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NUMBER OF POINTS SUPPORTING THE ZERO-SET OF A SECTION OF VECTOR BUNDLES ON SURFACES AND ON $\mathbf{P}^3$

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Abstract: Here we construct vector bundles on smooth surfaces and on $\mathbf{P}^3$ with zero-locus supported by a prescribed number of points (even over a finite field).

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1. Introduction

In the first part of this paper we will give a partial extension of [1] to arbitrary smmoth projective surfaces. As in [1] to apply [3] we work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. We will prove the following result.

Theorem 1. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Let X be a smooth and connected projective surface, $L, R \in \mathrm{Pic}(X)$ and integers c_2, x such that $0 < x \leq c_2$. Assume $h^0(X, R^*) = 0$, $c_2 > h^0(X, L \otimes R \otimes_X) + h^1(X, R^*)$, and that a general $D \in |L \otimes R \otimes \omega_X|$ is integral. Then there exist a rank 2 vector bundle E on X such that $c_1(E) = L$, $c_2(E) = c_2$, and $s \in H^0(X, E)$, $s \neq 0$, such that the zero locus $(s)_0$ of s is finite and $\sharp(((s)_0)_{red}) = x$.*

Then in Section 2 and Section 3 we will consider a similar, but different problem. We fix a prime-power q and take $\mathbb{K} := \bar{\mathbb{F}}_q$. To prove a result similar

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to Theorem 1 for this field it would be sufficient to avoid (using some added assumptions) any use of [3] (which is in general false in positive characteristic). However, we consider a different problem. We fix a geometrically integral projective variety X defined over $\mathbb{F}_q$. We look at vector bundles E on X defined over $\mathbb{F}_q$. We only consider sections s of E defined over $\mathbb{F}_q$. Hence both the scheme $(s)_0$ and the set $((s)_0)_{red}$ are defined over $\mathbb{F}_q$. Since $X(\mathbb{F}_q)$ is finite, obviously $((s)_0)_{red} \cap X(\mathbb{F}_q)$ is finite, even if $((s)_0)_{red} \cap X(\bar{\mathbb{F}}_q)$ is infinite (even if $s = 0$). Certainly, we will prescribe $s \in H^0(X, E)\backslash\{0\}$. But then two different questions may be asked:

(i) We impose $((s)_0)_{red} \cap X(\bar{\mathbb{F}}_q)$ finite, i.e. we impose that $(s)_0$ is a zero-dimensional scheme;

(ii) We only assume $s \neq 0$.

In Section 2 we will consider problem (ii) for rank 2 vector bundles on $\mathbf{P}^3$. In Section 3 we will consider both problems (i) and (ii) for certain smooth surfaces.

Proof of Theorem 1. Fix an integral $D \in |L \otimes R \otimes \omega_X|$ and a general x-ple $(P_1, \dots, P_x) \in D^x$. By the generality of each P_i we have $P_i \in D_{reg}$ for all i. Let $(c_2 - x + 1)P_x$ denote the effective degree $(c_2 - x + 1)$ Cartier divisor of D supported by the smooth point P_x. Set $Z := \{P_1, \dots, P_{x-1}\} \cup (c_2 - x + 1)P_x$. Since $P_i \in X_{reg}$ for all i, Z is a curvilinear zero-dimensional subscheme of D and hence of X and it has length c_2. Let E be the general extension of $\mathcal{I}_Z \otimes L$ by $\mathcal{O}_X$, i.e. the general sheaf fitting as the middle term in the following extension of coherent sheaves on X:

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_Z \otimes L \to 0\,. \tag{1}$$

Since length$(Z) = c_2$, E is a rank 2 torsion free sheaf, $c_1(E) \cong L$ and $c_2(E) = c_2$. Furthermore, E is equipped with a section s such that $(s)_0 = Z$. Since $\sharp(Z_{red}) = x$, to prove Theorem 1 it is sufficient to show that E is locally free, i.e. the so-called Cayley-Bacharach condition is satisfied. The inclusion $\mathcal{O}_X(-D) \to \mathcal{O}_X$ induced by D gives the following exact sequence of sheaves on X:

$$0 \to R^* \to L \otimes \omega_X \to L \otimes \omega_X|D \to 0\,. \tag{2}$$

Since $h^0(X, R^*) = 0$ we have an inclusion $H^0(X, L \otimes \omega_X) \subseteq H^0(D, L \otimes \omega_X|D)$. Furthermore $h^0(D, L \otimes \omega_X|D) \leq h^0(X, L \otimes \omega_X|D) + h^1(X, R^*) \leq c_2 - 1$. By [3] and the generality of the x-ple $(P_1, \dots, P_x) \in D^x$, we have $h^0(D, \mathcal{I}_{W,D} \otimes (L \otimes \omega_X|D))$ for every length $c_2 - 1$ closed subscheme W of Z. Thus $h^0(X, \mathcal{I}_W \otimes L \otimes \omega_X) = 0$ for any length $c_2 - 1$ closed subscheme W of Z. Hence Z satisfies the Cayley-Bacharach condition with respect to L. Hence E is locally free. $\square$

2. Rank 2 vector bundles on $\mathbf{P}^3$

Here we consider rank 2 vector bundles on $\mathbf{P}^3$. We distinguish the case c_1 even and the case c_1 odd. In the former case we may reduce to the case $c_1 = 0$. In the latter case we may reduce to the case $c_1 = -1$. Recall that $c_1(E)c_2(E) \equiv 0 \pmod 2$ for any rank 2 vector bundle E on $\mathbf{P}^3$ ([4], Corollary 2.2), and that $c_1(E)^2 - 4c_2(E) < 0$ for any rank 2 stable vector bundle E on $\mathbf{P}^3$ ([4], Lemma 3.2).

Theorem 2. *Fix a prime power q, and integers r, y, e such that $r \geq 2$, $0 \leq y \leq r$, $y \neq r-1$, $0 \leq e \leq r-y$, $e \neq 1$ and $e \neq r-y-1$. Set $x := (q+1)y + e$. Then there exist a rank 2 vector bundle E on $\mathbf{P}^3$ defined over $\mathbb{F}_q$ such that $c_1(E) = 2$ and $c_2(E) = r$, and $s \in H^0(\mathbf{P}^3, E)$ defined over $\mathbb{F}_q$ such that:*

(a) over $\bar{\mathbb{F}}_q$ the scheme $(s)_0$ is reduced and it is the disjoint union of r lines;

(b) $\sharp((s)_0 \cap \mathbf{P}^3(\mathbb{F}_q)) = x$.

Theorem 3. *Fix a prime power q, and integers r, y, e_1, e_2 such that $r \geq 2$, $0 \leq y \leq r$, $y \neq r-1$, $e_1 \geq 0$, $e_2 \geq 0$, $e_1 + e_2 \leq r - y$, $e_1 \neq 1$, $e_2 \neq 1$ and $e_1 + e_2 \neq r - y$. Set $x := (q+1)y + e_1 + 2e_2$. Then there exist a rank 2 vector bundle E on $\mathbf{P}^3$ defined over $\mathbb{F}_q$ such that $c_1(E) = 3$ and $c_2(E) = 2r$, and $s \in H^0(\mathbf{P}^3, E)$ defined over $\mathbb{F}_q$ such that:*

(a) over $\bar{\mathbb{F}}_q$ the scheme $(s)_0$ is reduced and it is the disjoint union of r lines;

(b) $\sharp((s)_0 \cap \mathbf{P}^3(\mathbb{F}_q)) = x$.

Remark 1. Take E as in the statement of Theorem 2. By [4], p. 236, we have $c_1(E(-1)) = 0$ and $c_2(E(-1)) = r - 1$. Take E as in the statement of Theorem 3. By [4], p. 236, we have $c_1(E(-2)) = -1$ and $c_2(E(-1)) = 2r - 2$.

Proof of Theorem 2. Take as $Z := (s)_0$ the disjoint union of r lines. Assume that Z is defined over $\mathbb{F}_q$. A smooth conic $D \subset \mathbf{P}^3$ is defined over $\mathbb{F}_q$ if and only if $\sharp(D \cap \mathbf{P}^3(\mathbb{F}_q)) \geq 3$ and in this case we have $\sharp(D \cap \mathbf{P}^3(\mathbb{F}_q)) = q+1$. If D is not defined over $\mathbb{F}_q$, then either $\sharp(D \cap \mathbf{P}^3(\mathbb{F}_q)) = 0$ or $\sharp(D \cap \mathbf{P}^3(\mathbb{F}_q)) = 1$ and both cases are possible. Take Z such that exactly y of the lines of Z are defined over $\mathbb{F}_q$. Fix an integer $f \geq 2$ and let $U \subset \mathbf{P}^3$ the disjiont union of f lines. Assume that U is defined over $\mathbb{F}_q$, but that the f lines $U_1, \dots, U_f$ of U are not defined over a smaller field, i.e. that these f lines are conjugate for the action of the Galois group of the field extension $[\mathbb{F}_{q^f} : \mathbb{F}_q]$. Notice that $\sharp(U_1 \cap \mathbf{P}^3(\mathbb{F}_q)) = 1$ if and only if $\sharp(U_1 \cap \mathbf{P}^3(\mathbb{F}_q)) = 1$. Take Z such that exactly e of its lines contain a point of $\mathbf{P}^3(\mathbb{F}_q)$. Then apply [4], Example 4.3.1. □

Proof of Theorem 3. Take as $Z := (s)_0$ the disjoint union of r smooth conics. Assume that Z is defined over $\mathbb{F}_q$. A smooth conic $D \subset \mathbf{P}^3$ is defined over $\mathbb{F}_q$ if and only if $\sharp(D\cap \mathbf{P}^3(\mathbb{F}_q))\ge 3$ and in this case we have $\sharp(D\cap \mathbf{P}^3(\mathbb{F}_q)) = q+1$. If D is not defined over $\mathbb{F}_q$, then either $\sharp(D\cap \mathbf{P}^3(\mathbb{F}_q)) = 0$ or $\sharp(D\cap \mathbf{P}^3(\mathbb{F}_q)) = 1$ or $\sharp(D\cap \mathbf{P}^3(\mathbb{F}_q)) = 2$ and all these cases are possible. Take$Z$ such that exactly y of the lines of Z are defined over $\mathbb{F}_q$. Fix an integer $f\ge 2$ and let $U\subset \mathbf{P}^3$ the disjiont union of f smooth conics $U_1,\dots,U_f$ of U are not defined over a smaller field, i.e. that these f smooth conics are conjugate for the action of the Galois group of the field extension $[\mathbb{F}_{q^f}:\mathbb{F}_q]$. Notice that $\sharp(U_i\cap \mathbf{P}^3(\mathbb{F}_q)) = \sharp(U_j\cap \mathbf{P}^3(\mathbb{F}_q))$ for all i,j. Assume that exactly e_i conics of Z contain i points of $\mathbf{P}^3(\mathbb{F}_q)$. Then apply [4], Example 4.3.2. □

3. Rank 2 Vector Bundles on surfaces over $\mathbb{F}_q$

We will first consider the case $X = \mathbf{P}^2$.

Theorem 4. *Fix a prime power q and integers $c_2\ge c_1+3\ge 4$, $c_2\ge x>0$ such that $q\ge x$. Then there exist a rank 2 vector bundle E on $\mathbf{P}^2$ defined over $\mathbb{F}_q$ such that $c_1(E)=c_1$ and $c_2(E)=c_2$ and $s\in H^0(\mathbf{P}^2,E)$ defined over $\mathbb{F}_q$ such that $\sharp(((s)_0)_{red}) = x$.*

Proof. Fix a line $D\subset \mathbf{P}^2$ defined over $\mathbb{F}_q$ and x distinct points $P_1,\dots,P_x\in D(\mathbb{F}_q)$. This is possible because $q\ge x$. Let W denote the zero-dimensional closed subscheme of D with length (c_2-x+1) such that $W_{red}=\{P_x\}$. Set $Z := W\cup\{P_1,\dots,P_{x-1}\}$. Thus Z is a length c_2 curvilinear scheme defined over $\mathbb{F}_q$. We will take as E a sheaf fitting in an exact sequence

$$0\to \mathcal{O}_{\mathbf{P}^2}\to E\to \mathcal{I}_Z(c_1)\to 0\,. \qquad (3)$$

Since $c_2 > c_1-2$, we have $h^0(\mathbf{P}^2,\mathcal{I}_{Z'}(c_1-3)) = h^0(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(c_1-4)) = h^0(\mathbf{P}^2,\mathcal{I}_Z(c_1-3))$ for every length (c_2-1) subscheme Z' of Z. Hence the pair (Z,c_1) satisfies the Cayley-Bacharach condition. Hence a general E fitting in (3) is locally free ([2]). Call Γ the $\bar{\mathbb{F}}_q$-vector space of all extensions (3) defined over $\bar{\mathbb{F}}_q$ and Γ_q the $\mathbb{F}_q$-vector space of all extensions (3) defined over $\mathbb{F}_q$. There are x lenght (c_2-1) subschemes Z' of Z and each of them is defined over $\mathbb{F}_q$. For each such Z' there is a codimension one linear subspace $B_{Z'}$ of Γ_q corresponding to the extensions (3) for which the Cayley-Bacharach condition is not satisfied by Z'. Since $q\ge x$ and any two linear subspaces of a vector space have $\{0\}$ in their common intersection, there is an extemsion $\epsilon\in\Gamma_q\backslash(\bigcup_{Z'}B_{Z'})$. The extension ϵ gives the locally free E defined over $\mathbb{F}_q$ we were looking for. □

Theorem 5. *Fix a prime power q. Let X be a smooth and connected projective surface defined over $\mathbb{F}_q$, $L,R\in \mathrm{Pic}(X)(\mathbb{F}_q)$ and integers c_2,x such* *that* $0 < x \le c_2$ *and* $q \ge x$. *Assume* $h^0(X, R^*) = 0$, $c_2 \ge 2 + L^2 + L \cdot R + L \cdot \omega_X$, *and the existence of an integral* $D \in |L \otimes R \otimes \omega_X|$ *defined over* $\mathbb{F}_q$ *and with* $D_{reg}(\mathbb{F}_q) \ne \emptyset$ *Set* $\alpha := \sharp(D(\mathbb{F}_q))$ *and assume* $x \le \alpha$. *Then there exists a rank 2 vector bundle* E *on* X *such that* $c_1(E) = L$, $c_2(E) = c_2$, *and* $s \in H^0(X, E)$, $s \ne 0$, *such that the zero locus* $(s)_0$ *of* s *is finite and* $\sharp(((s)_0)_{red}) = x$.

Proof. Fix $P_x \in D_{reg}((\mathbb{F}_q)$ and $x - 1$ distinct points of $D(\mathbb{F}_q)\backslash\{P_x\}$. Let $(c_2 - x - 1)P_x$ the effective degree $(c_2 - x + 1)$ Cartier divisor of D supported by P_x. Set $Z := \{P_1, \ldots, P_{x-1}\} \cup (c_2 - x - 1)P_x$. Z is a length c_2 zero-dimensional scheme of X. Since $P_x \in D_{reg}$, Z is curvilinear. Since $c_2 - 1 > \deg(L \otimes \omega_X|D)$, every $f \in H^0(D, (L \otimes \omega_X)|D)$ vanishing of a length $c_2 - 1$ closed subscheme of Z vanishes identically on D. Hence we may repeat the proof of Theorem 1 omitting the quotation of [3] and with the following added observation. Call Γ the $\bar{\mathbb{F}}_q$-vector space of all extensions (1) defined over $\bar{\mathbb{F}}_q$ and Γ_q the $\mathbb{F}_q$-vector space of all extensions (1) defined over $\mathbb{F}_q$. There are x lenght $(c_2 - 1)$ subschemes Z' of Z and each of them is defined over $\mathbb{F}_q$. For each such Z' there is a codimension one linear subspace $B_{Z'}$ of Γ_q corresponding to the extensions (1) for which the Cayley-Bacharach condition is not satisfied by Z'. Since $q \ge x$ and any two linear subspaces of a vector space have $\{0\}$ in their common intersection, there is an extension $\epsilon \in \Gamma_q\backslash(\bigcup_{Z'} B_{Z'})$. The extension ϵ gives the locally free E defined over $\mathbb{F}_q$ we were looking for. □

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