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SUBMANIFOLDS WITH NONZERO MEAN CURVATURE IN A EUCLIDEAN SPHERE

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Abstract: The purpose of this paper is to prove the following: Let M^n be a complete, connected and orientable submanifold with nonzero constant mean curvature H in $S^{n+2}(c)$ and S the squared norm of the second fundamental form. We put $|\phi|^2 = S - nH^2$ and B_H the square of the positive root of the equation $x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(H^2+c) = 0$ with respect to x. If $|\phi|^2$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then M^n lies in a totally geodesic hypersurface $S^{n+1}(c)$ of $S^{n+2}(c)$ and (1) either $|\phi|^2 \equiv 0$ and M^n is totally umbilic or $|\phi|^2 \equiv B_H$. (2) $|\phi|^2 \equiv B_H$ if and only if (B) $n \geq 3$ and $M^n = S^{n-1}(r_1) \times S^1(r_2) \subset S^{n+1}(c)$, where $r_1^2 + r_2^2 = \frac{1}{c}$ and $r_1^2 < \frac{n-1}{nc}$ or (C) $n = 2$ and $M^2 = S^1(r_1) \times S^1(r_2) \subset S^3(c)$, where $r_1^2 + r_2^2 = \frac{1}{c}$ and $r_1^2 \neq \frac{1}{2c}$.

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1. Introduction

Let $S^{n+p}(c)$ be an $(n+p)$-dimensional Euclidean sphere of constant sectional curvature c and M^n an n-dimensional, connected and orientable submanifold isometrically immersed in $S^{n+p}(c)$. We denote by h_{ij}^{α} the local component of the second fundamental form for each $i,\ j,\ \alpha$ $(1 \leq i, j \leq n,\ n+1 \leq \alpha \leq n+p)$. Let

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$$S := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2, \quad \text{and} \quad H := \frac{1}{n}\sqrt{\sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^{n} h_{ii}^{\alpha})^2}$$

be the squared norm of the second fundamental form and the mean curvature of M^n in $S^{n+p}(c)$, respectively. M^n is called *minimal* if the mean curvature H of M^n is equal to zero. In the case of $c = 1$, Chern, do Carmo and Kobayashi [3] proved that if M^n is minimal and $S \leq \frac{n}{2-\frac{1}{p}}$ for any point of M^n, then (1) either $S \equiv 0$ and M^n is totally geodesic or $S \equiv \frac{n}{2-\frac{1}{p}}$. (2) $S \equiv \frac{n}{2-\frac{1}{p}}$ if and only if $p = 1$ and M^n is a Clifford minimal hypersurface $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ in $S^{n+1}(1)$ or $n = p = 2$ and M^2 is a Veronese surface in $S^4(1)$.

Now, we denote by A_α the $n \times n$ matrix of h_{ij}^{α} respect to index $i,\ j$. Define linear maps $\phi_\alpha : T_xM \to T_xM$ by

$$\langle \phi_\alpha X, Y\rangle := H\langle X, Y\rangle - \langle A_\alpha X, Y\rangle \quad \text{for} \quad n+1 \leq \alpha \leq n+p,$$

where $\langle\ ,\ \rangle$ is the Riemannian metric of M^n. Moreover, we define the bilinear map $\phi : T_xM \times T_xM \to T_xM^{\perp}$ by

$$\phi(X, Y) = \sum_{\alpha=n+1}^{n+p} \langle \phi_\alpha X, Y\rangle e_\alpha,$$

where $\{e_{n+1}, \ldots, e_{n+p}\}$ denotes an orthonormal normal basis. It is easy to check that trace $\phi = 0$ and that

$$|\phi|^2 := \sum_{\alpha=n+1}^{n+p} \text{trace } \phi_\alpha^2 = S - nH^2.$$

Let

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + c)$$

be the polynomial for each real number $H \in \boldsymbol{R}$ and B_H the square of the positive root of $P_H(x) = 0$. Subjecting the condition that the mean curvature H of hypersurface M^n in $S^{n+1}(c)$ is constant, Alencar and do Carmo [1] generalized the result of Chern, do Carmo and Kobayashi [3]. The detail of their results is the following: Let M^n be a compact orientable hypersurface of $S^{n+1}(c)$. Assume that the mean curvature H is constant and that $|\phi|^2 \leq B_H$ for all points of M^n. Then (1) either $|\phi|^2 \equiv 0$ and M^n is totally umbilc or $|\phi|^2 \equiv B_H$. (2) $|\phi|^2 \equiv B_H$ if and only if: (a) M^n is a Clifford minimal hypersurface in $S^{n+1}(c)$. (b) $H \neq 0,\ n \geq 3$ and $M^n = S^{n-1}(r_1) \times S^1(r_2) \subset S^{n+1}(c)$, where $r_1^2 + r_2^2 = \frac{1}{c}$ and $r_1^2 < \frac{n-1}{nc}$. (c) $H \neq 0,\ n = 2$ and $M^2 = S^1(r_1) \times S^1(r_2) \subset S^3(c)$, where $r_1^2 + r_2^2 = \frac{1}{c}$ and $r_1^2 \neq \frac{1}{2c}$.

When M^n is an n-dimensional, complete, connected and orientable submanifold in an $(n+p)$-dimensional Euclidian space $\boldsymbol{E}^{n+p}$ Cheng [2] proved the following: If the mean curvature H is bounded nonzero and $S \leq \frac{n^2H^2}{n-1}$, then M^n lies in a totally geodesic submanifold $\boldsymbol{E}^{n+1}$ of $\boldsymbol{E}^{n+p}$ and M^n is isometric to the totally umbilical sphere $S^n(r)$, the totally geodesic Euclidian space $\boldsymbol{E}^n$ or the generalized cylinder $S^{n-1}(c) \times \boldsymbol{E}^1$.

The purpose of this paper is to study a generalization of the result of Alencar and do Carmo [1] by the similar way with Cheng [2]. We prove the following theorem.

Theorem. *Let M^n be a complete,connected and orientable submanifold with nonzero constant mean curvature H in $S^{n+2}(c)$. If $|\phi|$ satisfies $|\phi|^2 \leq B_H$ for all $x \in M^n$, then M^n lies in a totally geodesic hypersurface $S^{n+1}(c)$ of $S^{n+2}(c)$ and:*

(1) either $|\phi|^2 \equiv 0$ and M^n is totally umbilic, or $|\phi|^2 \equiv B_H$.

(2) $|\phi|^2 \equiv B_H$ if and only if: (B) $n \geq 3$ and $M^n = S^{n-1}(r_1) \times S^1(r_2) \subset S^{n+1}(c)$, where $r_1^2 + r_2^2 = \frac{1}{c}$ and $r_1^2 < \frac{n-1}{nc}$. (C) $n = 2$ and $M^2 = S^1(r_1) \times S^1(r_2) \subset S^3(c)$, where $r_1^2 + r_2^2 = \frac{1}{c}$ and $r_1^2 \neq \frac{1}{2c}$.

2. Preliminaries

Let $S^{n+p}(c)$ be an $(n+p)$-dimensional Euclidean sphere of constant curvature c and M^n an n-dimensional, complete, connected and orientable submanifold in $S^{n+p}(c)$. We choose a local field of orthonormal frames $\{e_1, \ldots, e_{n+p}\}$ adapted to the Riemannian metric of S^{n+p} and the dual coframes $\{\omega_1, \ldots, \omega_{n+p}\}$ in such a way that, restricted to the submanifold M^n, $\{e_1, \ldots, e_n\}$ are tangent to M^n. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \cdots \leq n+p; \quad 1 \leq i, j, k, \cdots \leq n;$$

$$n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p.$$

Then the structure equations of $S^{n+p}(c)$ are given by

$$d\omega_A = -\sum_{B=1}^{n+p} \omega_{AB} \wedge \omega_B, \omega_{AB} + \omega_{BA} = 0, \tag{1}$$

$$d\omega_{AB} = -\sum_{C=1}^{n+p} \omega_{AC} \wedge \omega_{CB} + \frac{1}{2} \sum_{C,D=1}^{n+p} K_{ABCD}\omega_C \wedge \omega_D, \tag{2}$$

with $K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$, where K_{ABCD} denotes the component of the curvature tensor of $S^{n+p}(c)$.

We restrict these forms to M^n. Then we get

$$\omega_\alpha = 0. \tag{3}$$

Since $0 = d\omega_\alpha = -\sum_{i=1}^n \omega_{\alpha i} \wedge \omega_i$, by Cartan's Lemma we may write

$$\omega_{\alpha i} = \sum_{j=1}^n h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{4}$$

From these formulas, we obtain

$$d\omega_i = -\sum_{j=1}^n \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{5}$$

$$d\omega_{ij} = -\sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l=1}^n R_{ijkl}\omega_k \wedge \omega_l, \tag{6}$$

and

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha=n+1}^{n+p} (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{7}$$

where R_{ijkl} is the component of the curvature tensor of M^n. Denoting by R_{jk} the component of Ricci curvature of M^n, we have

$$R_{jk} = (n-1)c\delta_{jk} + \sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^n h_{ii}^\alpha\ h_{jk}^\alpha - \sum_{i=1}^n h_{ik}^\alpha h_{ji}^\alpha) \tag{8}$$

from (7). We also have

$$d\omega_{\alpha\beta} = -\sum_{\gamma=1}^n \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2}\sum_{i,j=1}^n R_{\alpha\beta ij}\omega_k \wedge \omega_l, \tag{9}$$

$$R_{\alpha\beta ij} = \sum_{l=1}^n (h_{il}^\alpha h_{lj}^\beta - h_{jl}^\alpha h_{li}^\beta). \tag{10}$$

The Riemannian connection of M^n is defined by (ω_{ij}). The form $(\omega_{\alpha\beta})$ defines a connection in the normal bundle of M^n. The second fundamental form $\prod$ and the mean curvature vector h of M^n are defined by

$$\prod := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^n h_{ij}^\alpha \omega_i \omega_j e_\alpha \tag{11}$$

and

$$h := \frac{1}{n} \sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^{n} h_{ii}^{\alpha}) e_{\alpha}, \tag{12}$$

respectively.

On the other hand, the mean curvature H of M^n is defined by

$$H := \frac{1}{n} \sqrt{\sum_{\alpha=n+1}^{n+p} (\sum_{i=1}^{n} h_{ii}^{\alpha})^2} \tag{13}$$

(see Introduction).

We take the exterior differentiation of (4) and define h_{ijk}^{α} by

$$\sum_{k=1}^{n} h_{ijk}^{\alpha} \omega_k = dh_{ij} - \sum_{k=1}^{n} h_{ik}^{\alpha} \omega_{kj} - \sum_{k=1}^{n} h_{jk}^{\alpha} \omega_{ki} - \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta} \omega_{\beta\alpha}. \tag{14}$$

Then we obtain the Codazzi equation by straightforward computations,

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha}. \tag{15}$$

Similarly, we take the exterior differentiation of (14) and define h_{ijkl}^{α} by

$$\sum_{l=1}^{n} h_{ijkl}^{\alpha} \omega_l = dh_{ijk}^{\alpha} - \sum_{l=1}^{n} h_{ljk}^{\alpha} \omega_{li} - \sum_{l=1}^{n} h_{ilk}^{\alpha} \omega_{lj} - \sum_{l=1}^{n} h_{ijl}^{\alpha} \omega_{lk} - \sum_{\beta=n+1}^{n+p} h_{ijk}^{\beta} \omega_{\beta\alpha}. \tag{16}$$

Then the Ricci formula for the second fundamental form is given by

$$h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_{m=1}^{n} h_{mj}^{\alpha} R_{mikl} + \sum_{m=1}^{n} h_{im}^{\alpha} R_{mjkl} + \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta} R_{\beta\alpha kl}. \tag{17}$$

The Laplacian Δh_{ij} of h_{ij} is defined by

$$\Delta h_{ij}^{\alpha} := \sum_{k=1}^{n} h_{ijkk}^{\alpha}. \tag{18}$$

From the Codazzi equation (15) we obtain, for any $\alpha, n+1 \leq \alpha \leq n+p$,

$$\sum_{k=1}^{n} h_{ijkk}^{\alpha} = \sum_{k=1}^{n} h_{kijk}^{\alpha}.$$

Moreover, using the Ricci formula (17), we have

$$\Delta h_{ij}^{\alpha} = \sum_{m=1}^{n} h_{kijk}^{\alpha} = \sum_{m=1}^{n} h_{kikj}^{\alpha} + \sum_{k,m=n+1}^{n} h_{km}^{\alpha} R_{mijk}$$
$$+ \sum_{k,m=n+1}^{n} h_{mi}^{\alpha} R_{mkjk} + \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta} R_{\beta\alpha kl} = \sum_{m=1}^{n} h_{kkij}^{\alpha} + \sum_{k,m=n+1}^{n} h_{km}^{\alpha} R_{mijk}$$
$$+ \sum_{k,m=n+1}^{n} h_{mi}^{\alpha} R_{mkjk} + \sum_{\beta=n+1}^{n+p} h_{ij}^{\beta} R_{\beta\alpha kl}. \quad (19)$$

Let

$$S := \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2$$

denote the squared norm of the second fundamental form of M^n. Assuming the mean curvature vector $h \neq 0$ on M^n, we know that $e_{n+1} = h/H$ is a normal vector field defined globally on M^n. We define S_1 and S_2 by

$$S_1 := \sum_{i,j=1}^{n} (h_{ij}^{n+1} - H\delta_{ij})^2, \quad S_2 := \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2, \quad (20)$$

respectively. Then S_1 and S_2 are functions defined on globally and they are independent of the choice of the orthonormal frame $\{e_1, \ldots, e_n\}$. Also we have

$$|\phi|^2 = S - nH^2 = S_1 + S_2. \quad (21)$$

From the definition of the mean curvature vector h we know that

$$nH = \sum_{i=1}^{n} h_{ii}^{n+1} \ , \ \sum_{i=1}^{n} h_{ii}^{\alpha} = 0 \quad (22)$$

for $n+2 \leq \alpha \leq n+p$.

We establish the following lemmas for the proof of Theorem.

Lemma 1. (see Cheng [2]) *Let M^n be an n-dimensional submanifold with the mean curvature vector $h \neq 0$ in $S^{n+p}(c)$. Then we have*

$$\frac{1}{2}\Delta S_2 = \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h_{ijk}^{\alpha})^2 + nc \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2$$
$$+ nH \sum_{\alpha=n+2}^{n+p} \mathrm{trace}(A_{n+1}A_{\alpha}^2) - \sum_{\alpha=n+2}^{n+p} [\mathrm{trace}(A_{n+1}A_{\alpha})]^2$$
$$+ \sum_{\alpha,\beta=n+2}^{n+p} \mathrm{trace}(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha})^2 - \sum_{\alpha,\beta=n+2}^{n+p} [\mathrm{trace}\ (A_{\alpha}A_{\beta})]^2$$

$$+\sum_{\alpha=n+2}^{n+p}\text{trace}(A_{n+1}A_\alpha)^2-\sum_{\alpha=n+2}^{n+p}\text{trace}(A_{n+1}^2A_\alpha^2).$$

Lemma 2. (see Cheng [2]) *Let $a_1,\dots,a_n$ and b_{ij}, for $i,j=1,\dots,n$, be real numbers satisfying $\sum_{i=1}^n a_i=0$, $\sum_{i,j=1}^n b_{ij}^2=b$ and $b_{ij}=b_{ji}$ for $i,j=1,\dots,n$. Then we have*

$$-(\sum_{i=1}^n b_{ij}a_i)^2+\sum_{i,j=1}^n b_{ij}^2a_ia_j-\sum_{i,j=1}^n b_{ij}^2a_i^2\geq-\sum_{i=1}^n a_i^2b.$$

Lemma 3. (see Cheng [2]) *Let b_i, for $i=1,\dots,n$, be real numbers such that $\sum_{i=1}^n b_i=0$ and $\sum_{i=1}^n b_i^2=B$. Then we obtain:*

$$\sum_{i=1}^n b_i^4-\frac{B^2}{n}\leq\frac{(n-2)^2}{n(n-1)}B^2.$$

Lemma 4. (see Cheng [2]) *Let a_i and b_i, for $i=1,\dots,n$, be real numbers satisfying $\sum_{i=1}^n a_i=0$ and $\sum_{i=1}^n {a_i}^2=a$. Then we have*

$$\sum_{i=1}^n a_ib_i^2\geq-\sqrt{\sum_{i=1}^n {b_i}^4-\frac{1}{n}(\sum_{i=1}^n b_i^2)^2}\,\sqrt{a}.$$

The following generalized maximum principle due to Omori [5] and Yau [8] will be used in order to prove our theorem.

Generalized Maximum Principle. (see Omori [5] and Yau [8]) *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f\in C^2(M)$ a function bounded from above on M^n. Then, for any $\epsilon>0$, there exists a point $p\in M^n$ such that*

$$f(p)\geq\sup f-\epsilon,\quad \|\text{grad } f\|<\epsilon,\quad \Delta f(p)<\epsilon.$$

3. Proof of Theorem

We first compute the Laplacian ΔS_2 and show that $\Delta S_2\geq 0$. In case of $p=2$, according to Lemma 1, we obtain

$$\frac{1}{2}\Delta S_2=\sum_{i,j,k=1}^n(h_{ijk}^{n+2})^2+nc\sum_{i,j=1}^n(h_{ij}^{n+2})^2+nH\text{trace}(A_{n+1}A_{n+2}^2)$$

$$
\begin{aligned}
&-[\mathrm{trace}(A_{n+1}A_{n+2})]^2+\mathrm{trace}(A_{n+2}A_{n+2}-A_{n+2}A_{n+2})^2\\
&-[\mathrm{trace}\ (A_{n+2}^2)]^2+\mathrm{trace}(A_{n+1}A_{n+2})^2-\ \mathrm{trace}\ (A_{n+1}^2A_{n+2}^2)\\
&=\sum_{i,j,k=1}^{n}(h_{ijk}^{n+2})^2+nc\sum_{i,j=1}^{n}(h_{ij}^{n+2})^2+nH\mathrm{trace}\ (A_{n+1}A_{n+2}^2)\\
&-[\mathrm{trace}\ (A_{n+1}A_{n+2})]^2-S_2^2+\mathrm{trace}\ (A_{n+1}A_{n+2})^2-\mathrm{trace}\ (A_{n+1}^2A_{n+2}^2).
\end{aligned}
$$

Since $e_{n+1}=h/H$, we have trace $A_{n+2}=0$ and trace $A_{n+1}=nH$, we get

$$
\begin{aligned}
&-[\mathrm{trace}\ (A_{n+1}A_{n+2})]^2+\mathrm{trace}\ (A_{n+1}A_{n+2})^2-\mathrm{trace}\ (A_{n+1}^2A_{n+2}^2)\\
&=-[\mathrm{trace}\ (A_{n+1}-HI)A_{n+2}]^2+H\mathrm{trace}\ A_{n+2}\\
&+\mathrm{trace}\ \{(A_{n+1}-HI)A_{n+2}\}^2+2Htr(A_{n+1}A_{n+2}^2)-H^2\mathrm{trace}\ A_{n+2}^2\\
&-\mathrm{trace}\ \{(A_{n+1}-HI)^2A_{n+2}^2\}-2H\mathrm{trace}\ (A_{n+1}A_{n+2}^2)\ +\ H^2\mathrm{trace}\ A_{n+2}^2\\
&=-[\mathrm{trace}\ \{(A_{n+1}-HI)A_{n+2}\}]^2\\
&+\ \mathrm{trace}\ \{(A_{n+1}-HI)A_{n+2}\}^2-\mathrm{trace}\ \{(A_{n+1}-HI)^2A_{n+2}^2\},
\end{aligned}
$$

where I denotes the identity matrix.

For A_{n+2} we can take a local orthonormal frame field $e_1,\ldots,e_n$ such that $h_{ij}^{n+2}=\lambda_i^{n+2}\delta_{ij}$. Thus we have $\sum_{i=1}^n\lambda_i^{n+2}=0$ and $\mathrm{trace}A_{n+2}^2=\sum_{i=1}^n(\lambda_i^{n+2})^2$. Let $B:=A_{n+1}-HI=(b_{ij})$. Then we have $b_{ij}=b_{ji}$, $\sum_{i=1}^n\ b_{ii}=0$ and $\sum_{i,j=1}^n\ b_{ij}^2=S_1$ for any $i,j=1,\ldots,n$. So we get

$$
\begin{aligned}
&-[\mathrm{trace}\{(A_{n+1}-HI)A_{n+2}\}]^2+\mathrm{trace}\{(A_{n+1}-HI)A_{n+2}\}^2\\
&-\mathrm{trace}\{(A_{n+1}-HI)^2A_{n+2}^2\}\\
&=-[\mathrm{trace}\ (BA_{n+2})]^2+\mathrm{trace}\ (BA_{n+2})^2-\mathrm{trace}\ (B^2A_{n+2}^2)\\
&=-(\sum_{i=1}^{n}b_{ii}\lambda_i^{n+2})^2+\sum_{i,j=1}^{n}b_{ij}\lambda_i^{n+2}\lambda_j^{n+2}-\sum_{i,j=1}^{n}b_{ij}^2(\lambda_i^{n+2})^2.
\end{aligned}
$$

Since λ_i^{n+2} and b_{ij} for $i,j=1,\ldots,n$ satisfy the conditions in Lemma 2, we obtain

$$
\begin{aligned}
&-[\mathrm{trace}\{(A_{n+1}-HI)A_{n+2}\}]^2+\mathrm{trace}\{(A_{n+1}-HI)A_{n+2}\}^2\\
&-\mathrm{trace}\{(A_{n+1}-HI)^2A_{n+2}^2\}\geq-S_1\mathrm{trace}\ A_{n+2}^2=-S_1S_2\,,
\end{aligned}
$$

and

$$
\begin{aligned}
&nH\mathrm{trace}\ (A_{n+1}A_{n+2}^2)=nH\mathrm{trace}\ \{(A_{n+1}-HI)A_{n+2}^2\}\\
&+nH^2\mathrm{trace}\ A_{n+2}^2=nH\mathrm{trace}\ \{(A_{n+1}-HI)A_{n+2}^2\}+nH^2S_2.
\end{aligned}
$$

By making use of the same assertion as above we have

$$\text{trace } \{(A_{n+1} - HI)A_{n+2}^2\} = \sum_{i=1}^{n} b_{ii}(\lambda_i^{n+2})^2.$$

From Lemma 3 and Lemma 4, we obtain

$$\text{trace } \{(A_{n+1} - HI)A_{n+2}^2\} \geq \frac{n-2}{\sqrt{n(n-1)}}\sqrt{S_1}\text{trace } A_{n+2}^2.$$

Hence we conclude

$$nH\text{trace } \{(A_{n+1} - HI)A_{n+2}^2\} \geq nH^2 S_2 - \frac{n-2}{\sqrt{n(n-1)}}\sqrt{S_1}S_2.$$

In order to above inequality, we have

$$\begin{aligned}
&\frac{1}{2}\Delta S_2 \\
&\geq \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + ncS_2 + S_2(nH^2 - \frac{n-2}{\sqrt{n(n-1)}}H\sqrt{S_1}) - S_1S_2 - S_2^2 \\
&= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2(-S_1 - S_2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_1} + n(H^2 + c)) \\
&= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2(-(S - nH^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S - nH^2 - S_2} \\
&\qquad + n(H^2 + c)) \\
&\geq \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2(-(S - nH^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S - nH^2} \\
&\qquad + n(H^2 + c)) \\
&= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2(-(|\phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| - n(H^2 + c))) \\
&= \sum_{i,j,k=1}^{n} (h_{ijk}^{n+2})^2 + S_2(-P_H(|\phi|)).
\end{aligned}$$

Under our assumption $|\phi|^2 \leq B_H$, i.e. $-P_H(|\phi|) \geq 0$ we conclude

$$\frac{1}{2}\Delta S_2 \geq 0.$$

Next, we show that Ricci curvature R_{ii} of M^n is bounded from below and S_2 is bounded from above, when $|\phi|^2 \leq B_H$. From (8) and (21) we obtain

$$
\begin{aligned}
R_{ii} = (n-1)c + \sum_{\alpha=n+1}^{n+2}\sum_{k=1}^{n}(h_{kk}^{\alpha}h_{ii}^{\alpha} - (h_{ik}^{\alpha})^2)\\
&= (n-1)c + \sum_{\alpha=n+1}^{n+2}(h_{ii}^{\alpha}\sum_{k=1}^{n}h_{kk}^{\alpha}) - \sum_{\alpha=n+1}^{n+2}\sum_{k=1}^{n}(h_{ik}^{\alpha})^2\\
&= (n-1)c + h_{ii}^{n+1}\sum_{k=1}^{n}h_{kk}^{n+1} - \sum_{\alpha=n+1}^{n+2}\sum_{k=1}^{n}(h_{ik}^{\alpha})^2\\
&= (n-1)c + nHh_{ii}^{n+1} - \sum_{\alpha=n+1}^{n+2}\sum_{k=1}^{n}(h_{ik}^{\alpha})^2\\
&\geq (n-1)c + nHh_{ii}^{n+1} - \sum_{\alpha=n+1}^{n+2}\sum_{i,k=1}^{n}(h_{ik}^{\alpha})^2\\
&= (n-1)c + nHh_{ii}^{n+1} - |\phi|^2 - nH^2\\
&\geq (n-1)c + nHh_{ii}^{n+1} - nH^2 - B_H
\end{aligned}
$$

and $S_2 = |\phi|^2 - S_1 \leq \ \ B_H - S_1 \leq \ \ B_H$, respectivelly. Since H is nonzero constant there exist real numbers c_1, c_2 such that

$$
R_{ii} \geq c_1 \quad S_2 \leq c_2.
$$

Since Ricci curvature of M^n and S_2 satisfy above condition, we can apply the generalized maximum principle due to Omori [5] and Yau [8] to the function S_2. Then we have that there exists a sequence $\{p_l\} \subset M^n$ such that

$$
\lim_{l\to\infty} S_2(p_l) = \sup S_2\,, \qquad \lim_{l\to\infty}\sup \Delta S_2(p_l) \leq 0.
$$

Hence, $\sup S_2 = 0$, that is, $S_2 = 0$ on M^n. Thus we have

$$
\sum_{i,j,k=1}^{n}(h_{ijk}^{n+2})^2 = 0
$$

on M^n. On the other hand, from (13), we have, for $n+2$,

$$
\begin{aligned}
\sum_{i,k=1}^{n} h_{iik}^{n+2}\omega_k &= \sum_{i=1}^{n} dh_{ii}^{n+2} - \sum_{i,k=1}^{n} h_{ik}^{n+2}\omega_{ki}\\
&- \sum_{i,k=1}^{n} h_{ik}^{n+2}\omega_{ki} - \sum_{\beta=n+1}^{n+2}\sum_{i=1}^{n} h_{ii}^{\beta}\omega_{\beta n+2}\\
&= -2\sum_{i,k=1}^{n} h_{ik}^{n+2}\omega_{ki} - \sum_{i=1}^{n} h_{ii}^{n+1}\omega_{(n+1)(n+2)} = -nH\omega_{(n+1)(n+2)}\,.
\end{aligned}
$$

From $H \neq 0$ and above equality, we infer $\omega_{(n+1)(n+2)} = 0$. Thus, we know that e_{n+1} is parallel in the normal bundle $T^{\perp}M$ of M^n. Hence, if we denote by N_1 the normal subbundle spanned by e_{n+2} of the normal bundle of M^n, then M^n is totally geodesic with respsct to N_1. Since the e_{n+1} is parallel in the normal bundle, we know that the normal subbundle N_1 is invariant under parallel trnslation with respect to the normal connection of M^n. Then from Theorem 1 [7], we conclude that M^n lies in a totally geodesic hypersurface S^{n+1} of S^{n+2}. Furthermore, since H is constant and $S_2 = 0$ we can result in the discussion of Alencar and do Carmo [1] (see Introduction). This finished our proof of theorem.

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