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NUMBER OF POINTS OVER $\mathbb{F}_q$ OF PROJECTIVE VARIETIES: EXTREMALITY PROPERTIES OF UNIONS OF LINEAR SPACES

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Abstract: Here we study the following invariants. For every prime-power q and all positive integers n, m, d let $\tau(q, m, d, n)$ (respectively $\beta(q, m, d, n)$) denote the maximal integer $\sharp(X(\mathbb{F}_q))$ for some geometrically integral, respectively geometrically connected$m$-dimensional degree d variety $X \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$.

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1. Number of Points of Hyperplane Sections

For every prime-power q and all positive integers n, m, d let $\tau(q, m, d, n)$ (respectively $\beta(q, m, d, n)$) denote the maximal integer $\sharp(X(\mathbb{F}_q))$ for some geometrically integral (respectively geometrically connected$m$-dimensional degree d variety $X \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$. Here we will prove the following result.

Theorem 1. *Fix a prime power q and integers n, m, d such that $n \geq 2m + 2 \geq 4$.*

(a) Assume $q > d > 0$. Then $\tau(q, m, d, n) = \tau(q, m, d, 2m + 1)$.

(b) Assume $m \geq 2$ and take a geometrically connected degree d hypersurface $X \subset \mathbf{P}^{m+1}$ defined over $\mathbb{F}_q$. Let $\gamma_{X,q}$ denote the maximal integer $\sharp((V(\mathbb{F}_q) \cap X))$ for some $(m-1)$-dimensional linear subspace $V \subset \mathbf{P}^{m+1}$ defined over $\mathbb{F}_q$. Then $\sharp(X(\mathbb{F}_q)) \leq (q + 1) \cdots \beta(q, m - 1, d, m) - q \cdot \gamma_{X,q}$.

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We will also prove results when X is not geometrically integral. In this case we get a sharp upper bound for the integer $\sharp(X(\mathbb{F}_q))$ and that the upper bound is an equality if and only if X is a suitable union of linear spaces (see Lemma 1 and Proposition 1 and Proposition 2).

Lemma 1. *Fix a prime-power q and integers d, n such that $q > d > 0$ and $n \geq 3$. Let $C \subset \mathbf{P}^n$ be a degree d curve defined over $\mathbb{F}_q$. Then $\sharp(C(\mathbb{F}_q)) \leq d(q+1)$ and $\sharp(C(\mathbb{F}_q)) = d(q+1)$ if and only if C is the union of d disjoint lines, each of them defined over $\mathbb{F}_q$.*

Proof. Let A be the union of all lines of C (defined over $\bar{\mathbb{F}}_q$) and B the union of all the other geometric irreducible components of C. A and B are defined over $\mathbb{F}_q$ and $\deg(C) = \deg(A)+\deg(B)$. Since the lemma is obvious when C is a union of lines, to prove the lemma it is suficient to get a contradiction under the assumption $A = \emptyset$. Assume the existence of a linear subspace V defined over $\mathbb{F}_q$ such that $\dim(V) = n-3$ and $V \cap C = \emptyset$. Let $\phi : \mathbf{P}^n \backslash V \to \mathbf{P}^1$. The degree d morphism $\phi|C$ is defined over $\mathbb{F}_q$. Thus $\sharp(C(\mathbb{F}_q)) \leq d \cdot \sharp(\mathbf{P}^1(\mathbb{F}_q)) = d(q+1)$. Now assume that no such V exists, but that there is linear subspace M defined over $\mathbb{F}_q$ such that $\dim(M) = n-3$ and $M \cap C(\mathbb{F}_q) = \emptyset$. Again, since $\phi|(C\backslash C \cap M)$ is a degree d morphism defined over $\mathbb{F}_q$ we get $\sharp(C(\mathbb{F}_q)) \leq d(q+1)$. Since C has degree d and t is defined over $\mathbb{F}_q$ there is a hypersurface $\Gamma \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$, with $\deg(\Gamma) \leq d$ and $C \subset \mathbf{P}^n$ (take a cone with as vertex a suitable $(n-3)$-dimensional linear space). Since $\deg(\Gamma) \leq q$, there is $P \in \mathbf{P}^n(\mathbb{F}_q)$ such that $P \notin \Gamma$. Hence $P \notin C$. Take the linear projection from P. The image $C' \subset \mathbf{P}^{n-1}$ is a degree $d'|d$ (and hence $d' \leq d$) curve defined over $\mathbb{F}_q$. Repeating the construction $n-3$ times we get the existence of the linear space M. □

Remark 1. Fix an integer d, a prime-power q such that $q > d > 0$ and a degree d curve $C \subset \mathbf{P}^2$ defined over $\mathbb{F}_q$. The proof of Lemma 1 shows $\sharp(C(\mathbb{F}_q)) \leq 1+dq$ and $\sharp(C(\mathbb{F}_q)) = 1+dq$ if and only if there is $P \in \mathbf{P}^2(\mathbb{F}_q)$ such that C is the union of d lines through P, each of them defined over $\mathbb{F}_q$.

Lemma 2. *Fix a prime-power q and integers d, n such that $q > d > 0$ and $n \geq 3$. Let $C \subset \mathbf{P}^n$ be a geometrically connected degree d curve defined over $\mathbb{F}_q$. Then $\sharp(C(\mathbb{F}_q)) \leq 1 + dq$ and $\sharp(C(\mathbb{F}_q)) = 1 + dq$ if and only if there is $P \in \mathbf{P}^n(\mathbb{F}_q)$ such that C is the union of d lines through P, each of them defined over $\mathbb{F}_q$. If C is not a union of lines defined over $\mathbb{F}_q$, then $\sharp(C(\mathbb{F}_q)) \leq (q+1)d-q$.*

Proof. We will only check the last assertion. If C contains a geometric irreducible component not defined over $\mathbb{F}_q$ or a geometric irreducible component D defined over $\mathbb{F}_q$, but with $D(\mathbb{F}_q)$, then we easily win. Hence we may assume the existence of $P \in C(\mathbb{F}_q)$ such that no line contained in C and defined over $\mathbb{F}_q$ contains P. As in the proof of Lemma 1 the degree $d-1$ linear projection of $C \backslash \{P\}$ from P gives $\sharp(C(\mathbb{F}_q)) \leq 1 + (d-1)(q+1) = (q+1)d - q$. $\square$

Proposition 1. *Fix a prime-power q, and integers n, d such that $n \geq 3$ and $q > d \geq 2$, and a hypersurface $X \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$. Then $\sharp(X(\mathbb{F}_q)) \leq (q^n - 1)/(q-1) + d(q^{n+1}-1)/(q-1)$ and $\sharp(X(\mathbb{F}_q)) = (q^n-1)/(q-1) + d(q^{n+1}-1)/(q-1)$ if and only if there are an $(n-2)$-dimensional linear subspace $L \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$ and d distinct hyperplanes $A_i \subset \mathbf{P}^n$, $1 \leq i \leq d$, defined over $\mathbb{F}_q$ such that $X = \cup_{i=1}^d A_i$ and $L \subset A_i$ for all i.*

Proof. Decreasing if necessary d we reduce to the case " X reduced ". Let A be the union of all hyperplanes of $\mathbf{P}^n$ (defined over $\bar{\mathbb{F}}_q$) contained in X and B the union of all the other geometric irreducible components of X. A and B are defined over $\mathbb{F}_q$ and $\deg(X) = \deg(A) + \deg(B)$. Since the proposition is obvious when X is a union of hyperplanes, to prove the proposition it is suficient to get either the strict inequality or a contradiction under the assumption $A = \emptyset$. By Lemma 2 we may use induction on the integer n. Fix $P \in X(\mathbb{F}_q)$ and let E (respectively F) be the union of all geometric irreducible components of X which are (respectively which are not) cones with vertex containing P. Both E and F are defined over $\mathbb{F}_q$ and $\deg(X) = \deg(E) + \deg(F)$. Take the linear projection from P and count the points. By the inductive assumption on n we have $\sharp(E(\mathbb{F}_q)) \leq 1 + q \cdot (\deg(E)))(q^n-1)/(q-1))$ if $E \neq \emptyset$. Since $\sharp(\mathbf{P}^{n-1}(\mathbb{F}_q)) = (q^n-1)/(q-1)))$, the linear projection from P gives $\sharp(F(\mathbb{F}_q)) \leq 1 + q(q^n-1)/(q-1))$ if $E \neq \emptyset$. $\square$

In a similar way we get the following result.

Proposition 2. *Fix a prime-power q, and integers n, m, d such that $n \geq 2m+1 \geq 3$ and $q > d \geq 2$, and a geometricallu connected $X \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$ and with pure dimension m. Then $\sharp(X(\mathbb{F}_q)) \leq d(q^{m+1}-1)/(q-1) + 1 - d$ and $\sharp(X(\mathbb{F}_q)) = d(q^{m+1}-1)/(q-1) + 1$ if and only if there are $P \in \mathbf{P}^n$ defined over $\mathbb{F}_q$ and d distinct m-dimensional linear spaces $A_i \subset \mathbf{P}^n$, $1 \leq i \leq d$, defined over $\mathbb{F}_q$ such that $X = \cup_{i=1}^d A_i$ and $P \in A_i$ for all i.*

Let $X \subset \mathbf{P}^n$ be a union of m-dimensional linear spaces as in Proposition 2. Notice that X may be non-degenerate only if $n \leq 1 + md$.

Proof of Theorem 1. Since we do not assumed in the definition of $\tau(q,m,d,n)$ that X is non-degenerate, obviously $\tau(q,m,d,n) \geq \tau(q,m,d,2m+1)$. To check the other inequality in part (a) for all $n \geq 2m+2$ it is sufficient to prove $\tau(q,m,d,n) \leq \tau(q,m,d,n-1)$ for all $n \geq 2m+2$. Fix n and $X \subset \mathbf{P}^n$

occuring in the definition of $\tau(q,m,d,n)$. Since $\sharp(D(\mathbb{F}_q)) = q+1$ for every line D, Proposition 2 implies the existence of $P \in \mathbf{P}^m(\mathbb{F}_q)$ such that for all $A, B \in X(\mathbb{F}_q)$ with $A \neq B$ the line $\langle\{A,B\}\rangle$ does not contain P. Take the linear projection from P and the image of X in $\mathbf{P}^{n-1}$ by this projection to get $\tau(q,m,d,n) \le \tau(q,m,d,n-1)$. For part (b) first note that every hypersurface of a projective space is locally Cohen-Macaulay and geometrically connected. Take $V \subset \mathbf{P}^{m+1}$ computing $\gamma_{X,q}$. Use the pencil of all hyperplanes H containing V and that each hypersurface $X \cap H$ of H is geometrically connected. □

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