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ON THE ADJACENT VERTEX-DISTINGUISHING EDGE CHROMATIC NUMBER OF $F_m \vee W_n$

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Abstract: The adjacent vertex-distinguishing edge chromatic number of join graph of wheel Wm and wheel Wn graph, where W_m, W_n denote wheel with order $m+1$ and wheel with order $n+1$ respectively, is obtained in this paper.

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1. Introduction

The coloring problem of graphs is widely applied in practice. In [8], some conditional coloring problems are introduced. Some network problems can be converted to the strong edge coloring (see [7], [6], [5], [3], [1], [2]) and adjacent strong edge coloring.

Definition 1. (see [7], [6], [5], [3], [1], [2]) For a graph $G(V, E)$, if a proper coloring f is satisfied with $C(u) \neq C(v)$ for $\forall u, v \in V(G)(u \neq v)$, then f is called *k-strong edge coloring of G*, is abbreviated k-SEC, and

$$\chi'_s(G) = \min\{k|k - SEC\ of\ G\}$$

is called the strong edge chromatic number of G. For $\forall uv \in E(G)$,$C(u) \neq C(v)$,

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then f is called k-adjacent strong edge coloring of G, is abbreviated k-ASEC, and

$$\chi_{as}^{'}(G) = \min\{k | k - ASEC\ of\ G\}$$

is called the adjacent strong edge chromatic number of G (see [6]), where

$$C(u) = \{f(uv) | uv \in E(G)\}.$$

Conjecture. (see [2]) *Let G be a connected graph with $|G| \geq 3$, and $G \neq C_5$(5-cycle), then*

$$\chi_{as}^{'}(G) \leq \Delta(G) + 2\,,$$

where $\Delta(G)$ is the maximum degree of graph G.

Definition 2. (see [4]) Let G and H be two disconnect simple graphs, namely $V(G) \cap V(H) = E(G) \cap E(H) = \phi$. The graph $G \vee H$ is called *join graph*, where $V(G \vee H) = V(G) \cup V(H)$, $E(G \vee H) = E(G) \cup E(H) \cup \{uv | u \in E(G), v \in E(H)\}$.

2. Main Result

In order to describe simply, we mark it as:

$V(F_m) = \{u_i | i = 0, \cdots, m\}$;

$E(F_m) = \{u_0 u_i | i = 1, 2, \cdots, m\} \cup \{u_i u_{i+1} | i = 1, 2, \cdots, m-1\}$;

$V(W_n) = \{v_i | i = 0, 1, \cdots, n\}$;

$E(W_n) = \{v_0 v_i | i = 1, 2, \cdots, n\} \cup \{v_i v_{i+1} | i = 1, 2, \cdots, n-1\} \cup \{v_n v_1\}$;

Lemma 1. $\Delta(F_m \vee W_n) = m + n + 1$.

Lemma 2. (see [7], [2]) *For complete graph K_n with order $n \geq 3$*

$$\chi_{as}^{'}(K_n) = \begin{cases} n, & n \equiv 1(\ mod\ 2); \\ n+1, & n \equiv 0(\ mod\ 2). \end{cases}$$

Lemma 3. (see [2]) *For a connected graph G with $|V(G)| \geq 3$ and two adjacent vertices with maximum degree, then*

$$\chi_{as}^{'}(G) \geq \Delta(G) + 1.$$

Lemma 4. (see [4]) *For bipartite graph G,*

$$\chi^{'}(G) = \Delta(G).$$

Lemma 5. (see [2]) *For a wheel W_n with order $n \geq 4$,*

$$\chi'_{as}(W_n) = n.$$

Theorem 1. *For $m \geq n = 3$, then*

$$\chi'_{as}(F_m \vee W_3) = \begin{cases} 9, & m = 3; \\ m+5, & m \geq 4. \end{cases}$$

Proof. When $m = 3$, $F_3 \vee W_3$ is regard as $K_8 \setminus \{v_n v_1\}$, by Lemma 1, the Theorem 1 is true.

When $m = 4$, in order to prove the $F_4 \vee W_3$ is $m+5$-ASEC, we can prove the graph $w_4 \vee W_3$ is $m+5$-ASEC firstly. As $w_4 \vee W_3$ is regard as $K_9 \setminus \{e_1, e_2\}$, where e_1, e_2 are edge which are not adjacent. By Lemma 3, let f be a 9-ASEC of K_9, supposing e_1 and e_1 are coloring with similar color, then f be a 9-ASEC of $K_9 \setminus \{e_1, e_2\}$, each number of set of colors assigned to the edges e_1 and e_2 incident to four vertices is 7, for f of K_9, $\overline{C}(u_i) = \{i+1\}, i = 1, 2, 3, 4$, then for f of $K_9 \setminus \{e_1, e_2\}$, $\overline{C}(u_i) = \{1, i+1\}, i = 1, 2, 3, 4$. Here $\overline{C}(u) = \{1, 2, \cdots, 9\} \setminus C(u)$, so f is 9-ASEC of $W_4 \vee W_3$. Then $F_4 \vee W_3$ is regard as $w_4 \vee W_3 \setminus \{v_n v_1\}$, it cannot change the chromatic number, so $F_4 \vee W_3$ is 9-ASEC.

When $m \geq 5$, in order to prove Theorem 1, we need to prove that $W_m \vee W_3$ exits $(m+5)$-ASEC firstly, then can prove that $F_m \vee W_3$ exits $(m+5)$-ASEC.

By Lemma 2, let f_1 be a 5-ASEC, color edges with colors $m+1, m+2, m+3, m+4, m+5$ incident to vertices of u_0, v_0, v_1, v_2, v_3, supposing:

$$\overline{C}(u_0) = \{m+1\},\ \overline{C}(v_0) = \{m+2\},\quad \overline{C}(v_1) = \{m+3\},\ \overline{C}(v_2) = \{m+4\},$$

$$\overline{C}(v_3) = \{m+5\},$$

where $\overline{C}(v) = \{m+1, m+2, m+3, m+4, m+5\} \setminus C(v)$.

By Lemma 4, let f_2 be a m-ASEC.

We color edges with colors $1, 2, \cdots, m$ incident to u_0, v_0, v_1, v_2, v_3 and $u_1, u_2, \cdots, u_m$. Supposing $u_0 = w_1, v_0 = w_2, v_1 = w_3, v_2 = w_4, v_3 = w_5$.

$$f_2(u_i w_j) = i + j - 1, \quad i = 1, 2, \cdots, m; \quad j = 1, 2, 3, 4, 5.$$

Case 1. If $m \equiv 0(\bmod 3)$, coloring edges $u_1u_2, u_2u_3, \cdots, u_{m-1}u_m, u_mu_1$ with colors $m+1, m+2, m+3$ circularly.

Case 2. If $m \equiv 1(\bmod 3)$, coloring edges $u_1u_2, u_2u_3, u_3u_4, u_4u_5$ with colors $m+1, m+2, m+3, m+4$ firstly, then color edges $u_5u_6, u_6u_7, \cdots, u_{m-1}u_m$, u_mu_1 with colors $m+1, m+2, m+3$ circularly.

Case 3. If $m \equiv 2(\bmod 3)$, coloring edges $u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6$ with colors $m+1, m+2, m+3, m+4, m+5$ firstly, then color edges $u_6u_7, u_7u_8, \cdots, \cdots, u_{m-1}u_m, u_mu$ with colors $m+1, m+2, m+3$ circularly.

Obviously, the combination of f_1, f_2 is a $(m+5)$-ASEC of $W_m \vee W_3$.

Similar with above, $F_m \vee W_3$ is regard as $W_m \vee W_3 \setminus \{v_nv_1\}$, so $F_m \vee W_3$ is $(m+5)$-ASEC

Thus Theorem 1 is true. □

Theorem 2. *For* $m \geq n \geq 4$, *then*

$$\chi'_{as}(F_m \vee W_n) = m + n + 2.$$

Proof. Obviously, $d(u_0) = d(v_0) = m + n + 1$, $u_0v_0 \in E(F_m \vee W_n)$, by Lemma 3, we have $\chi'_{as}(F_m \vee W_n) \geq m + n + 2$. Now, we can use the way as above to prove that $F_m \vee W_n$ exits $(m+n+2)$-ASEC. We prove $W_m \vee W_n$ exits $(m+n+2)$-ASEC firstly, then can prove $F_m \vee W_n$ exits $(m+n+2)$-ASEC by deleting one edge. $F_m \vee W_n$ is regard as $W_m \vee W_n$

u_i is denoted by x_{i+1}, and v_j by y_{j+1} $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

Case 1. For $m = n$ suppose $C = \{1, 2, \cdots, n+1, 0, n+3, \cdots, 2n+2\}, \overline{C}(u) = C \backslash C(u)$, let f be

$$f(x_1y_j) = j, j = 1, 2, \cdots, n+1;$$

$$f(x_iy_j) = i + j(\bmod (n+2)), i = 2, 3, \cdots, n+1; j = 1, 2, \cdots, n+1;$$

$$f(v_0v_j) = n + 2 + j, j = 1, 2, \cdots, n;$$

$$f(v_iv_{i+1}) = n + 1 + i, i = 1, 2, \cdots, n-1;$$

$$f(u_0u_j) = n + 2 + j, j = 1, 2, \cdots, n;$$

$$f(u_iu_{i+1}) = n + 1 + i, i = 2, 3 \cdots, n-1;$$

$$f(u_nu_1) = 2n + 1, \ f(u_1u_2) = 2n + 2.$$

For f,

$$\overline{C}(u_0) = \{2\}; \quad \overline{C}(v_0) = \{1\};$$

$$C(v_i) = \{i+2, i+3, \cdots, i+n+2\}(\bmod (n+2)) \\ \vee \{n+i, n+i+1, n+i+2\}, \quad i = 3, 4, \cdots, n;$$

$$C(v_1) = \{3, 4, \cdots, n+1, 0, 1, n+3, 2n+1, 2n+2\};$$

$$C(v_2) = \{4, 5, \cdots, n+1, 0, 1, 2, n+3, n+4, 2n+2\};$$

$$C(u_i) = \{i+1, i+3, , i+4, \cdots, i+n+2\}(\bmod (n+2))$$

$$\vee \{n+i, n+i+1, n+i+2\}, \quad i = 3, 4, \cdots, n;$$

$$C(u_1) = \{2, 4, , 5, \cdots, n+1, 0, 1, n+3, 2n+1, 2n+2\};$$

$$C(u_2) = \{3, 5, 6, \cdots, n+1, 0, 1, 2, n+3, n+4, 2n+2\}.$$

Obviously, the f is $(2n+2)$-ASEC of $W_n \vee W_n$ $(m = n \geq 4)$.

Case 2. For $m > n \geq 4$, supposing $C = \{1, 2, \cdots, m-1, 0, m+1, m+2, \cdots, m+n+2\}, \overline{C}(u) = C \backslash C(u)$.

Let f be

$$f(y_i x_j) = i + j - 1(\bmod m), i = 1, 2, \cdots, n+1; j = 1, 2, \cdots, n+1;$$

$$f(v_0 v_j) = m + j + 1, j = 1, 2, \cdots, n;$$

$$f(v_i v_{i+1}) = m + i, i = 2, 3, \cdots, n-1;$$

$$f(v_n v_1) = m + n;\ f(v_1 v_2) = m + n + 2;$$

$$f(u_0 u_i) = n + 2 + i, i = 1, 2, \cdots, m;$$

$$f(u_i u_{i+1}) = n + 1 + i, i = 2, 3 \cdots, n-1;$$

$$f(u_m u_1) = m + n + 1,\ f(u_1 u_2) = m + n + 2.$$

Because of $d(u_i) \neq d(v_j)$, $\overline{C}(v_0) = \{m+n+2\}$ and $\overline{C}(u_0) = \{0\}$, the f is $(m+n+2)$-ASEC of $W_m \vee W_n$ $(m = n \geq 4)$.

From all of above, f is $(m+n+2)$-ASEC of $W_m \vee W_n (m = n \geq 4)$. So we prove f is $(m+n+2)$-ASEC of $F_m \vee W_n$ $(m = n \geq 4)$. $\square$

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