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HYERS-ULAM-RASSIAS STABILITY OF A LINEAR DIFFERENTIAL EQUATION

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Abstract: In this paper, we generalize a theorem of Alsina and Ger [1] by proving the Hyers-Ulam-Rassias stability of linear differential equations, $y'(t) = \lambda y(t)$.

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1. Introduction

In 1940, S.M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [12]). Among those was the question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

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If the answer is affirmative, the functional equation for homomorphisms is said to have the Hyers-Ulam stability because the first result concerning the stability of functional equations was presented by D.H. Hyers. Indeed, he has answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces (see [5]).

In 1978, Th.M. Rassias [10] addressed the Hyers's Stability Theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference $h(xy) - h(x)h(y)$ and proved a considerably generalized result of Hyers (see also [3]). Since then, the stability problems of various functional equations have been investigated by many authors (see [2, 4, 6, 7, 8, 9, 11]).

By regarding the large influence of D.H. Hyers, S.M. Ulam, and Th.M. Rassias on the study of stability problems of functional equations, the stability phenomenon that was proved by Th.M. Rassias is called the Hyers-Ulam-Rassias stability.

Let $I = (a, b)$ be an open real interval, where we assume that a and b satisfy $-\infty \le a < b \le +\infty$.

C. Alsina and R. Ger were the first authors who investigated the Hyers-Ulam stability of a differential equation (see [1]). In fact, they proved that if a differentiable function $y : I \to \mathbb{R}$ satisfies $|y'(t) - y(t)| \le \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to \mathbb{R}$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \le 3\varepsilon$ for every $t \in I$.

The aim of this paper is to study the Hyers-Ulam-Rassias stability of the following linear differential equation

$$y'(t) = \lambda y(t). \tag{1}$$

More precisely, we prove that if λ is a non-zero real number and $I = (a, b)$ is an arbitrary open interval, and further if a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the inequality

$$|y'(t) - \lambda y(t)| \le \sum_{k=0}^{n} a_k t^k$$

for all $t \in I$, then there exist real numbers c and α_k, $k \in \{0, 1, \ldots, n\}$, such that

$$\left|y(t) - ce^{\lambda t}\right| \le \begin{cases} \left|\displaystyle\sum_{k=0}^{n} \alpha_k t^k - \lim_{s \to b^-} \sum_{k=0}^{n} \alpha_k s^k e^{\lambda(t-s)}\right| & (\text{for } \lambda > 0), \\ \left|\displaystyle\sum_{k=0}^{n} \alpha_k t^k - \lim_{s \to a^+} \sum_{k=0}^{n} \alpha_k s^k e^{\lambda(t-s)}\right| & (\text{for } \lambda < 0), \end{cases}$$

for any $t \in I$, where we refer to (2) and (3) for the a_k's and α_k's.

2. Preliminaries

Following an idea of C. Alsina and R. Ger [1], we can easily prove the following lemma. So, we omit the proof.

Lemma 1. *Let I be an arbitrary non-degenerate open interval. Assume that $z : I \to \mathbb{R}$ is a continuously differentiable function and λ is a real number.*

(a) The inequality $\lambda z(t) \leq z'(t)$ holds true for all $t \in I$ if and only if there exists an increasing continuously differentiable function $i : I \to \mathbb{R}$ such that

$$z(t) = i(t)e^{\lambda t}$$

for all $t \in I$;

(b) The inequality $\lambda z(t) \geq z'(t)$ holds true for any $t \in I$ if and only if there exists a decreasing continuously differentiable function $d : I \to \mathbb{R}$ such that

$$z(t) = d(t)e^{\lambda t}$$

for all $t \in I$.

Before going through the main theorems, we prove a lemma concerning an equality of finite double series that helps to simplify the proof of Theorem 4.

Lemma 2. *For any non-zero real number λ and for any real sequence $\{a_i\}_{i=0,1,2,\ldots}$, the equality*

$$\sum_{k=0}^{m}\sum_{i=k}^{m} \frac{i!a_i}{k!\lambda^{i+1-k}} t^k = \sum_{k=0}^{m} a_k \sum_{i=0}^{k} \frac{k!t^{k-i}}{(k-i)!\lambda^{i+1}} \quad (t \in \mathbb{R})$$

holds for any non-negative integer m.

Proof. We use mathematical induction on m to prove the assertion. It is trivial to show the validity of the assertion for $m = 0$. Now, assume that the assertion is true for some $m \geq 0$.

Consider the case for $m + 1$ as follows:

$$\begin{aligned}\sum_{k=0}^{m+1}\sum_{i=k}^{m+1} \frac{i!a_i}{k!\lambda^{i+1-k}} t^k &= \sum_{k=0}^{m}\sum_{i=k}^{m} \frac{i!a_i}{k!\lambda^{i+1-k}} t^k + \sum_{k=0}^{m+1} \frac{(m+1)!a_{m+1}}{k!\lambda^{m+2-k}} t^k \\ &= \sum_{k=0}^{m} a_k \sum_{i=0}^{k} \frac{k!t^{k-i}}{(k-i)!\lambda^{i+1}} + a_{m+1} \sum_{i=0}^{m+1} \frac{(m+1)!t^{m+1-i}}{(m+1-i)!\lambda^{i+1}}\end{aligned}$$

$$= \sum_{k=0}^{m+1} a_k \sum_{i=0}^{k} \frac{k! t^{k-i}}{(k-i)! \lambda^{i+1}},$$

where we obtain the last term in the second equality by reversing the order of summation of the last term in the first equality. We proved the validity of our assertion for $m+1$ and the proof is complete. □

Let $I = (a, b)$ be an open real interval, where a and b satisfy $-\infty \le a < b \le \infty$. Assume that real numbers $a_0, a_1, \ldots, a_n$ are given with the property:

$$\sum_{k=0}^{n} a_k t^k \ge 0 \quad (t \in I), \tag{2}$$

where n is a fixed non-negative integer. Set

$$\alpha_k := \sum_{i=k}^{n} \frac{i! a_i}{k! \lambda^{i+1-k}} \tag{3}$$

for $k = 0, 1, \ldots, n$ and for some non-zero real number λ. This definition yields an equality

$$\sum_{k=0}^{n} \lambda \alpha_k t^k - \sum_{k=1}^{n} k \alpha_k t^{k-1} = \sum_{k=0}^{n} a_k t^k \quad (t \in \mathbb{R}) \tag{4}$$

which turns out to be useful in the proof of the following theorem.

Theorem 3. *Let I be an arbitrary non-degenerate open interval and let λ be a non-zero real number. A continuously differentiable function $y : I \to \mathbb{R}$ satisfies the following inequality*

$$|y'(t) - \lambda y(t)| \le \sum_{k=0}^{n} a_k t^k, \tag{5}$$

for all $t \in I$, if and only if there exists an increasing continuously differentiable function $i : I \to \mathbb{R}$ such that

$$y(t) = i(t) e^{\lambda t} + \sum_{k=0}^{n} \alpha_k t^k \tag{6}$$

and

$$0 \le i'(t) \le 2 \sum_{k=0}^{n} a_k t^k e^{-\lambda t} \tag{7}$$

for any $t \in I$.

Proof. First, assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the inequality (5). Equivalently, y satisfies

$$\lambda y(t) - \sum_{k=0}^{n} a_k t^k \le y'(t) \le \lambda y(t) + \sum_{k=0}^{n} a_k t^k \tag{8}$$

for each $t \in I$.

Define a function $z_1 : I \to \mathbb{R}$ to be

$$z_1(t) = y(t) + \sum_{k=0}^{n} \alpha_k t^k.$$

Then, z_1 is continuously differentiable on I. If we differentiate $z_1(t)$ with respect to t, then

$$z_1'(t) = y'(t) + \sum_{k=1}^{n} k\alpha_k t^{k-1}.$$

Subtract $\lambda z_1(t)$ from the above equality and make use of (4) and the second inequality in (8):

$$z_1'(t) - \lambda z_1(t) = y'(t) - \lambda y(t) - \sum_{k=0}^{n} a_k t^k \le 0.$$

Thus, $z_1'(t) \le \lambda z_1(t)$ holds for any $t \in I$.

According to Lemma 1 (b), there exists a decreasing continuously differentiable function $d : I \to \mathbb{R}$ such that

$$z_1(t) = y(t) + \sum_{k=0}^{n} \alpha_k t^k = d(t)e^{\lambda t} \tag{9}$$

for all $t \in I$.

Now, let $z_2 : I \to \mathbb{R}$ be a continuously differentiable function defined by

$$z_2(t) = y(t) - \sum_{k=0}^{n} \alpha_k t^k.$$

By a similar way to the first part, we conclude that $z_2'(t) \ge \lambda z_2(t)$ is true for all $t \in I$. Hence, Lemma 1 (a) implies that there exists an increasing continuously differentiable function $i : I \to \mathbb{R}$ such that

$$z_2(t) = y(t) - \sum_{k=0}^{n} \alpha_k t^k = i(t)e^{\lambda t} \tag{10}$$

for each $t \in I$. Hence, we have the equality (6).

It follows from (9) and (10) that

$$y(t) = d(t)e^{\lambda t} - \sum_{k=0}^{n} \alpha_k t^k = i(t)e^{\lambda t} + \sum_{k=0}^{n} \alpha_k t^k.$$

By differentiating the above equalities and by using (9) and (10) again, we have

$$\begin{aligned} d'(t)e^{\lambda t} + \lambda y(t) + \sum_{k=0}^{n} \lambda \alpha_k t^k - \sum_{k=1}^{n} k\alpha_k t^{k-1} \\ = i'(t)e^{\lambda t} + \lambda y(t) - \sum_{k=0}^{n} \lambda \alpha_k t^k + \sum_{k=1}^{n} k\alpha_k t^{k-1} \end{aligned}$$

and further

$$d'(t) = i'(t) + 2\left(\sum_{k=1}^{n} k\alpha_k t^{k-1} - \lambda \sum_{k=0}^{n} \alpha_k t^k \right) e^{-\lambda t} \le 0,$$

since d is decreasing. Hence, considering the equality (4), we get the inequalities of (7) from the last inequality.

Conversely, assume that a continuously differentiable function $y : I \to \mathbb{R}$ is given by the representation in (6), where $i : I \to \mathbb{R}$ is a continuously differentiable function satisfying (7). Differentiate $y(t)$ and then subtract $\lambda y(t)$ from $y'(t)$ to get

$$y'(t) - \lambda y(t) = i'(t)e^{\lambda t} + \sum_{k=1}^{n} k\alpha_k t^{k-1} - \lambda \sum_{k=0}^{n} \alpha_k t^k$$

for all $t \in I$. Finally, by using (4) and (7), we obtain

$$-\sum_{k=0}^{n} a_k t^k \le y'(t) - \lambda y(t) \le \sum_{k=0}^{n} a_k t^k$$

for any $t \in I$. The last inequalities are equivalent to (5). □

3. Hyers-Ulam-Rassias Stability of Differential Equation (1)

In the following theorem, we prove the Hyers-Ulam-Rassias stability of the differential equation (1) which obviously improves a result of Alsina and Ger (see Remark of [1]).

In the proof of the following theorem, we make use of the formula concerning integration

$$\int \tau^k e^{-\lambda\tau} d\tau = -\sum_{i=0}^{k} \frac{k!\tau^{k-i}}{(k-i)!\lambda^{i+1}} e^{-\lambda\tau}, \tag{11}$$

for $k = 0, 1, 2, \ldots$.

Theorem 4. *Let λ be a non-zero real constant. Assume that $I = (a, b)$ is an arbitrary open interval with $-\infty \le a < b \le \infty$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the inequality* (5) *for all $t \in I$, then there exists a real number c such that*

$$\left| y(t) - ce^{\lambda t} \right| \le \begin{cases} \left| \sum_{k=0}^{n} \alpha_k t^k - \lim_{s \to b^-} \sum_{k=0}^{n} \alpha_k s^k e^{\lambda(t-s)} \right| & (\text{for } \lambda > 0), \\ \left| \sum_{k=0}^{n} \alpha_k t^k - \lim_{s \to a^+} \sum_{k=0}^{n} \alpha_k s^k e^{\lambda(t-s)} \right| & (\text{for } \lambda < 0), \end{cases} \tag{12}$$

for all $t \in I$.

Proof. First, we will prove our theorem for $\lambda > 0$. Define

$$c := \lim_{s \to b^-} \left\{ i(s) + \sum_{k=0}^{n} \alpha_k s^k e^{-\lambda s} \right\},$$

where $i : I \to \mathbb{R}$ is an increasing continuously differentiable function which is given in Theorem 3 (see the inequalities just below for the existence of c).

Due to the fundamental theorem of calculus, if we integrate the inequalities in (7) from t to b and take the integration formula (11) into account, then

$$\begin{aligned} 0 \le \lim_{s \to b^-} i(s) - i(t) &= \int_t^b i'(\tau) d\tau \le 2 \sum_{k=0}^{n} a_k \int_t^b \tau^k e^{-\lambda\tau} d\tau \\ &= -2 \lim_{s \to b^-} \sum_{k=0}^{n} a_k \sum_{i=0}^{k} \frac{k! s^{k-i}}{(k-i)!\lambda^{i+1}} e^{-\lambda s} + 2 \sum_{k=0}^{n} a_k \sum_{i=0}^{k} \frac{k! t^{k-i}}{(k-i)!\lambda^{i+1}} e^{-\lambda t}. \end{aligned}$$

By Lemma 2 and (3), we further obtain

$$0 \le \lim_{s \to b^-} i(s) - i(t) \le -2 \lim_{s \to b^-} \sum_{k=0}^{n} \alpha_k s^k e^{-\lambda s} + 2 \sum_{k=0}^{n} \alpha_k t^k e^{-\lambda t}.$$

In view of the definition of c, the last inequalities are changed into

$$0 \leq c - \lim_{s \to b^-} \sum_{k=0}^{n} \alpha_k s^k e^{-\lambda s} - i(t) \leq -2 \lim_{s \to b^-} \sum_{k=0}^{n} \alpha_k s^k e^{-\lambda s} + 2 \sum_{k=0}^{n} \alpha_k t^k e^{-\lambda t}.$$

Multiply the last inequalities by $e^{\lambda t}$ and add $\lim\limits_{s \to b^-} \sum\limits_{k=0}^{n} \alpha_k s^k e^{\lambda(t-s)} - \sum\limits_{k=0}^{n} \alpha_k t^k$ to the resulting inequalities. Then, we take (6) into account to get the inequality (12).

Now, let λ be a negative real constant and define

$$c := \lim_{s \to a^+} \left\{ i(s) + \sum_{k=0}^{n} \alpha_k s^k e^{-\lambda s} \right\}$$

(see Theorem 3 for the increasing continuously differentiable function $i : I \to \mathbb{R}$). We integrate the inequalities in (7) from a to t and then we can prove our theorem for $\lambda < 0$ by a similar way as before. $\square$

Corollary 5. *Let λ be a non-zero real constant and define*

$$I = \begin{cases} (a, \infty) & (\text{for } \lambda > 0), \\ (-\infty, b) & (\text{for } \lambda < 0), \end{cases}$$

where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ are fixed. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the inequality (5) *for all $t \in I$, then there exists a unique real number c such that*

$$\left| y(t) - ce^{\lambda t} \right| \leq \left| \sum_{k=0}^{n} \alpha_k t^k \right| \tag{13}$$

for all $t \in I$.

Proof. According to Theorem 4, there exists a real number c such that the inequality (13) holds for any $t \in I$. Hence, we only need to prove the uniqueness of c.

Suppose c_1 is another real number with which the inequality (13) is satisfied. In view of the triangle inequality and (13), we get

$$\begin{aligned} |c - c_1| &\leq e^{-\lambda t} |y(t) - c_1 e^{\lambda t}| + e^{-\lambda t} |ce^{\lambda t} - y(t)| \\ &\leq 2e^{-\lambda t} \left| \sum_{k=0}^{n} \alpha_k t^k \right| \end{aligned}$$

$$\to \quad 0 \quad \text{as either } t \to \infty \text{ for } \lambda > 0 \text{ or } t \to -\infty \text{ for } \lambda < 0,$$

which completes our proof. □

Theorem 4 and Corollary 5 state that each solution of the inequality (5) can be approximated by a solution of the differential equation (1) within a distance depending on t. Unfortunately, there is no efficient way to find out the constant c occurring in (12) or in (13), even though we can get some information on the behavior of $y(t)$ from Theorem 3.

However, if an initial condition, namely $y(a)$, is known, then the following corollary may help to evaluate the lower and upper bounds for $y(t)$.

Theorem 6. *Let a be a real constant and let $I = (a, b)$ be an arbitrary non-degenerate open interval. Assume that a function $y : I \cup \{a\} \to \mathbb{R}$ is continuously differentiable on I and continuous at a on the right. If y satisfies the inequality (5) for any $t \in I$, then*

$$\left(y(a) - \sum_{k=0}^{n} \alpha_k a^k \right) e^{\lambda(t-a)} + \sum_{k=0}^{n} \alpha_k t^k \le y(t)$$
$$\le \left(y(a) + \sum_{k=0}^{n} \alpha_k a^k \right) e^{\lambda(t-a)} - \sum_{k=0}^{n} \alpha_k t^k \,,$$

for all $t \in I$.

Proof. Define $c := \lim_{s \to a^+} i(s)$, where $i : I \to \mathbb{R}$ is given in Theorem 3. Let us integrate the inequalities in (7) from a to t by using the formula (11). Then,

$$0 \le i(t) - c \le 2\sum_{k=0}^{n} a_k \int_a^t \tau^k e^{-\lambda\tau} d\tau = -2\sum_{k=0}^{n} \alpha_k t^k e^{-\lambda t} + 2\sum_{k=0}^{n} \alpha_k a^k e^{-\lambda a},$$

where the last equality follows from Lemma 2 and (3).

Multiply the inequalities by $e^{\lambda t}$ and add $\sum_{k=0}^{n} \alpha_k t^k$ to the resulting inequalities and then take the equality in (6) into account. Then, we get

$$\sum_{k=0}^{n} \alpha_k t^k \le y(t) - ce^{\lambda t} \le -\sum_{k=0}^{n} \alpha_k t^k + 2\sum_{k=0}^{n} \alpha_k a^k e^{\lambda(t-a)} \,,$$

or

$$ce^{\lambda t} + \sum_{k=0}^{n} \alpha_k t^k \le y(t) \le \left(ce^{\lambda a} + 2\sum_{k=0}^{n} \alpha_k a^k \right) e^{\lambda(t-a)} - \sum_{k=0}^{n} \alpha_k t^k.$$

Furthermore, if we let $t \to a^+$ in (6), then we have

$$y(a) = ce^{\lambda a} + \sum_{k=0}^{n} \alpha_k a^k.$$

The above inequalities together with this equality yield the validity of our assertion. □

4. Some Examples

In this section, let λ be a non-zero real number and define

$$I = \begin{cases} (a, \infty) & (\text{for } \lambda > 0), \\ (-\infty, b) & (\text{for } \lambda < 0), \end{cases}$$

where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ are arbitrarily given.

If we let $a_k = 0$ for $k = 1, 2, \ldots, n$, then Corollary 5 deals with the Hyers-Ulam stability of the differential equation (1).

Example 1. Assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the inequality

$$|y'(t) - \lambda y(t)| \le \varepsilon$$

for all $t \in I$. Let $a_0 = \varepsilon$ and $a_k = 0$ for $k = 1, 2, \ldots, n$. Then, $\alpha_0 = \varepsilon\lambda^{-1}$ and $\alpha_k = 0$ for $k = 1, 2, \ldots, n$. According to Corollary 5, there exists a unique real number c such that $|y(t) - ce^{\lambda t}| \le \varepsilon|\lambda|^{-1}$ for any $t \in I$.

Example 2. Let a_0 and a_1 be given with $a_0 + a_1 t \ge 0$ for any $t \in I$. Then, $\alpha_0 = a_0\lambda^{-1} + a_1\lambda^{-2}$ and $\alpha_1 = a_1\lambda^{-1}$. Assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies the inequality

$$|y'(t) - \lambda y(t)| \le a_0 + a_1 t$$

for all $t \in I$. Due to Corollary 5, there exists a unique real number c such that $|y(t) - ce^{\lambda t}| \le |a_0\lambda^{-1} + a_1\lambda^{-2} + a_1\lambda^{-1}t|$ for all $t \in I$.

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