \int_\alpha^\beta \frac{\psi(u)\,du}{\beta_\mu - u} \;=\; 2(\beta_\mu - 2\xi)\;, \qquad \mu > 0\,,$$

we deduce that

$$\lim_{\mu\to 0+} \mu H'(-i\beta_\mu)\beta'_\mu \;=\; 2(\beta - 2\xi) \;>\; 0\;.$$

Since $\lim_{\mu\to 0+} \mu\beta'_\mu = 0$, it follows that $\lim_{\mu\to 0+} H'(-i\beta_\mu) = \infty$, as required.

The corresponding results for α_μ are proved in an identical manner. □

Thus the terms $H'(-i\alpha_\mu)^{-1}e^{-i\alpha_\mu t}$ and $H'(-i\beta_\mu)^{-1}e^{-i\beta_\mu t}$ are small when μ is small, and so the dominating term in the expression for φ is the multiple of the Fourier transform of Ξ. Since Ξ belongs to both $L^1[\alpha,\beta]$ and $L^2[\alpha,\beta]$, its Fourier transform $\mathcal{F}\Xi$ belongs to $L^2(\mathbb{R})$. Consequently, insofar as we may ignore the periodic components of the solution, on probabilistic grounds, say, the remaining expression in φ has finite lifetime and, moreover, can be uniquely distinguished from the periodic terms. That we might feel able to ignore these periodic terms in the sharp cut-off solution is an idea further reinforced by the fact that $\alpha_\mu \to \alpha$ and $\beta_\mu \to \beta$ extremely rapidly as $\mu \to 0+$. The precise rate of convergence is given in this next proposition.

Proposition 56. *The following limits hold:*

$$\lim_{\mu\to 0+} \mu^2 \log(\alpha - \alpha_\mu) \;=\; \frac{\alpha - 2\xi}{\psi(\alpha)}\;, \tag{16.2a}$$

$$\lim_{\mu\to 0+} \mu^2 \log(\beta_\mu - \beta) \;=\; \frac{2\xi - \beta}{\psi(\beta)}\;. \tag{16.2b}$$

Proof. Let $0 < \eta < \min\big(\psi(\beta), \frac{1}{3}(\beta - \alpha)\big)$. Since

$$\mu^2 \int_\alpha^\beta \frac{\psi(u) - \psi(\beta)}{\beta_\mu - u}\,du - \psi(\beta)\mu^2 \log(\beta_\mu - \beta) \;=\; \beta_\mu - 2\xi - \psi(\beta)\mu^2 \log(\beta_\mu - \alpha)\;,$$

we deduce that we can find $\mu_1 > 0$ such that

$$0 < \mu < \mu_1 \;\Rightarrow\; \left| \mu^2 \int_\alpha^\beta \frac{\psi(u) - \psi(\beta)}{\beta_\mu - u}\,du - \psi(\beta)\mu^2 \log(\beta_\mu - \beta) - \beta + 2\xi \right| \;<\; \eta\;.$$

We can find $0 < \delta < \beta - \alpha$ such that $|\,\psi(u) - \psi(\beta)\,| \le \eta$ for $\beta - \delta \le u \le \beta$, and then find $\mu_2 > 0$ such that $2\mu_2^2 \,\|\,\psi\,\|_\infty\,(\beta - \alpha) \le \delta\eta$ and

$$0 < \mu < \mu_2 \;\Rightarrow\; \big|\,\mu^2 \log(\beta_\mu - \beta + \delta)\,\big| \le 1\;.$$

Thus

$$
\begin{aligned}
&\left| \mu^2 \int_\alpha^\beta \frac{\psi(u)-\psi(\beta)}{\beta_\mu - u}\, du \right| \\
\leq\ & 2\mu^2 \,\|\, \psi \,\|_\infty \int_\alpha^{\beta-\delta} \frac{du}{\beta_\mu - u} + \mu^2 \int_{\beta-\delta}^\beta \frac{|\,\psi(u)-\psi(\beta)\,|}{\beta_\mu - u}\, du \\
\leq\ & \frac{2\mu^2(\beta-\alpha)}{\beta_\mu - \beta + \delta} \,\|\, \psi \,\|_\infty + \eta\mu^2 \log\Big(\frac{\beta_\mu - \beta + \delta}{\beta_\mu - \beta}\Big) \\
\leq\ & 2\eta - \eta\mu^2 \log(\beta_\mu - \beta)\ ,
\end{aligned}
$$

for $0 < \mu < \mu_2$, and hence

$$
\frac{\beta - 2\xi - 3\eta}{\psi(\beta)+\eta} \ \leq\ -\mu^2 \log(\beta_\mu - \beta) \ \leq\ \frac{\beta - 2\xi + 3\eta}{\psi(\beta)-\eta}\ ,
$$

for $0 < \mu < \min(\mu_1, \mu_2)$, which establishes the desired limit for $\mu^2 \log(\beta_\mu - \beta)$ as $\mu \to 0+$. The matching result for $a(\mu)$ is proved in an identical manner. $\square$

This result has the following important consequences:

Corollary 57. *We have the following asymptotic estimates as* $\mu \to 0+$*:*

$$
\begin{aligned}
\alpha_\mu &= \alpha + \mathrm{O}\big(\exp(-(2\xi - \alpha - \eta)\mu^{-2})\big)\ , && (16.3a)\\
\beta_\mu &= \beta + \mathrm{O}\big(\exp(-(\beta - 2\xi - \eta)\mu^{-2})\big)\ , && (16.3b)
\end{aligned}
$$

$$
\begin{aligned}
H'(-i\alpha_\mu)^{-1} &= \mathrm{O}\big(\mu^{-2} \exp(-(2\xi - \alpha - \eta)\mu^{-2})\big)\ , && (16.4a)\\
H'(-i\beta_\mu)^{-1} &= \mathrm{O}\big(\mu^{-2} \exp(-(\beta - 2\xi - \eta)\mu^{-2})\big)\ , && (16.4b)
\end{aligned}
$$

for any $0 < \eta < \min(2\xi - \alpha, \beta - 2\xi)$.

Proof. Since $\mu^2 \log(\beta_\mu - \beta)$ converges to $2\xi - \beta$ as $\mu \to 0+$, we can find $\tilde{\mu} > 0$ such that

$$
\begin{aligned}
0 < \mu < \tilde{\mu} \quad &\Rightarrow \quad \mu^2 \log(\beta_\mu) - \beta) < -(\beta - 2\xi - \eta) \\
&\Rightarrow \quad \beta < \beta_\mu < \beta + \exp(-(\beta - 2\xi - \eta)\mu^{-2})\ .
\end{aligned}
$$

This establishes (16.3b). Moreover, we can assume that $\exp(-(\beta - 2\xi - \eta)\tilde{\mu}^{-2}) < \beta - \alpha$ and so, if $0 < \mu < \tilde{\mu}$, then

$$
\begin{aligned}
\int_\alpha^\beta \frac{\psi(u)\, du}{(\beta_\mu - u)^2} \quad &\geq \quad \gamma \int_\alpha^\beta \frac{du}{(\beta - e^{-(\beta-2\xi-\eta)\mu^{-2}} - u)^2} \\
&= \quad \gamma e^{(\beta-2\xi-\eta)\mu^{-2}} \frac{\beta - \alpha}{\beta - \alpha + e^{-(\beta-2xi-\eta)\mu^{-2}}} \\
&\geq \quad \frac{1}{2} \gamma e^{(\beta-2\xi-\eta)\mu^{-2}}\ ,
\end{aligned}
$$

and so $H'(-i\beta_\mu) > \frac{1}{2}\gamma\mu^2 \exp((\beta - 2\xi - \eta)\mu^{-2})$, which inequality establishes (16.4b).

The results for α_μ and $H'(-i\alpha_\mu)$ are proved in an identical manner. □

Thus we have shown, in a precise sense, that for small μ, the periodic terms in φ are extremely small, and hence the square-integrable component of φ is an extremely good uniform approximation for φ. Moreover, the solution φ in this sharp cut-off case must be a reasonable uniform approximation to a comparable smooth solution for small μ, since its coupling function can be approximated in $L^2(\mathbb{R})$-norm by a regulated coupling function. As μ becomes larger, however, the periodic terms in our solution become significant. This is paralleled by the same behaviour in the smooth solution, which develops periodic terms once μ has reached a critical value.

We thus feel justified in introducing the following quantity.

Definition 58. For sharp cut-off models, we define the *effective lifetime* of the states $\mathcal{X} = e[+] \otimes \Omega$, $e_{\mathrm{S}} \otimes \Omega$, to be the quantity

$$\tau_{\mathrm{eff}}(\mathcal{X}) = \frac{1}{\pi} \int_0^\infty |\, \mathcal{F}[\Xi](t)\,|^2 \, dt \tag{16.5}$$

Since Ξ is a real-valued function, this implies that

$$\tau_{\mathrm{eff}}(\mathcal{X}) = \frac{1}{2\pi} \,\|\, \Xi \,\|_2^2 \ . \tag{16.6}$$

This definition depends on the detailed form of $\mathcal{X}_t$ for the two states under consideration. If we could find a principle (perhaps invariance under some symmetry) that would distinguish $\mathcal{F}\Xi$ from any periodic terms *a priori*, we could offer a definition of effective lifetime which was applicable to other, perhaps all, states and coupling functions. Unfortunately we have not found any such selection principle.

In this regard, the time evolution of 1-qubit state vectors $\Psi = (\eta e[+] + \zeta e[-]) \otimes \Omega$ can be calculated by simple linear combination, as we did for the regulated model. It is also clear, as before, that any state vector with $\zeta \neq 0$ has infinite lifetime, due to the nontrivial multiple of $e[-] \otimes \Omega$ which appears in Ψ_t.

In the same way that the effective lifetime supplements the lifetime for the states Ψ in the cut-off model, a notion of *effective threshold time* $T_{\mathrm{eff}}(\Psi)$ will supplement the threshold time.

Definition 59. For state vectors $\Psi = (\eta e[+] + \zeta e[-]) \otimes \Omega$ with $|\,\zeta\,| <$

$e^{-1/4}$, the effective threshold time $T_{\text{eff}}(\Psi)$ is that extended positive positive real for which

$$\left| e^{i\xi t} \left(\left| \sqrt{2/\pi}\eta \right|^2 \mathcal{F}[\Xi](t) + |\zeta|^2 \right) \right|^2 < e^{-1} \tag{16.7}$$

whenever $t > T(\Psi)$.

What we have done here is replace $\varphi(t)$ of equation (11.5) by the simpler expression $\sqrt{2/\pi}\mathcal{F}[\Xi](t)$ and substitute it into Definition 38. This replacement is the analogue of the change from lifetime to effective lifetime.

Unlike the threshold time for the regulated models, we do not know that $T_{\text{eff}}(\Psi)$ is finite. To determine whether it is or not is a matter of studying the asymptotic behaviour of $\mathcal{F}[\Xi](t)$ as $t \to \infty$, not a particularly easy task. The general conditions that we have assumed in the cut-off case do not ensure differentiability of Ξ nor the continuity of the derivative if it exists; and it was the continuous differentiability of Ξ in the n-regulated or hyperregulated model that enabled us to evaluate the threshold time there. As can be seen from inspection of the functions Ξ displayed in the following section, that Ξ cannot be assumed to have continuous derivative is a real problem. In both cases below, Ξ is differentiable on the open interval (α, β), but its derivative cannot be extended to a function continuous on $[\alpha, \beta]$. Indeed, $\Xi'(t)$ diverges to infinity as t approaches either α or β.

We do not see this as a problem, however. Our opinion is that the n-regulated, or more likely the hyperregulated, model, is the appropriate one to consider on physical grounds. The fact that the mathematical properties of the idealized sharp cut-off model are imperfect reflects the imperfections of the model rather than any physics, since it introduces an unnecessary discontinuity in the Hamiltonian.

17. The Top Hat Solution

In this section we shall give the solution when the coupling function λ is the characteristic function of a compact interval in $(0, \infty)$, and the dispersion law is $\varepsilon(k) = c\,|\,k\,|$. The details of the calculation are routine and we omit them.

- The coupling function ψ is

$$\psi(u) = \begin{cases} 2c^{-1}, & \alpha \le u \le \beta, \\ 0, & \text{otherwise}, \end{cases} \tag{17.1}$$

where $\alpha < 2\xi < \beta$, as usual.

- The Laplace transform of ψ is

$$L[\psi](\delta - iy) = \frac{2}{c}\Big[\tan^{-1}\big(\frac{\beta - y}{\delta}\big) + \tan^{-1}\big(\frac{y-\alpha}{\delta}\big)\Big] - \frac{i}{c}\log\Big(\frac{\delta^2 + (\beta - y)^2}{\delta^2 + (y-\alpha)^2}\Big) , \quad (17.2)$$

for $\delta > 0$ and $\alpha < y < \beta$.

- Its Hilbert transform is

$$(\mathcal{H}[\psi])(y) = -\frac{2}{\pi c}\log\big(\frac{\beta - y}{y - \alpha}\big) , \qquad \alpha < y < \beta . \quad (17.3)$$

- The amplitudes of the periodic terms are

$$H'(-i\alpha_\mu) = \frac{c(\alpha - \alpha_\mu)(\beta - \alpha_\mu) + 2\mu^2(\beta - \alpha)}{c(\alpha - \alpha_\mu)(\beta - \alpha_\mu)} , \quad (17.4a)$$

$$H'(-i\beta_\mu) = \frac{c(\beta_\mu - \alpha)(\beta_\mu - \beta) + 2\mu^2(\beta - \alpha)}{c(\beta_\mu - \alpha)(\beta_\mu - \beta)} . \quad (17.4b)$$

- Finally, the function φ has the complicated but explicit form

$$\varphi(t) = \frac{c(\alpha - \alpha_\mu)(\beta - \alpha_\mu)}{c(\alpha - \alpha_\mu)(\beta - \alpha_\mu) + 2\mu^2(\beta - \alpha)} e^{-i\alpha_\mu t} + \frac{c(\beta_\mu - \alpha)(\beta_\mu - \beta)}{c(\beta_\mu - \alpha)(\beta_\mu - \beta) + 2\mu^2(\beta - \alpha)} e^{-i\beta_\mu t} + 2c\mu^2 \int_\alpha^\beta \frac{e^{-iyt}\, dy}{4\pi^2\mu^4 + [cy - 2c\xi + 2\mu^2\log(\frac{\beta - y}{y - \alpha})]^2} , \quad (17.5)$$

for $t > 0$, where α_μ and β_μ are the roots of the equations

$$\alpha_\mu + \frac{2\mu^2}{c}\log\big(\frac{\beta - \alpha_\mu}{\alpha - \alpha_\mu}\big) = \beta_\mu + \frac{2\mu^2}{c}\log\big(\frac{\beta_\mu - \beta}{\beta_\mu - \alpha}\big) = 2\xi . \quad (17.6)$$

It is evidently a formidable, if not impossible, task to determine the characteristic time T and the effective lifetime of $\mathcal{X}$ in closed form.

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