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CONTROLLABILITY OF AFFINE SYSTEMS FOR THE GENERALIZED HEISENBERG LIE GROUPS

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Abstract: Affine control systems on the generalized Heisenberg Lie groups are studied. Controllability of this kind of class of systems on the generalized Heisenberg Lie group is established by relating to their associated bilinear part.

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1. Introduction

In this paper, controllability property of affine control systems on the generalized Heisenberg Lie groups is studied by relating to the controllability of the associated bilinear control system. In [2], the authors Jurdjevic and Sallet studied the controllability of affine systems in Euclidean spaces. They consider the associated bilinear system on $\mathbb{R}^n$ is controllable for the controllability of the affine system on $\mathbb{R}^n$. In this work, the approach of [2] is extended to the study of affine control systems on a generalized Heisenberg Lie group.

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In [1], the authors Ayala and San Martin study the controllability problem through the subalgebra of the Lie algebra of the affine group of G generated by the vector fields of the linear control system where the drift vector field X is an infinitesimal automorphism, i.e., $(X_t)_{t\in \mathbb{R}}$ is a one-parameter subgroup of $\mathrm{Aut}(G)$; lift the system itself to a right-invariant control system on the Lie group $\mathrm{Af}(G)$ and give controllability results for compact connected and noncompact semi-simple Lie groups cases.

This paper is organized in three sections. In Section 2, affine control systems on Lie groups and in Section 3, the generalized Heisenberg Lie group which is an important model of nilpotent simply connected Lie groups are presented. In Section 3, an automorphism of the Lie algebra of the generalized Heisenberg Lie group such that it shrinks the elements of the Lie algebra to the neutral element is found and that the $\mathrm{Aut}(H)$-orbit of the generalized Heisenberg Lie group is dense is proved. The characterization of the controllability of affine control systems on the generalized Heisenberg Lie group associated to their bilinear control systems is given and this result is proved. By the controllability property, we mean that the positive orbit is equal to the state space, i.e., it is possible to reach every state by the trajectories with positive time.

2. Affine Control Systems on Lie Groups

Let G be a connected Lie group with Lie algebra $L(G)$. The affine group $\mathrm{Af}(G)$ of G is the semi-direct product of $\mathrm{Aut}(G)$ with G itself, i.e., $\mathrm{Af}(G) = \mathrm{Aut}(G) \times_s G$. The multiplication in $\mathrm{Af}(G)$ is defined by

$$(\phi, g_1) \cdot (\psi, g_2) = (\phi \circ \psi, g_1\phi(g_2)).$$

Denote by 1 the identity element of $\mathrm{Aut}(G)$ and by e the identity element of G. The group identity of $\mathrm{Af}(G)$ is $(1, e)$ and $(\phi^{-1}, \phi^{-1}(g^{-1}))$ is the inverse of $(\phi, g) \in \mathrm{Af}(G)$. Then, $g \to (1, g)$ and $\phi \to (\phi, e)$ embed G into $\mathrm{Af}(G)$ and $\mathrm{Aut}(G)$ into $\mathrm{Af}(G)$, respectively. Therefore, G and $\mathrm{Aut}(G)$ are subgroups of $\mathrm{Af}(G)$. There is a natural action

$$\mathrm{Af}(G) \times G \to G$$

defined by

$$(\phi, g_1) \cdot g_2 \to g_1\phi(g_2),$$

where $(\phi, g_1) \in \mathrm{Af}(G)$ and $g_2 \in G$. This action is transitive. Indeed, if it is taken $g_2 = e$, then $(\phi, g_1) \cdot e = g_1$ since $\phi(g_2) = e$.

The groups G and $\mathrm{Aut}(G)$ are closed subgroups of $\mathrm{Af}(G)$. Denote by $\mathrm{Aut}(L(G))$ the automorphism group of $L(G)$ which is a Lie group and its Lie algebra is $\mathrm{Der}(L(G))$, the Lie algebra of derivations of $L(G)$. If G is simply connected, then $\mathrm{Aut}(L(G))$ and $\mathrm{Aut}(G)$ are isomorphic. In fact, there is an isomorphism Φ which assigns to each authomorphism ϕ of G its differential $d\phi\mid_1$ at the identity. Any authomorphism of $L(G)$ extends to an authomorphism of G, therefore Φ is indeed an isomorphism between $\mathrm{Aut}(L(G))$ and $\mathrm{Aut}(G)$. Thus, in this case, the Lie algebra of $\mathrm{Aut}(G)$ is $\mathrm{Der}(L(G))$.

The Lie algebra $\mathrm{af}(G)$ of $\mathrm{Af}(G)$ is the semi-direct product $\mathrm{Der}(L(G)) \times_s L(G)$. Its Lie bracket is given by

$$[(D_1, X_1), (D_2, X_2)] = ([D_1, D_2], D_1X_2 - D_2X_1 + [X_1, X_2])$$

An affine control system $\Sigma = (G, \mathcal{D})$ on a Lie group $G \subset \mathrm{Af}(G)$ is determined by the family of differential equations:

$$\dot{x} = (D + X)(x) + \sum_{j=1}^{d} u_j(t)(D^j + Y^j)(x)$$

parametrized by U, family of piecewise constant real valued funtions, where $x \in G$; $D, D^1, \ldots, D^d \in \mathrm{Der}(L(G))$ and $X, Y^1, \ldots, Y^d \in L(G)$. Then, the dynamic is given by

$$\mathcal{D} = \{D + X + \sum_{j=1}^{d} u_j(D^j + Y^j) \mid u \in \mathbb{R}^d\}.$$

If the affine control system is considered on an Abelian Lie group, then it becomes a linear control system. In fact, for the Abelian Lie group case since any bracket between the elements of the Lie algebra is null the affine system turns to the form of the linear control system.

If it is considered $X = 0$ and $Y^1 = Y^2 = \cdots = Y^d = 0$ for affine control system on a Lie group, then it becomes a bilinear control system. In general, affine control systems define richer class of systems than the bilinear class, and their controllability properties on generalized Heisenberg Lie group are essentially governed by their bilinear parts $D, D^1, \ldots, D^d$.

3. The Generalized Heisenberg Lie Group

The Heisenberg Lie Groups are obtained by the following construction, [3]: Let V and Z be finite dimensional real vector spaces and $\beta : V \times V \to Z$ be a symplectic map (i.e., a nondegenerate skew-symmetric bilinear map). The vector space $V \times Z$ becomes a Lie algebra $L(H) =: h(V, Z, \beta)$ with bracket $[(v_1, z_1), (v_2, z_2)] = (0, \beta(v_1, v_2))$. The corresponding simply connected Lie group $H =: H(V, Z, \beta)$ is $V \times Z$ itself endowed with the Campbell-Hausdorff multiplication

$$(v_1, z_1) * (v_2, z_2) = (v_1 + v_2, z_1 + z_2 + \frac{1}{2}\beta(v_1, v_2)).$$

Lemma 1. *For the generalized Heisenberg Lie group $H =: H(V, Z, \beta)$, the map $\varphi_\lambda = \sqrt{\lambda}\mathrm{Id} \times \lambda\mathrm{Id}$, i.e., $\varphi_\lambda(v, z) = (\sqrt{\lambda}v, \lambda z)$ is an automorphism.*

Proof. The mapping φ_λ is 1-1 and onto its image.

$$\varphi_\lambda((v_1, z_1) * (v_2, z_2)) = \varphi_\lambda(v_1 + v_2, z_1 + z_2 + \frac{1}{2}\beta(v_1, v_2))$$

$$= (\sqrt{\lambda}\mathrm{Id}v_1 + \sqrt{\lambda}\mathrm{Id}v_2, \lambda\mathrm{Id}z_1 + \lambda\mathrm{Id}z_2 + \frac{\lambda\mathrm{Id}}{2}\beta(v_1, v_2))$$

by bilinearity of β

$$= (\sqrt{\lambda}\mathrm{Id}v_1 + \sqrt{\lambda}\mathrm{Id}v_2, \lambda\mathrm{Id}z_1 + \lambda\mathrm{Id}z_2 + \frac{1}{2}\beta(\sqrt{\lambda}v_1, \sqrt{\lambda}v_2))$$

$$= (\sqrt{\lambda}\mathrm{Id}v_1, \lambda\mathrm{Id}z_1) * (\sqrt{\lambda}\mathrm{Id}v_2, \lambda\mathrm{Id}z_2)$$

$$= \varphi_\lambda(v_1, z_1) * \varphi_\lambda(v_2, z_2).$$

This proves that φ_λ is an authomorphism. □

Corollary. *Let H be a generalized Heisenberg Lie group. Then there exists a 1-parameter family of automorphisms φ_λ such that $\varphi_\lambda(v, z) \to 0$ as $\lambda \to 0$.*

Lemma 2. *Let H be a generalized Heisenberg Lie group. Then there exits a dense $\mathrm{Aut}(H)$-orbit.*

Proof. The set

$$\mathcal{O} =: \exp(L(H) - [L(H), L(H)]) = H - [H, H]$$

is an $\mathrm{Aut}(H)$-orbit of H. In fact, the exponential map is a global diffeomorphism for simply connected nilpotent Lie groups. Moreover, taken any two elements

$X, Y \in \mathcal{O}$ such that the line segment $\mid XY \mid$ parallel to $[H,H]$, they can be connected via a line segment by taking once X as a initial point so that the function of that connection $f_p : \mathcal{O} \to \mathcal{O}$ defined by $X \to t_1 X + t_2 = Y$, where $t_1, t_2 \in \mathrm{IR}$, is an automorphism. Actually, it is possible to connect these segments with the perpendicular segments to each other via the same way. That $\mathrm{Aut}(H)$-orbit of H is $\mathcal{O}$ is open. In fact, if $\dim Z = 1$ the center $[H,H]$ forms a line. Indeed, for any Heisenberg group $[X,Y] = Z$, $X, Y, Z \in L(H)$. For the density, any $x \in [H,H]$ every ball $B(x,\delta)$

$$B(x,\delta) \cap H - [H,H] \neq \emptyset.$$

Thus, $\overline{H - [H,H]} = H$. □

Theorem. *An affine control system $\Sigma = (H, \mathcal{D})$ on the generalized Heisenberg Lie group H is controllable if Σ does not have equilibrium point and the induced bilinear control system $\Sigma_b = (H, \mathcal{D}_b)$, where*

$$\mathcal{D}_b = \{D + \sum_{j=1}^{d} u_j D^j \mid D, D^j \in \mathrm{Der}(L(H)); u \in \mathrm{IR}^d\},$$

is controllable in the $\mathrm{Aut}(H)$*-orbit of* H*.*

Proof. For the affine system, to not have equalibrium point is a necessary condition for controllability, since the set of reachable points from a fixed point consists of a single point. Define the automorphism $\xi_\lambda : \mathrm{af}(H) \to \mathrm{af}(H)$ such that $\xi_\lambda(D + X) = D + \varphi_\lambda(X)$, where $\varphi_\lambda = (\sqrt{\lambda}\mathrm{Id}, \lambda \mathrm{Id})$ and so $\varphi_\lambda(X) \to 0$ as $\lambda \to 0$, for all $X \in L(H)$. Then, $\xi_\lambda(\mathcal{D}) \to \mathcal{D}_b$ as $\lambda \to 0$. Therefore, $\xi_\lambda(\Sigma) \to \Sigma_b$ as $\lambda \to 0$. By the hypothesis, Σ_b is controllable on the $\mathrm{Aut}(H)$-orbit of H and by Lemma 2, the $\mathrm{Aut}(H)$-orbit of H is dense. Hence, Σ_b is controllable on every orbit.

Consider the unit sphere $S(1,1)$ centered at 1 which is the boundary of the unit ball $B(1,1)$ centered at 1. For λ sufficiently small, $\xi_\lambda(\Sigma)$ is controllable on $S(1,1) - [H,H]$ since complete controllability is preserved under small perturbations, [4]. Then $\xi_\lambda(\Sigma)$ is controllable on $B(1,1) - [H,H]$. Indeed, finite systems normally controllable on $S(1,1)$ are open, [4]. Therefore, Σ is controllable on $B(1_{\varphi^{-1}}, 1) - [H,H]$, where $1_{\varphi^{-1}} = (1, (\frac{\mathrm{Id}}{\sqrt{\lambda}}e, \frac{\mathrm{Id}}{\lambda}e))$. Then, the positive orbit of affine system through $(1, (\frac{\mathrm{Id}}{\sqrt{\lambda}}e, \frac{\mathrm{Id}}{\lambda}e))$ is open and interior is nonempty since it contains $B(1_{\varphi^{-1}}, 1) - [H,H]$.

Then, normally accessible from $1_{\varphi^{-1}}$. Since the state space is connected, Σ is controllable on H. $\square$

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