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CAMBRIDGE RING: STABILITY AND ASYMPTOTIC BEHAVIOR

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Abstract: A ring of N cells rotates in discrete steps past N queues, moving customers (messages) from their queues of arrival to randomly chosen distination. Each message is assembled into a fixed-length packet. Packets already on the ring have priority over waiting packets. The model has applications in communication systems, processor intercommunication networks, and flexible manufacturing. Under Gready policy, independent identical Bernoulli variables with rate parameter $\frac{\lambda}{N}$ model message generation at the processors, and i.i.d geometric random variables with rate parameter $\frac{\mu}{N}$ model the transit time. Our main result is an upper bound of $\mathbb{E}(Q)$, where Q is the common stationary distribution of sizes of the N queues.

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1. Introduction

The Cambridge ring is a system of N queues arranged around a ring of N cells, each queue giving access to one cell, and each cell being able to carry one customer. The queues are numbered from $i = 0$ to $i = N-1$, and for $j \in \mathbf{Z}$, $[j]$ will denote the unique integer $i \in \{0, ..., N-1\}$ such that $j = i \quad \text{modulo} \quad N$. The ring rotates continuously past the queues so that the cell facing queue i at time $n \in \mathbf{N}$ will be in front of queue $[i+1]$ at time $n+1$. Between n and $n+1$, queue i receives $\delta_i(n)$ new customers.

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Customer p in the order of arrival is asking for a travel of length $\sigma_i(p) \in \{1, 2, 3, ...\}$ on the ring. At time $n+1$, customers arrived at their destinations are discharged, and empty cells pick up the customer in front of their queues if these are not empty.

The Cambridge ring was introduced in a paper by Coffman et al [1] as a generic model with applications in communication systems, car traffic, flexible manufacturing and processor interconnection networks (see [1] for more details and numerous references). In such paper, the authors assume that the numbers of new arrivals $\delta_i(n),\ 0 \le i \le N-1, n \in \mathbb{N}$, are i.i.d. with mean λ, and similary the customer travel lengths $\sigma_i(p),\ 0 \le i \le N-1, p \ge 1$, which are i.i.d. with mean $1/\mu$, and independents of the arrivals. The system evolution is then described by a Markov chain, and a partial study of its asymptotics is presented. In particular, the authors identify the necessary ergodicity condition $\lambda/\mu < 1$, and conjecture that it should also be sufficient; In the case when no travel length exceeds N, they prove the system to be stable (i.e., ergodic) under condition $\lambda N < 1$. In the case when the travel length are geometrically distributed, they show that the expected condition $\lambda/\mu < 1$ is actually sufficient; various results and conjectures on the system asymptotics when $N \to \infty$ and λN remains bounded are also stated.

In [3], the author relaxes the statistical assumptions imposed in [1] for the stability analysis of the Cambridge ring in two ways. First, he allows the number of arrivals and travel length distributions to depend on the queues; that is, he only assumes that, for $0 \le i \le N-1$, the arrivals $\delta_i(n), n \in \mathbb{N}$, are i.i.d. with mean λ_i, and the $\sigma_i(p), p \ge 1$, are i.i.d. with mean $1/\mu_i$, and independent of the arrivals; besides, arrivals and travel lengths at different queues are assumed independent. Second, he allows any distributions, not necessarily geometric, for the travel lengths.

In this context, it is not even trivial to guess the right stability conditions. The elementary argument used in [1] to obtain condition $\lambda/\mu < 1$, which consists in considering the mean number of customers on the ring in steady state, would only yield the necessary condition:

$$\sum_{i=0}^{N-1} \frac{\lambda_i}{\mu_i} < N,$$

which is far from being sufficient in general. In fact, his analysis shows that stability does not only depend on the values of the parameters λ_i, μ_i with $0 \le i \le N-1$, but more deeply on the specific distributions of the travel lengths; that is, given two Cambridge rings with the same parameter values, one may be stable and the other unstable (i.e., transient). This point markedly distinguishes the Cambridge ring from most queueing networks.

In Section 2, we recall two important results: Theorem 2.1 is due to V. Dumas [4], it gives a necessary and sufficient condition of stability of the Cambridge ring, thanks to the fluid limit approch; Theorem 2.2 due to Coffman et al [2], gives an asymptotic behavior of the average number of customers in each queue comprising the network. It shows that if the number of cells is very high, then the average number of customers in each queue tends towards zero.

Our result (Section 3) is rather different. Using probabilistic technics we give an upper bound for $\mathbb{E}(Q)$, and this for any number of cells.

Notations. For $x \in \mathbb{R}$, $\lceil x \rceil$ (respectively $\lfloor x \rfloor$) denotes the only integer n such that $n-1 < x \leq n$ (respectively $n \leq x < n+1$).

2. Markovian Structure-Traffic Intensities

Consider the system at time n, just after the pickups, and call the cell in front of queue i "cell i". Let $Q_i(n)$ be the size of queue i and $L_i(n)$ the occupant of cell i's residual travel length ($L_i(n) = 0$, means that cell i is about to go off again empty).

Set $Q(n) = (Q_i(n))_{0 \leq i \leq N-1}$, and $L(n) = (L_i(n))_{0 \leq i \leq N-1}$. The system state at time n is defined by $X(n) = (Q(n), L(n))$. It is easy to check that the sequence $X = (X(n))_{n \in \mathbf{N}}$ forms a discrete time Markov chain with (discrete) state space $\mathbf{Z}_+^N \times \mathbf{Z}_+^N$. This chain is not necessarily irreducible nor aperiodic in general. The above introduction of "cell i" is not only a natural trick to define the system state, but proves to be essential for the stability analysis of the Cambrige ring; that is, the right point of view consists in considering that each queue i is assigned a virtual server, namely cell i, and that each customer, once admitted on the ring, sequentially visits a random number of servers, each time asking for a service which lasts exactly one unit of time.

What is the total service time required at cell j by the p-th customer of queue i? It is given by

$$\sigma_{ij}(p) = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\sigma_i(p) > [j-i] + nN\}} = \lceil \frac{\sigma_i(p) - [j-i]}{N} \rceil.$$

Its expectation will be denoted $1/\mu_{ij}$.

With each queue j, we may now associate the cumulative service time required at cell j by all the customers present in the system at time n. We call it the potential load of cell j, and denote it by $W_j(n)$. This is a random function of $X(n)$, but by the law of large numbers:

$$\frac{1}{n}\left(W_j(n) - \sum_{i=0}^{N-1} \frac{Q_j(n)}{\mu_{ij}}\right) \longrightarrow 0 \text{ a.s. when } n \longrightarrow \infty. \tag{1}$$

There is a natural decomposition of $W_j(n)$ as:

$$W_j(n) = \sum_{i=0}^{N-1} Q_{ij}(n) - B_j(n), \tag{2}$$

where $Q_{ij}(n)$ is the cumulative service time required at cell j by customers arrived to the queue i up to time n (including the possible customer in cell i at time 0), and $B_j(n)$ is the occupation time of cell j up to time n. By the law of large numbers again:

$$\frac{1}{n}\left(Q_{ij}(n) - \frac{N_i(n)}{\mu_{ij}}\right) \longrightarrow 0 \text{ and } \frac{N_i(n)}{n} \longrightarrow \lambda_i \text{ a.s. When } n \longrightarrow \infty,$$

where $N_i(n) = \sum_{m=0}^{n-1} \delta_i(m)$. In consequence

$$\frac{Q_{ij}(n)}{n} \longrightarrow \frac{\lambda_i}{\mu_{ij}} \text{ a.s. when } n \longrightarrow \infty\,. \tag{3}$$

On the other hand,

$$B_j(n) = \sum_{m=0}^{n-1} \mathbf{1}_{\{L_j(m)>0\}}. \tag{4}$$

There is now a natural way to define traffic intensities for the Cambridge ring.

Definition 2.1. We call

$$\rho_i = \sum_{i=0}^{N-1} \frac{\lambda_i}{\mu_{ij}}$$

the traffic intensity of queue j, $0 \leq j \leq N-1$.

Theorem 2.1. (see [4]) *If we assume that*

$$\mathbb{P}(\delta_i(n) = 0) > 0,\ \ 0 \leq i \leq N-1,$$

then the Cambridge ring is stable if and only if

$$\rho_i < 1,\ \ 0 \leq i \leq N-1.$$

Let Q_j be the size of queue j , $0 \leq j \leq N-1$ in stationary regime, then we have the following result.

Theorem 2.2. (see [2]) *Let us assume that* $\delta_j(n) \sim \mathcal{B}(\frac{\lambda}{N})$ *(Bernoulli) and* $\sigma_j(p) \sim \mathcal{G}(\frac{\mu}{N})$ *(geometric).*

If $\lambda < \mu$, *then*

$$\boldsymbol{E}(Q) = \Theta(\frac{1}{N}) \text{ when } \quad N \longrightarrow \infty.$$

3. Upper Bound for the Average Number of Customers in a Queue

Now we arrive to our principal result; Theorem 3.1 gives an upper bound for the average number of customers in each queue, moreover it allows us to have an improvement of Theorem 2.2 ; Corollary 3.1 gives an explicit formula for $\mathbb{E}(Q)$ when N becomes infinite.

Theorem 3.1. *Under the assumptions of the preceding theorem, we have*

$$\mathbb{E}(Q) \leq \lambda \frac{(1-\frac{\mu}{N})}{(\mu-\lambda)} \frac{\lambda}{\mu}.$$

Proof. The function f is defined by

$$f : \{0, 2, ..., N-1\} \longrightarrow \{0, 2, ..., N-1\}$$
$$j \longrightarrow f(j) = \begin{cases} j+1, & \text{if } j < N-1, \\ 0, & \text{if } j = N-1. \end{cases}$$

Let $f^0 = I_{\{0,2,\dots,N-1\}}$ (identical function), and for all $n \geq 1$,

$$f^n = f \circ f^{n-1} \text{ thus } f^0 = f^{N-1}.$$

For all n, $n \geq 1$, let us define the events A_k, B_k, $k \in \mathbb{N}$ by

$$A_k = \{Q_j(n-1) = k,\ L_{f^{N-2}(j)}(n-1) = 0\},$$

$$B_k = \{Q_j(n-1) = k,\ L_{f^{N-2}(j)}(n-1) > 0\}.$$

For all j, $0 \leq j \leq N-1$ we have:

$$\{Q_j(n) = 0\} = \{Q_j(n) = 0, Q_j(n-1) = 0\} \cup \{Q_j(n) = 0, Q_j(n-1) = 1\}.$$

Conditionning with respect to

$$\{L_{f^{N-2}(j)}(n-1)>0\} \text{ and } \{L_{f^{N-2}(j)}(n-1)=0\}$$

yields

$$(Q_j(n)=0)=(Q_j(n)=0)\cap(A_0\cup B_0\cup A_1\cup B_1),$$

and

$$\begin{aligned} q_0=\mathbb{P}(Q_j(n)=0)&=\mathbb{P}\left((Q_j(n)=0)\cap(A_0\cup B_0\cup A_1\cup B_1)\right)\\ &=\sum_{i=0}^{1}\mathbb{P}(Q_j(n)=0/A_i)\mathbb{P}(A_i)+\sum_{i=0}^{1}\mathbb{P}(Q_j(n)=0/B_i)\mathbb{P}(B_i). \end{aligned}$$

Let

$$\begin{aligned} q_0=\mathbb{P}(A_0)+[(1-\frac{\lambda}{\mu})+\frac{\lambda}{N}\frac{\mu}{N}]\mathbb{P}(B_0)+(1-\frac{\lambda}{N})\mathbb{P}(A_1)&\\ +(1-\frac{\lambda}{\mu})\frac{\mu}{N}\mathbb{P}(B_1).& \quad (5) \end{aligned}$$

In stationary regime the law of Q_j does not depend on time, i.e.

$$\mathbb{P}\left(Q_j(n)=0\right)=\mathbb{P}\left(Q_j(n-1)=0\right).$$

Thus

$$\begin{aligned} q_0=\mathbb{P}(Q_j(n-1)=0,L_{f^{N-2}(j)}(n-1)=0)&\\ +\mathbb{P}(Q_j(n-1)=0,L_{f^{N-2}(j)}(n-1)>0)=\mathbb{P}(A_0)+\mathbb{P}(B_0).& \quad (6) \end{aligned}$$

From equations (5) and (6) we have

$$(\frac{\mu}{N}-\frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_1)-(\frac{\lambda}{N}-\frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_0)+(1-\frac{\lambda}{N})\mathbb{P}(A_1)=0. \quad (7)$$

If $k\geq 1$, then

$$\begin{aligned} (Q_j(n)=k)&=(Q_j(n)=k,Q_j(n-1)=k-1)\\ &\cup(Q_j(n)=k,Q_j(n-1)=k)\cup(Q_j(n)=k,Q_j(n-1)=k+1)\\ &=((Q_j(n)=k)\cap(A_{k-1}\cup A_k\cup A_{k+1}))\\ &\cup((Q_j(n)=k)\cap(B_{k-1}\cup B_k\cup B_{k+1})), \end{aligned}$$

and

$$q_k = \mathbb{P}(Q_j(n) = k) = \sum_{i=k-1}^{k+1} \mathbb{P}((Q_j(n)) = k)/A_i)\mathbb{P}(A_i)$$
$$+ \sum_{i=k-1}^{k+1} \mathbb{P}((Q_j(n) = k)/B_i)\mathbb{P}(B_i) = (\frac{\lambda}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_{k-1}) + \frac{\lambda}{N}\mathbb{P}(A_k)$$
$$+ (1 - \frac{\lambda}{N} - \frac{\mu}{N} + 2\frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_k) + (1 - \frac{\lambda}{N})\mathbb{P}(A_{k+1})$$
$$+ (\frac{\mu}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_{k+1})\,. \quad (8)$$

In stationary regime we have

$$\mathbb{P}(Q_j(n) = k) = \mathbb{P}(Q_j(n-1) = k)), \text{ for all } k \geq 1$$

and

$$q_k = \mathbb{P}(Q_j(n-1) = k, L_{f^{N-2}(j)}(n-1) = 0)$$
$$+ \mathbb{P}(Q_j(n-1) = k, L_{f^{N-2}(j)}(n-1) > 0).$$

Let

$$q_k = \mathbb{P}(A_k) + \mathbb{P}(B_k). \quad (9)$$

Equations (8) and (9) give

$$(\frac{\lambda}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_{k-1}) + (\frac{\lambda}{N} - 1)\mathbb{P}(A_k) + (-\frac{\lambda}{N} - \frac{\mu}{N} + 2\frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_k)$$
$$+ (1 - \frac{\lambda}{N})\mathbb{P}(A_{k+1}) + (\frac{\mu}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_{k+1}) = 0, \quad (10)$$

Let

$$(\frac{\mu}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_{k+1}) - (\frac{\lambda}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_k) + (1 - \frac{\lambda}{N})\mathbb{P}(A_{k+1})$$
$$= (\frac{\mu}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_k) - (\frac{\lambda}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_{k-1}) + (1 - \frac{\lambda}{N})\mathbb{P}(A_k)\,. \quad (11)$$

By iteration in equation (11) and from equation (7), we have

$$(\frac{\mu}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_k) - (\frac{\lambda}{N} - \frac{\lambda}{N}\frac{\mu}{N})\mathbb{P}(B_{k-1}) + (1 - \frac{\lambda}{N})\mathbb{P}(A_k) = 0. \quad (12)$$

Finally, from equations (9) and (12), we have

$$q_k = \frac{\lambda}{N}\frac{(1-\frac{\mu}{N})}{(1-\frac{\lambda}{N})}\mathbb{P}(B_{k-1}) + (1-\frac{\mu}{N})\mathbb{P}(B_k).$$

The average number of customers in each queue is given by

$$\begin{aligned}\mathbb{E}(Q_j) = \sum_{k=1}^{\infty} kq_k &= \frac{\lambda}{N}\frac{(1-\frac{\mu}{N})}{(1-\frac{\lambda}{N})}\sum_{k=1}^{\infty} k\mathbb{P}(B_{k-1}) + (1-\frac{\mu}{N})\sum_{k=1}^{\infty} k\mathbb{P}(B_k) \\ &= [(1-\frac{\mu}{N}) + \frac{\lambda}{N}\frac{(1-\frac{\mu}{N})}{(1-\frac{\lambda}{N})}]\sum_{k=1}^{\infty} k\mathbb{P}(B_k) + \frac{\lambda}{N}\frac{(1-\frac{\mu}{N})}{(1-\frac{\lambda}{N})}\sum_{k=0}^{\infty}\mathbb{P}(B_k).\end{aligned}$$

Let Y_n be the number of occupied cells at time n. In stationary regime, the application of Little's formula shows that the average number of customers in the cells is given by

$$\mathbb{E}(Y_{n-1}) = N\mathbb{P}(L_{f^{N-2}(j)}(n-1) > 0) = \lambda\mathbb{E}(\sigma) = \frac{\lambda}{\mu}N,$$

and

$$\mathbb{P}(L_{f^{N-2}(j)}(n-1) > 0) = \frac{\lambda}{\mu}.$$

Since

$$\mathbb{P}(B_k) = \mathbb{P}(Q_j(n-1) = k, L_{f^{N-2}(j)}(n-1) > 0),$$

the average number of customers in each queue is given by

$$\begin{aligned}\mathbb{E}(Q_j) = \frac{(1-\frac{\mu}{N})}{(1-\frac{\lambda}{N})}\sum_{k=1}^{\infty} k(Q_j(n-1) = k, L_{f^{N-2}(j)}(n-1) > 0) \\ + \frac{\lambda}{N}\frac{(1-\frac{\mu}{N})}{(1-\frac{\lambda}{N})}\frac{\lambda}{\mu}. \qquad (13)\end{aligned}$$

On the other hand, the sum in (13) is bounded from above by $\mathbb{E}(Q_j)$, thus

$$\mathbb{E}(Q_j) \leq \frac{\lambda(1-\frac{\mu}{N})}{(1-\frac{\lambda}{N})}\frac{\lambda}{\mu}.$$

Using equation (13) and the result of Theorem 2.2, and noticing that

$$\frac{(1-\frac{\mu}{N})}{(1-\frac{\lambda}{\mu})}\sum_{k=1}^{\infty} k\mathbb{P}(Q_j(n-1), L_{f^{N-2}(j)}(n-1) > 0) < \mathbb{E}(Q_j),$$

we obtain the following result.

Corollary 3.1. *Under the assumptions of the preceding theorem, and when N becomes infinite we have*

$$\mathbb{E}(Q_j) \simeq \frac{\lambda}{N} \frac{(1 - \frac{\mu}{N})}{(1 - \frac{\mu}{N})} \frac{\lambda}{\mu}.$$

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