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# ADJACENT VERTEX-DISTINGUISHING TOTAL CHROMATIC NUMBERS OF $C_n^3$ ($n \geq 4$)

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**Abstract:** For a simple graph $G$, let $f$ be an proper total coloring of $G$. If for any edge $e = uv$, we have $C(u) \neq C(v)$, then $f$ is called an adjacent vertex-distinguishing total coloring of $G$, denoted by AVDTC, where $C(u) = \{f(u)\} \cup \{f(uv)|uv \in E(G)\}$. For a simple graph $G$, the adjacent vertex-distinguishing total chromatic number, denoted by $\chi_{at}(G)$, is the minimum number of colors required in an adjacent vertex-distinguishing total coloring of $G$. In this paper, we obtain the adjacent vertex-distinguishing total chromatic numbers of $C_n^3$.

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**Key Words:** total coloring, adjacent vertex-distinguishing total coloring, adjacent vertex-distinguishing total chromatic number

## 1. Introduction

Vertex-distinguishing edge coloring (or strong coloring) of graphs is studied in

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paper [2], introduced from the theory of network. The adjacent strong edge coloring, or adjacent vertex-distinguishing edge coloring of graphs is introduced in paper [5] and adjacent vertex-distinguishing total coloring of graphs introduced in paper [6]. All of the graphs concerned in this paper are simple, finite and undirected graph. We denote by $V(G)$, $E(G)$ and $\Delta(G)$ the set of vertices, edges and the maximum degree of graph $G$, respectively. The other terminologies and marks refer to [3], [4], [1].

**Definition 1.1.** (see [6]) Let $G(V,E)$ be a simple graph, $k$ be a positive integer and $f$ be a mapping from $V(G)\cup E(G)$ to $\{1,2,\cdots,k\}$. For all $u \in V(G)$, the set $\{f(u)\}\cup\{f(uv)|uv \in E(G)\}$ is denoted by $C(u)$. If:

1) for any $uv, vw \in E(G), u \neq w$, we have $f(uv) \neq f(vw)$;

2) for any $uv \in E(G), u \neq v$, we have $f(u) \neq f(v), f(u) \neq f(uv), f(v) \neq f(uv)$, then $f$ is called a $k$-proper-total-coloring. If $f$ is a $k$-proper-total-coloring, and

3) for any edge $uv \in E(G)$, we have $C(u) \neq C(v)$, then $f$ is called a $k$-adjacent vertex-distinguishing total coloring of graph $G$($k$-AVDTC of $G$ in brief) and the number

$$\chi_{at}(G) = \min\{k \mid G \text{ has a } k\text{-AVDTC }\}$$

is called the adjacent vertex-distinguishing total chromatic number of $G$.

**Lemma 1.1.** (see [6]) *For graph $G$, if $uv \in G$ and $d(u) = d(v) = \Delta(G)$, then*

$$\chi_{at}(G) \geq \Delta(G) + 2.$$

**Definition 1.2.** If $f$ is a mapping from $V(C_n^k)\cup E(C_n^k)$ to $\{1,2,\cdots,l\}$, let matrix $A = (a_{ij})_{n\times n}$, where:

1. When $v_iv_j \in E(C_n^k)$, if $i<j, j-i \leq k$, then $a_{ij} = f(v_iv_j)$, otherwise $a_{ij} = 0$.

2. When $v_iv_j \in E(C_n^k)$,$i>j$, if $a_{ji}=0$, then $a_{ij} = f(v_iv_j)$, otherwise $a_{ij}$.

3. $a_{ii} = f(v_i)$ $(i = 1,2,\cdots,n)$.

4. $a_{ij} = 0, v_iv_j \notin E(C_n^k)$.

Then $A$ is called the coloring matrix of $C_n^k$ (denoted by ${}_n^kA_l$). If $f$ is a $k$-AVDTC of $C_n^k$, then ${}_n^kA_l$ is called the $k$-adjacent vertex-distinguishing total coloring matrix of $C_n^k$(denoted by $k_{at}{-}{}_n^kA_l$).

For any $i \in \{1,2,\cdots,l\}$, let

$$\alpha(i)_k = \begin{cases} 1, & \text{if color } i \text{ occurs in the } k\text{-th line,} \\ 0, & \text{otherwise.} \end{cases}$$

It is obviously that $C(v_i) = \alpha_i \cup \beta_i - \{0\}$, where $\alpha_i, \beta_i$ are the sets of elements of the $i$-th line vector and the $i$-th column vector of ${}^k_n A_l$, respectively.

**Lemma 1.2.** *For graph $C_n^k$ and positive integer $l$, let $f$ be a total coloring of $G$, ${}^k_n A_l$ be the coloring matrix of $G$. $f$ is a $l$-AVDTC of $C_n^k$, if and only if:*

1. $\alpha_i \cap \beta_i - \{0\} = \{f(v_i)\}$;
2. $C(v_i) \neq C(v_j)$, *if* $v_i v_j \in E(C_n^k)$;
3. *for any* $i, j \in \{1, 2, \cdots, l\}$, $|\sum_{t=1}^{n} \alpha(i)_t - \sum_{t=1}^{n} \alpha(j)_t| \leq 1$.

In this paper, we only consider $C_n^3$.

## 2. Main Results

**Theorem 2.1.** *For $n = 4$, $\chi_{aet}(C_n^3) = 4$.*

*Proof.* It is obviously that $\chi_{aet}(C_4^3) \geq 4$. Now only to give a 4-AVDTC of $C_n^3$.

Let $f$ be a mapping from $V(C_4^3) \cup E(C_4^3)$ to $\{1, 2, 3, 4\}$, where

$$ {}^3_4 A_4 = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 \\ 3 & 0 & 0 & 4 \end{pmatrix}. $$

By Lemma 1.2. we know that $f$ is a 4-AVDTC of $C_4^3$. So $\chi_{at}(C_4^3) = 4$. The result is true. □

**Theorem 2.2.** (see [6]) *For $C_5^3$, $\chi_{at}(C_5^3) = 4$.*

**Theorem 2.3.** *For $n = 6$, $\chi_{at}(C_n^3) = 5$.*

*Proof.* By Lemma 1.1., we obtain that $\chi_{at}(C_6^3) \geq 5$. Now only to give a 5-AVDTC of $C_6^3$.

Let $f$ be a mapping from $V(C_6^3) \cup E(C_6^3)$ to $\{1, 2, 3, 4, 5\}$, where

$$ {}^3_6 A_5 = \begin{pmatrix} 1 & 2 & 0 & 5 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 \\ 4 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}. $$

By Lemma 1.2. we know that $f$ is a 5-AVDTC of $C_6^3$.
So $\chi_{at}(C_6^3) = 5$.

We define six matrixes as follows:

$$A_1 = \begin{pmatrix} 1 & 5 & 0 & 3 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 & 5 \\ 6 & 0 & 0 & 0 & 5 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 6 \\ 2 & 0 & 4 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 5 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 5 & 3 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 6 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 5 & 3 \\ 6 & 0 & 4 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 4 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 6 & 4 \\ 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 & 3 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 6 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 1 & 5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 4 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 4 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 5 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix},$$

$$A_6 = \begin{pmatrix} 1 & 5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 4 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 4 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 5 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}.$$

**Lemma 2.1.** *For $C_7^3$, $\chi_{at}(C_7^3) = 6$.*

*Proof.* By Lemma 1.1., we obtain $\chi_{at}(C_7^3) \geq 6$. Now only to give a 6-AVDTC of $C_7^3$.

Let $f$ be a mapping from $V(C_7^3) \cup E(C_7^3)$ to $\{1, 2, 3, 4, 5, 6\}$, where ${}_7^3A_6 = A_1$. By Lemma 1.2. we know that $f$ is a 6-AVDTC of $C_7^3$. So $\chi_{at}(C_7^3) = 6$. $\square$

**Lemma 2.2.** *For $C_8^3$, $\chi_{at}(C_8^3) = 6$.*

*Proof.* By Lemma 1.1, we obtain that $\chi_{at}(C_8^3) \geq 6$. Now only to give a 6-AVDTC.

Let $f$ be a mapping from $V(C_8^3) \cup E(C_8^3)$ to $\{1, 2, 3, 4, 5, 6\}$, where ${}_8^3A_6 = A_2$. By Lemma 1.2. we know that $f$ is a 6-AVDTC of $C_8^3$. So $\chi_{at}(C_8^3) = 6$. $\square$

**Lemma 2.3.** *For $C_9^3$, $\chi_{at}(C_9^3) = 6$.*

*Proof.* By Lemma 1.1, we obtain $\chi_{at}(C_9^3) \geq 6$. Now only to give a 6-AVDTC of $C_9^3$.

Let $f$ be a mapping from $V(C_9^3) \cup E(C_9^3)$ to $\{1,2,3,4,5,6\}$, where ${}_9^3A_6 = A_3$.

By Lemma 1.2. we know that $f$ is a 6-AVDTC of $C_9^3$. So $\chi_{at}(C_9^3) = 6$. □

**Lemma 2.4.** *For $C_{10}^3$, $\chi_{at}(C_{10}^3) = 6$.*

*Proof.* By Lemma 1.1, we obtain $\chi_{at}(C_{10}^3) \geq 6$. Now only to give a 6-AVDTC of $C_{10}^3$.

Let $f$ be a mapping from $V(C_{10}^3) \cup E(C_{10}^3)$ to $\{1,2,3,4,5,6\}$, where ${}_{10}^3A_6 = A_4$.

By Lemma 1.2. we know that $f$ is a 6-AVDTC of $C_{10}^3$. So $\chi_{at}(C_{10}^3) = 6$. □

**Lemma 2.5.** *For $C_{11}^3$, $\chi_{at}(C_{11}^3) = 6$.*

*Proof.* By Lemma 1.1, we obtain $\chi_{at}(C_{11}^3) \geq 6$. Now only to give a 6-AVDTC of $C_{11}^3$.

Let $f$ be a mapping from $V(C_{11}^3) \cup E(C_{11}^3)$ to $\{1,2,3,4,5,6\}$, where ${}_{11}^3A_6 = A_5$.

By Lemma 1.2. we know that $f$ is a 6-AVDTC of $C_{11}^3$. So $\chi_{at}(C_{11}^3) = 6$. □

**Lemma 2.6.** *For $C_{12}^3$, $\chi_{at}(C_{12}^3) = 6$.*

*Proof.* By Lemma 1.1, we obtain $\chi_{at}(C_{12}^3) \geq 6$. Now only to give a 6-AVDTC of $C_{12}^3$.

Let $f$ be a mapping from $V(C_{12}^3) \cup E(C_{12}^3)$ to $\{1,2,3,4,5,6\}$, where ${}_{12}^3A_6 = A_6$.

By Lemma 1.2. we know that $f$ is a 6-AVDTC of $C_{12}^3$. So $\chi_{at}(C_{12}^3) = 6$. □

**Theorem 2.4.** *For $n \geq 13$, $\chi_{at}(C_n^3) = 6$.*

*Proof.* By Lemma 1.1, we obtain $\chi_{at}(C_n^3) \geq 6$. Now only to give a 6-AVDTC of $C_n^3$. Let $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$.

*Case 1.* When $n \equiv 1(\bmod 6)$:

let $n = 6l+7(1 \leq l)$.Let $f$ be a mapping from $V(C_n^3) \cup E(C_n^3)$ to $\{1,2,3,4,5,6\}$. To $f$, ${}_n^3A_6$ is the coloring matrix of $C_n^3$, where the element $a_{ij} = b_{dm}$ of ${}_n^3A_6$ satisfy the following:

(a) when $i = 1, 2, \cdots, 6l$:

1.1. when $j = i$: $a_{ij} = b_{dm}$, $m = 1$; if $i \equiv k(\bmod 6)$, and if $k = 0$, then $d = 6$, otherwise $d = k$;

1.2. when$j = i + 1$: $a_{ij} = b_{dm}$, $m = 2$; if $i \equiv k(\bmod 6)$, and if $k = 0$, then $d = 6$, otherwise $d = k$.

1.3. when $j = i + 3$: $a_{ij} = b_{dm}$, $m = 3$; if $i \equiv k(\bmod 6)$, and if $k = 0$, then $d = 6$, otherwise $d = k$;

1.4. otherwise, $a_{ij} = 0$.

(b) when $i = 6l+1, 6l+2, \cdots, 6l+7$:

2.1. $a_{ij} \in A$, constructed by the 7 latest lines and the $1, 2, 3, 6l+4, 6l+5, 6l+6, 6l+7$ rows of ${}^3_nA_6$ as Lemma 2.1.

2.2. otherwise, $a_{ij} = 0$.

It is obviously that ${}^3_nA_6$ is the $k_{at}$-${}^k_nA_6$ of $C_n^3$. So $f$ is a 6-AVDTEC of $C_n^3$.

*Case 2.* When $n \equiv 2, 3, 4, 5, 0(\mod 6)$, construct the coloring matrix of $C_n^3$ similarly as in Case 1. and according to Lemmas 2.2, 2.3, 2.4, 2.5 and 2.6, respectively.

Summing up all of above, the theorem is true. □

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