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TORSIONAL OSCILLATIONS OF A DISK IN A SECOND GRADE FLUID BOUNDED BY A POROUS MEDIUM

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Abstract: The flow due to torsional oscillations of an infinite disk at a small distance from the unbounded porous medium has been discussed when the entire space between the disk and the porous medium is filled with a second grade fluid. It is assumed that the flow between the disk and the porous medium is governed by the equation of motion of the second grade fluid and that in the porous medium by modified Brinkman equation. The solution has been obtained by expanding all the entities in the powers of the amplitude of oscillations which is assumed to be small. It is observed that with the increase of non-Newtonian parameter the steady radial and the axial velocity components increase and the amplitude of the transverse component of the velocity decreases in the entire region. The oscillations of the disk induce a steady radial-axial flow in both the regions in such a way that there is a steady axial flow of the fluid from the porous medium to the free flow region, i.e. the fluid is expelled out from the porous medium. The effect of non-Newtonian parameter is to increase the steady axial flow from porous medium to free flow region.

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1. Introduction

The flow of a viscous fluid over and through a porous medium has been subject of intensive studies because of its importance in many engineering problems and its natural occurence in the flow of rivers through porous banks. In many cases the moving fluid is non-Newtonian, such as the flow of crude oil through porous rocks, the motion of synovial fluid bounded by porous cartilage in joints of a human body and the flow of muddy water near the banks of the river. Non-Newtonian behaviour is explained by the following constitutive equation of a second grade fluid proposed by Truesdell and Rajgopal [16].

$$\boldsymbol{\tau} \;=\; -\,pI + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2{\mathbf{A}_2}^2 \tag{1}$$

where $\boldsymbol{\tau}$ is the stress tensor, p is the pressure, μ is the coefficient of viscosity, α_1 and α_2 are material moduli, $\mathbf{A}_1$ and $\mathbf{A}_2$ are kinematical tensors defined by:

$$\begin{aligned} \mathbf{A}_1 &= (\mathrm{grad}\,\mathbf{V}) + (\mathrm{grad}\,\mathbf{V})^{Trans}, \\ \mathbf{A}_2 &= \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\mathrm{grad}\,\mathbf{V}) + (\mathrm{grad}\,\mathbf{V})^{Trans}\,\mathbf{A}_1\ , \end{aligned} \tag{2}$$

where $\mathbf{V}$ is the velocity vector. Dunn and Fosdick [2] have found that is the fluid is compatible with thermodynamics in the sense that Clasius-Duhem inequality by met in all the motions then

$$\mu \;\geq\; 0 \;\;\text{and}\;\; \alpha_1 + \alpha_2 \;=\; 0. \tag{3}$$

They also found that $\alpha \geq 0$. A discussion on the constants is found in a review article by Dunn and Rajgopal [3]. Cauchy's equation of motion can be written as:

$$\rho\left[\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] \;=\; \nabla\cdot\boldsymbol{\tau}, \tag{4}$$

where ρ is the density of the fluid. The equations of motion of a second grade fluid is given by substituting the stress $\boldsymbol{\tau}$ from (1) and (2) in (4). The torsional oscillations of a plate in Newtonian fluids have been discussed by Rosenblat [8]. He obtained the solution by expanding velocity components and the pressure in the powers of the amplitude of oscillations of the plate. Rosenblat [9] has also discussed the case when the fluid is confined between two torsionally plates. Srivastava [12] has studied these problems for a second order fluid. Srivastava [14] has also discussed the flow of a viscous fluid confined between a torsionally oscillating disk and a porous medium fully saturated with the fluid. In this paper we have studied the torsional oscillations of a disk in a second grade fluid bounded by a porous medium and obtained the solution by expanding velocity components and the pressure in powers of the amplitude of the oscillations. The flow in the region between the disk and porous interface is assumed to

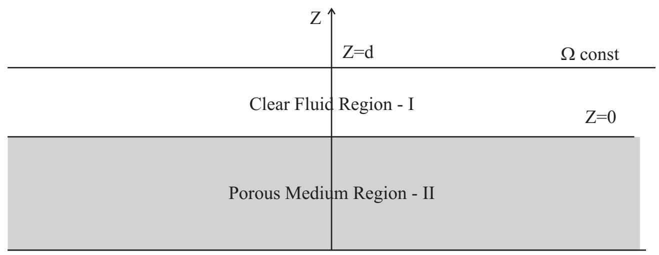

Figure 1:

be governed by the equations of a motion of a second grade fluid and that in the porous region by modified Brinkman equation [1]. The two flows are matched at the interface by the matching conditions proposed by Ochoa-Tapia and Whitaker [6]. Such coupled boundary value problems of this type have been solved by many scientists [4, 5, 7, 10].

2. Statement of the Problem

Consider the flow of a second grade fluid confined between an impervious disk performing torsional oscillations of frequency n with angular speed Ω and a porous medium of infinite extent. Let (r, θ, z) be a set of cylindrical polar coordinates and let the oscillating disk be represented by the plane $z = d$, the interface by $z = 0$ and the porous region by $z \leq 0$ (see Figure 1). The region between $z = 0$ and $z = d$ is called zone I in which fluid flows freely and its motion is governed by constitutive equations (1) and (2). Zone II is the region $z \leq 0$ in which the fluid flows through the pores of the porous medium and the motion is governed by modified Brinkman equation proposed in this paper. Let the superscript in the bracket of an entity $\chi^{(i)}$, $i = 1, 2$ denote the zone to which the entity belongs. Assuming u, v, w as the velocity components in the direction of r, θ, z respectively the boundary conditions of the problem in this notation are

$$u^{(1)} = 0, \quad v^{(1)} = r\Omega \cos nt, \quad w^{(1)} = 0 \quad \text{on} \quad z = d, \tag{5}$$

$$u^{(2)} \to 0, \quad v^{(2)} \to 0 \quad \text{on} \quad z \to -\infty. \tag{6}$$

Matching conditions of two flows at the interface of clear fluid and porous medium have been investigated by Ochoa-Tapia and Whitaker [6] who have shown that the equations require a discontinuity in shearing stresses but continuity of the velocity components and normal stress at the interface. These conditions in our notation can be written as:

$$u^{(1)} \;=\; u^{(2)} \;\; v^{(1)} \;=\; v^{(2)}, \;\; w^{(1)} \;=\; w^{(2)} \;\; \text{at} \;\; z \;=\; 0, \tag{7}$$

$$\tau_{zz}^{(1)} \;=\; \tau_{zz}^{(2)} \;\; \text{at} \;\; z = 0, \tag{8}$$

$$\tau_{z\theta}^{(1)} - \tau_{z\theta}^{(2)} \;=\; \frac{\beta\mu}{\sqrt{k}}\, v^{(1)} \;\; \text{at} \;\; z = 0, \tag{9}$$

$$\tau_{zr}^{(1)} - \tau_{zr}^{(2)} \;=\; \frac{\beta\mu}{\sqrt{k}}\, u^{(1)} \;\; \text{at} \;\; z = 0, \tag{10}$$

where τ_{zz}, $\tau_{z\theta}$, τ_{zr} are stress components and β is a dimensionless constant of order one whose sign may be positive or negative.

3. Equations of Motion

The flow in zone I will be discussed first and in the zone II later. In zone I we assume the velocity components in the following form:

$$u^{(1)} \;=\; r\Omega f'(\eta, T), \;\; v^{(1)} \;=\; r\Omega g(\eta, T), \;\; w^{(1)} \;=\; -\,2d\Omega f(\eta, T), \tag{11}$$

where

$$\eta \;=\; z/d, \;\; T \;=\; nt, \;\; \Omega \;=\; n\epsilon \tag{12}$$

and ϵ is the amplitude of oscillations of the plate. Here a prime denotes partial differentiation with respect to η. Writing (1) and (2) in cylindrical polar coordinates and substituting the expressions of the velocity components from (11), stress components can be written as:

$$\tau_{zz}^{(1)} = -p^{(1)} + \mu\Omega \times \left[-4f' + \alpha\epsilon\left\{-4\,\frac{\partial f'}{\partial T} + 16(f')^2 + 8ff'' + \left(\frac{r}{d}\right)^2((f'')^2 + (g')^2)\right\}\right], \tag{13}$$

$$\tau_{zr}^{(1)} = \mu\Omega\left(\frac{r}{d}\right)\left[f'' + \alpha\left\{\frac{\partial f''}{\partial T} + 2\epsilon(f'f''' - ff^{IV} + 2(f'')^2)\right\}\right], \tag{14}$$

$$\tau_{z\theta}^{(1)} = \mu\Omega\left(\frac{r}{d}\right)\left[g' + \alpha\left\{\frac{\partial g'}{\partial T} + 2\epsilon(f'g' - fg'')\right\}\right], \tag{15}$$

$$\tau_{rr}^{(1)} = -p^{(1)} + \mu\Omega\left[2f' + \alpha\epsilon\left\{2\,\frac{\partial f'}{\partial T} - \epsilon\left(ff'' + \left(\frac{r}{d}\right)^2(f'')^2\right)\right\}\right], \tag{16}$$

$$\tau_{\theta\theta}^{(1)} = -p^{(1)} + \mu\Omega\left[2f' + \alpha\epsilon\left\{2\,\frac{\partial f'}{\partial T} - \epsilon\left(ff'' + \left(\frac{r}{d}\right)^2(g')^2\right)\right\}\right], \tag{17}$$

$$\tau_{r\theta}^{(1)} = -\mu\Omega\alpha\epsilon\left(\frac{r}{d}\right)^2 f''g', \tag{18}$$

where $\alpha = \alpha_1 n/\mu$ is a dimensionless non-Newtonian parameter representing the effect of second grade fluid. These expression for stress components substituted in the equations of motion suggest the following form for the pressure $p^{(1)}$ (see Srivastava [12]):

$$p^{(1)} = \mu\Omega\left[-p_1(z) + \alpha\epsilon\left(\frac{r}{d}\right)^2((f'')^2 + (g')^2)\right]. \tag{19}$$

Writing (4) in cylindrical polar coordinates and then substituting velocity components from (11), stress components from (13)-(18) and pressure from (19) we get the following equations of motion in the direction of r, θ and z respectively:

$$R_e\left[\frac{\partial f'}{\partial T} + \epsilon((f')^2 - g^2 - 2ff'')\right]$$
$$= f''' + \alpha\left[\frac{\partial f'''}{\partial T} - \epsilon(2ff^{IV} + (f'')^2 + (g')^2 - 2ff''')\right], \tag{20}$$

$$R_e\left[\frac{\partial g}{\partial T} + 2\epsilon(f'g - fg')\right] = g'' + \alpha\left[\frac{\partial g''}{\partial T} + 2\epsilon(f'g'' - fg''')\right], \tag{21}$$

$$R_e\left[-2\frac{\partial f}{\partial T} + 4\epsilon ff'\right] = \frac{dp_1}{d\eta} - 2f'' + 2\alpha\left[-2\frac{\partial f''}{\partial T} + 2\epsilon(ff''' + 4f'f'')\right], \tag{22}$$

where $R_e = \rho n d^2/\mu$ is the Reynolds number.

Now we will consider the motion of fluid in zone II. Assume the following form of the velocity components and pressure for this zone:

$$u^{(2)} = r\Omega F'(\eta, T), \quad v^{(2)} = r\Omega G(\eta, T), \quad w^{(2)} = -2d\Omega F(\eta, T), \tag{23}$$

$$p^{(2)} = -\mu\Omega P_1(\eta, T). \tag{24}$$

In the porous medium the magnitudes of the oscillating components of velocity are small. Replacing f, g, h by F, G, H respectively and neglecting the squares of velocity except the terms giving the centrifugal force and adding a term accounting for the resistance of flow in porous medium the equations (20)-(22) yield the following equations of motion in r, θ and z directions respectively for this zone:

$$R_e\frac{\partial F'}{\partial T} - R_e G^2 = F''' + \alpha\left[\frac{\partial F'''}{\partial T} - (G')^2\right] - \sigma^2 F', \tag{25}$$

$$R_e\frac{\partial G}{\partial T} = G'' + \alpha\frac{\partial G''}{\partial T} - \sigma^2 G, \tag{26}$$

$$-R_e \frac{\partial F}{\partial T} = -2F'' - \alpha \frac{\partial F''}{\partial T} + 2\sigma^2 F + \frac{dP_1}{d\eta}. \quad (27)$$

The equation (25)-(27) may be taken as the modified form of Brinkman equation for the flow of second grade fluid in the porous medium in which rotation is involved. To solve the problem we adopt the complex notation with the convention that only the real parts of the complex quantities represent physical quantities. In this notation the boundary conditions (5) and (6) become:

$$f = f' = 0, \quad g = e^{iT} \quad \text{at} \quad \eta = 1, \quad (28)$$

$$F' \to 0, \qquad G \to 0 \qquad \text{as} \quad \eta \to \infty. \quad (29)$$

Matching conditions (7) and (10) using (11), (23) and the expression of stress from (14) at the interface are:

$$f = F \quad f' = F' \quad \text{at} \quad \eta = 0,$$
$$f'' - F'' = \beta\sigma F' \qquad \text{at} \quad \eta = 0. \quad (30)$$

The stress components in the porous region have been obtained by replacing f, g by F, G respectively and neglecting the square of velocities in the expressions for stress components from (13)-(18). Now a solution of the equations is sought by expanding all the entities of the amplitude of oscillation ϵ which is assumed to be small as:

$$\chi_i = \chi_0 + \epsilon\chi_1 + \epsilon^2\chi_2 + \dots,$$

where χ stands for f, g, p_1, F, G, P_1.

4. First Approximate Solution

For the first approximation the following form of the functions are assumed:

$$f_0 = f_0^*(\eta)e^{iT}, \quad g_0 = g_0^*(\eta)e^{iT}, \quad p_{10} = p_{10}^*(\eta)e^{iT}, \quad (31)$$

$$F_0 = F_0^*(\eta)e^{iT}, \quad G_0 = G_0^*(\eta)e^{iT}, \quad P_{10} = P_{10}^*(\eta)e^{iT}. \quad (32)$$

Substituting functions in the equations of motion and using boundary and matching conditions from (28), (29) and (30) respectively we find that (see Srivastava [14])

$$f_0^* = F_0^* = 0. \quad (33)$$

Substituting (31), (32) in equations (21) and (26) respectively, we get the following equations for g_0^* and G_0^*

$$\frac{d^2 g_0^*}{d\eta^2} - \frac{iR_e g_0^*}{1+i\alpha} \;=\; 0. \tag{34}$$

$$\frac{d^2 G_0^*}{d\eta^2} - \frac{\sigma^2 + iR_e}{1+i\alpha}\, G_0^* \;=\; 0. \tag{35}$$

Substituting (31) and (32) in (28) and (29) respectively we get the following boundary conditions:

$$g_0^* \;=\; 1 \;\;\text{at}\;\; \eta \;=\; 1\,, \tag{36}$$

$$G_0^* \to 0 \;\;\text{as}\;\; \eta \to \infty. \tag{37}$$

Substituting (31) and (32) in $\tau_{r\theta}^{(1)}$ and the corresponding stress component in the porous medium the matching conditon (9) yields the following equations:

$$g_0^* \;=\; G_0^* \;\;\text{at}\;\; \eta \;=\; 0\,, \tag{38}$$

$$\frac{dg_0^*}{d\eta} - \frac{dG_0^*}{d\eta} \;=\; \frac{\beta\sigma(1-i\alpha)}{1+\alpha^2}\, g_0^* \;\;\text{at}\;\; \eta \;=\; 0. \tag{39}$$

Solutions of (34) and (35) satisfying the boundary conditions (36), (37) and the matching conditions (38) and (39) are given by

$$g_0^*(\eta) = \frac{(1+\alpha^2)(A_1+iA_2)\cosh(A_1+iA_2)\eta + (K_1+iK_2)\sinh(A_1+iA_2)\eta}{(1+\alpha^2)(A_1+iA_2)\cosh(A_1+iA_2) + (K_1+iK_2)\sinh(A_1+iA_2)}, \tag{40}$$

$$G_0^*(\eta) = \frac{(1+\alpha^2)(A_1+iA_2)e^{(B_1+iB_2)\eta}}{(1+\alpha^2)(A_1+iA_2)\cosh(A_1+iA_2) + (K_1+iK_2)\sinh(A_1+iA_2)}, \tag{41}$$

where

$$\left.\begin{aligned}
A_1 &= \left[\frac{R_e}{2} + \frac{\alpha}{2(1+\alpha^2)}\right]^{1/2}, \quad A_2 = \left[\frac{R_e}{2} - \frac{\alpha}{2(1+\alpha^2)}\right]^{1/2}, \\
B_1 &= \left[\frac{\sigma^4 + R_e^2}{(1+\alpha^2)}\right]^{1/2} \cos\frac{\theta}{2}, \quad B_2 = \left[\frac{\sigma^4 + R_e^2}{(1+\alpha^2)}\right]^{1/2} \sin\frac{\theta}{2}, \\
\tan\theta &= \frac{R_e - \sigma^2\alpha}{\sigma^2 + \alpha R_e}, \\
K_1 &= B_1(1+\alpha^2) - \sigma\beta, \quad K_2 = B_2(1+\alpha^2) + \sigma\beta\alpha.
\end{aligned}\right\} \tag{42}$$

5. Second Approximate Solution

To calculate the second approximation g_1 and G_1 we have to substitute f_0 and F_0 respectively in the equation (21) and (26) but they are zero hence calculations give:

$$g_1(\eta, T) \;=\; G_1(\eta, T) \;=\; 0. \tag{43}$$

To calculate $f_1(\eta, T)$ we have to substitute $[\mathrm{Real}\, g_0(\eta, T)]^2$ which is a complex function and consists of the following two parts:

$$[\mathrm{Real}(g_0^* e^{iT})]^2 \;=\; \frac{1}{2} g_0^* \bar{g}_0^* + \mathrm{Real}(\frac{1}{2}\, g_0^* e^{2iT}), \tag{44}$$

where $\bar{g}_0^*$ is the complex conjugate of g_0^*. The equation (44) suggests that we should decompose the radial and the axial velocity components into steady and unsteady parts, hence we write

$$f_1(\eta, T) \;=\; f_{1s}(\eta) + f_{1t}(\eta) e^{2iT}. \tag{45}$$

Flow induced in the porous medium will have to match the flow in the clear fluid at the interface hence we assume the following form of $F_1(\eta, T)$ in the porous medium:

$$F_1(\eta, T) \;=\; F_{1s}(\eta) + F_{1t}(\eta) e^{2iT}. \tag{46}$$

Substituting $f_1(\eta, T)$ from (45) and $g_0(\eta, T)$ from (31) in (20) we get the following differential equations for f_{1s} and f_{1t}:

$$\frac{d^3 f_{1s}}{d\eta^3} \;=\; -\frac{R_e}{2} g_0^* \bar{g}_0^* + \frac{\alpha}{2} \frac{dg_0^*}{d\eta} \frac{d\bar{g}_0^*}{d\eta}, \tag{47}$$

$$(1 + 2i\alpha) f_{1t}''' \;=\; 2iR_e f_{1t}' + \frac{R_e}{2} (g_0^*)^2 + \frac{\alpha}{2} \frac{d^2 g_0^*}{d\eta^2}. \tag{48}$$

Substituting $F_1(\eta, T)$ from (46) and $G_0(\eta, T)$ from (32) in (25) we get the following differential equations for the flow in the porous region:

$$\frac{d^3 F_{1s}}{d\eta^3} - \sigma^2 \frac{dF_{1s}}{d\eta} \;=\; -\frac{R_e}{2} G_0^* \bar{G}_0^* + \frac{\alpha}{2} \frac{dG_0^*}{d\eta} \frac{d\bar{G}_0^*}{d\eta}, \tag{49}$$

$$(1 + 2i\alpha) F_{1t}''' - (\sigma^2 + 2iR_e) F_{1t}' \;=\; 0. \tag{50}$$

Boundary condition (28) for f_{1s} and f_{1t} are:

$$\frac{df_{1s}}{d\eta} \;=\; f_{1s} \;=\; 0 \qquad \text{at} \;\; \eta \;=\; 1, \tag{51}$$

$$f_{1t}' \;=\; f_{1t} \;=\; 0 \qquad \text{at} \;\; \eta \;=\; 1. \tag{52}$$

Matching condition (30) for f_{1s} and f_{1t} are:

$$f_{1s} = F_{1s} \quad \text{at} \quad \eta = 0, \tag{53}$$

$$\frac{df_{1s}}{d\eta} = \frac{dF_{1s}}{d\eta} \quad \text{at} \quad \eta = 0, \tag{54}$$

$$\frac{d^2 f_{1s}}{d\eta^2} - \frac{d^2 F_{1s}}{d\eta^2} = \beta\sigma \frac{df_{1s}}{d\eta} \quad \text{at} \quad \eta = 0, \tag{55}$$

$$f_{1t} = F_{1t} \quad \text{at} \quad \eta = 0, \tag{56}$$

$$f'_{1t} = F'_{1t} \quad \text{at} \quad \eta = 0, \tag{57}$$

$$(1+2i\alpha)f''_{1t} - F''_{1t} = \beta\sigma F'_{1t} \quad \text{at} \quad \eta = 0. \tag{58}$$

The solution of (47) is given as:

$$\begin{aligned}
&-f_{1s}(\eta) = \\
&\frac{R_e}{2}\left[\left(\frac{A_1^2+A_2^2}{2}\right)\left(\frac{\sinh 2A_1\eta}{8A_1^3} - \frac{\sin 2A_2\eta}{8A_2^3}\right) + \left(\frac{K_1^2+K_2^2}{2}\right)\left(\frac{\sinh 2A_1\eta}{8A_1^3}\right.\right. \\
&\left.\left.+\frac{\sin 2A_2\eta}{8A_2^3}\right) + \left(\frac{A_2K_1 - A_1K_2}{8A_2^3}\right)\cos 2A_2\eta + \left(\frac{A_1K_1 + A_2K_2}{8A_1^3}\right)\cosh 2A_1\eta\right] \\
&\qquad - \frac{\alpha}{2}\left[\frac{(A_1^2+A_2^2)^2}{2}\left(\frac{\sinh 2A_1\eta}{8A_1^3} + \frac{\sin 2A_2\eta}{8A_2^3}\right)\right. \\
&\qquad \left\{\frac{(A_1K_1 - A_2K_2)^2}{2} + \frac{(A_1K_2 + A_2K_1)^2}{2}\right\} \\
&\left(\frac{\sinh 2A_1\eta}{8A_1^3} - \frac{\sin 2A_2\eta}{8A_2^3}\right) + 2(A_1^2 - A_2^2)\left\{\left(\frac{A_1K_1 - A_2K_2}{8A_1^3}\right)\cosh 2A_1\eta\right. \\
&\left.+\left(\frac{A_2K_1 + A_1K_2}{8A_2^3}\right)\cos 2A_2\eta\right\} + 2A_1A_2\left\{\left(\frac{A_2K_1 + A_1K_2}{8A_1^3}\right)\cosh 2A_1\eta\right. \\
&\qquad\qquad \left.-\left(\frac{A_1K_1 - A_2K_2}{8A_2^3}\right)\cos 2A_2\eta\right\} + c_1\eta^2 + c_2\eta + c_3\,,
\end{aligned} \tag{59}$$

where c_1, c_2 and c_3 are constants of integration. The solution of equation (49) is given as:

$$\begin{aligned}
-F_{1s}(\eta) = &\left\{\frac{R_e(A_1^2+A_2^2)}{2B_1N(4B_1^2-\sigma^2)} + \frac{\alpha(A_1^2+A_2^2)B_1^2}{2B_1N(4B_1^2-\sigma^2)}\right\} e^{2B_1\eta} \\
&+ D_1 e^{\sigma\eta} + D_2\,,
\end{aligned} \tag{60}$$

where D_1, D_2 are constants of integration. In writing the solution (60) the constant of integration which makes the velocity infinite at large distance from the interface is taken to be zero. The five constants c_1, c_2, c_3, D_1, D_2 are determined by using two boundary and three matching conditions given in (53)-(55) respectively. Using expressions of f_{1s} and F_{1s} from (59) and (60) respectively the boundary condition (51) and the matching conditions (53)-(55) can be written as follows:

$$c_1 + c_2 + c_3 = E_1 \,, \tag{61}$$

$$2c_1 + c_2 = E_2 \,, \tag{62}$$

$$Q_1 + D_1 + D_2 = c_3 + E_3 \,, \tag{63}$$

$$2B_1 Q_1 + \sigma D_1 = c_2 + E_4 \,, \tag{64}$$

$$4B_1^2 Q_1 + \sigma^2 D_1 - 2c_1 = E_5 + (2B_1 Q_1 + D_1) \,. \tag{65}$$

The expressions for $E_1, E_2, E_3, E_4, E_5, Q_1$ are given in Appendix. Using equations (61)-(65) the values of the constants c_1, c_2, c_3, D_1, D_2 can be determined.

The solution of (48) is given as:

$$\begin{aligned} f'_{1t} = {} & L_1 \cosh(M_1 + iM_2)\eta + L_2 \sinh(M_1 + iM_2)\eta + \frac{1}{2} R_e\{1 - \alpha(A_1 + iA_2)^2\} \\ & [\{(A_1 + iA_2)^2 + (K_1 + iK_2)^2\} \cosh 2(A_1 + iA_2)\eta \\ & + \{(A_1K_1 - A_2K_2) + i(A_1K_2 + A_2K_1)\} \\ & + \{(A_1 + iA_2)^2 - (K_1 + iK_2)^2\} \sinh 2(A_1 + iA_2)\eta]/\text{denom}, \end{aligned} \tag{66}$$

where

$$\left.\begin{aligned} \text{denom} = {} & M[4R_e\{(A_1A_2 + K_1K_2) - i(K_1^2 - K_2^2)\} + 4R_e\{(A_1K_1 - A_2K_2) \\ & + i(A_1K_2 + A_2K_1) + (K_1^2 - K_2^2) + 2i(K_1K_2 - A_1A_2)\}] \\ M_1 = {} & \left(\frac{4R_e^2}{1 + 4\alpha^2}\right)^{1/4}, \quad M_2 = \left(\frac{4R_e^2}{1 + 4\alpha^2}\right)^{1/4}, \quad \tan\theta = 2\alpha \\ M = {} & \frac{1}{2}\{(A_1 + iA_2)^2 + (K_1 + iK_2)^2\} \cosh 2(A_1 + iA_2) + \{(A_1K_1 - A_2K_2) \\ & i(K_1K_2 - A_1A_2)\} \sinh 2(A_1 + iA_2) - \frac{1}{2}\{(K_1 + iK_2)^2 - (A_1 + iA_2)^2\} \end{aligned}\right\} \tag{67}$$

and L_1, L_2 are constants of integration.

The solution of equation (50) is given as:

$$F'_{1t} = J_1 e^{s\eta} + J_2 e^{-s\eta} \,, \tag{68}$$

where

$$s = s_1 + is_2, \quad s_1 = \left[\frac{\sigma^4 + 4R_e^2}{(1+4\alpha^2)}\right]^{1/2} \cos\frac{\theta}{2},$$
$$s_2 = \left[\frac{\sigma^4 + 4R_e^2}{(1+4\alpha^2)}\right]^{1/2} \sin\frac{\theta}{2}, \quad \tan\theta = \frac{2R_e - 2\sigma^2\alpha}{\sigma^4 + 4\alpha R_e} \tag{69}$$

and J_1, J_2 are constants of integration.

The constant J_2 is taken zero because it makes the velocity infinite at large distance and after integrating the equations (66) and (68) we get five constants which can be calculated using boundary conditions (52) and three matching conditions (56)-(58). We shall not calculate these constants here because it becomes too lengthy and has not much physical significance.

6. Transverse Component of the Velocity

Substituting (40) in (31) and using (11) we get the transverse component of velocity in the first approximation as follows:

$$\begin{aligned}\frac{\nu}{r\Omega} = \frac{1}{N}[&(A_1^2 + A_2^2 + K_1^2 + K_2^2)\{\cosh A_1(1+\eta)\cos A_2(1-\eta)\cos nt \\ &+ \sinh A_1(1+\eta)\sin A_2(1-\eta)\sin nt\} + [(A_1^2 + A_2^2 - K_1^2 - K_2^2)\{\cosh A_1(1-\eta) \\ &\cos A_2(1+\eta)\cos nt + \sinh A_1(1-\eta)\sin A_2(1+\eta)\sin nt\} + [(A_1K_1 + A_2K_2) \\ &\{\sinh A_1(1+\eta)\cos A_2(1-\eta)\cos nt + \cosh A_1(1+\eta)\cos A_2(1-\eta)\sin nt\} \\ &+ (A_1K_2 - A_2K_1)\{\cosh A_1(1-\eta)\sin A_2(1+\eta)\cos nt \\ &+ \sinh A_1(1-\eta)\cos A_2(1+\eta)\sin nt\}\,,\end{aligned} \tag{70}$$

where

$$\begin{aligned}N = &(A_1^2 + A_2^2 + K_1^2 + K_2^2)\cosh 2A_1 + (A_1^2 + A_2^2 - K_1^2 - K_2^2)\cosh 2A_2 \\ &+ (A_1K_1 + A_2K_2)\sinh 2A_1 + (A_2K_1 - A_1K_2)\sin 2A_2\,.\end{aligned} \tag{71}$$

For large values of R_e the equation (70) gives the behaviour of the transverse component of velocity as:

$$\frac{v^{(1)}}{r\Omega} = e^{-A_1(1-\eta)}\cos[nt - A_2(1-\eta)]\,. \tag{72}$$

The amplitude of this component of velocity is

$$\frac{|v|}{r\Omega} = e^{-A_1(1-\eta)}\,. \tag{73}$$

The values of A_1 are given in Table 1 for various values of non-Newtonian parameter α. Using the values of A_1 in the expression (73) it follows that the effect of non-Newtonian parameter α is to increase the amplitude of oscillations of transverse velocity.

Substituting (41) in (32) and using (23) we get the transverse component of the velocity in the porous medium as:

$$\begin{aligned}\frac{v^{(2)}}{r\Omega} &= \frac{e^{B_1\eta}}{N}[\{(1+\alpha^2)A_1^2 + A_2^2\}\cosh A_1 \cos A_2 + (A_1K_1 + A_2K_2)\sinh A_1 \cos A_2 \\ &+ (A_1K_2 - A_2K_1)\cosh A_1 \sin A_2]\cos(B_2\eta + nt) + [\{(1+\alpha^2)A_2^2 + A_1^2\}\sinh A_1 \sin A_2 \\ &+ (A_1K_1 + A_2K_2)\cosh A_1 \sin A_2 + (A_1K_2 - A_2K_1)\sinh A_1 \cos A_2]\sin(B_2\eta + nt)\,. \end{aligned} \tag{74}$$

The amplitude of oscillations of this component of velocity is given by

$$\frac{|v^{(1)}|}{r\Omega} = \sqrt{\frac{2(1+\alpha^2)(A_1^2+A_2^2)}{N}}\,e^{B_1\eta}. \tag{75}$$

This shows that the amplitude of oscillations of the transverse component of the velocity in the porous region is maximum at the interface $\eta = 0$ and is given by

$$Amp. = \sqrt{\frac{2(1+\alpha^2)(A_1^2+A_2^2)}{N}}. \tag{76}$$

Using the expression (76) the values of amplitude have been calculated by taking $\sigma = 3.0, 5.0,\ \beta = 0.2,\ R_e = 1.0, 2.0, 3.0$ and $\alpha = 0, 0.1, 0.2, 0.3, 0.4$ and are given in Table 2, which shows that the amplitude decreases with the increase of σ as well as with increase of non-Newtonian parameter α.

The oscillations of the transverse component of the velocity at the interface has a phase lag δ with the oscillation of the disk. The expression of δ is given as:

$$\begin{aligned}\delta = \tan^{-1}&[\{(1+\alpha^2)A_2^2 + A_1^2\}\sinh A_1 \sin A_2 + (A_1K_1 + A_2K_2)\cosh A_1 \sin A_2 \\ &+ (A_1K_2 - A_2K_1)\sinh A_1 \cos A_2]/ \\ &[\{(1+\alpha^2)A_1^2 + A_2^2\}\cosh A_1 \cos A_2 + (A_1K_2 + A_2K_1)\sinh A_1 \cos A_2 \\ &- (A_1K_2 - A_2K_1)\cosh A_1 \sin A_2]\,. \end{aligned} \tag{77}$$

Using expression (77) the value of δ have been calculated by taking $\sigma = 3.0, 4.0, 5.0,\ \beta = 0.2, -0.2,\ R_e = 3.0$ and $\alpha = 0, 0.1, 0.2, 0.3, 0.4$ and are given in Table 3, which reveals that δ decreases with the increase of σ as well as with increase of non-Newtonian parameter α.

7. Discussions and Conclusions

Now we discuss the behaviour of transverse component of velocity. From Table 2 we observe that the amplitude of oscillations of transverse component of velocity decreases with the increase of σ as well as with increase of non-Newtonian parameter α. Phase lag of oscillations of transverse component of velocity with that of disk decreases with the increase of σ as well as with increase of non-Newtonian parameter α.

The steady radial velocity component at the interface is given by substituting (60) in (23)

$$u^{(2)}(0) \;=\; r\Omega F'(0) \;=\; -\,r\Omega[Q_1 + D_1], \tag{78}$$

where the expression of Q_1 is given in Appendix. The flow in the porous medium in the radial direction has maximum value at the interface given by equation (78). The differentiation of equation (60) shows that radial flow in the porous medium decays exponentially as we enter inside it and vanish at a large distance from the interface.

Now we discuss the steady axial flow of a second grade fluid induced in the porous medium by a disk oscillating torsionally at a small distance from it. The equation (60) gives

$$F_1(\eta) \to -D_2 \quad \text{as} \quad \eta \to \infty, \tag{79}$$

which shows that the axial velocity component at large distance from the interface does not vanish. This component at large distance from the interface is given by

$$w^{(2)}(-\infty) \;=\; 2d\Omega D_2. \tag{80}$$

The values of (D_2/R_e) are given in Table 4 for $\sigma = 3.0, 4.0, 5.0$, $\beta = 0.2$ and $\alpha = 0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4$. The inspection of this table reveals that the axial velocity component in the porous medium is towards the interface (as D_2 is positive) and it decreases with the increase of σ but increases with the increase of non-Newtonian parameter α. For small values of R_e the graph of (F_{1s}/R_e) against the distance from the interface has been drawn in Figure 2 for $\sigma = 3.0, 4.0, 5.0$ and $\alpha = 0, 0.1, 0.2$. The graph reveals: (i) the axial velocity component increases as we enter the porous medium and attains a constant value at a large distance from the interface; (ii) the magnitude of axial velocity decreases with the increase of σ but increases with the increase of non-Newtonian parameter α. Hence it may be concluded that torsional oscillations of a disk near a porous medium fully saturated with the fluid extracts the fluid from the porous medium. This fact may be used by the geologists to extract

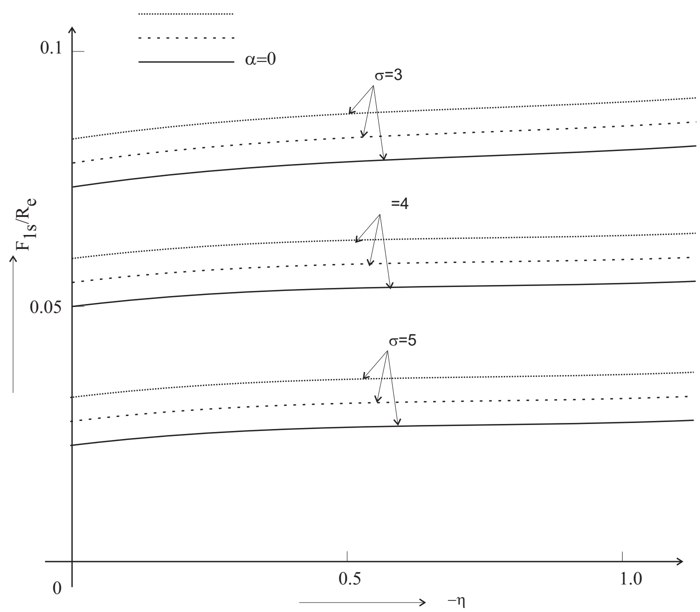

Figure 2:

fluid from porous ground or rocks and in this process the non-Newtonian fluid should be used as its efficiency is more. Motion of the water near the banks of a river may be rotatory as well as oscillatory. The erosion of the land will be more probable if the water near the banks of a river behaves as a non-Newtonian fluid by presence of mud and leaves. Hence it is suggested that the banks of rivers must be cleaned to avoid the land erosion.

For small values of Reynolds number R_e the shearing stress $\tau_{r\theta}^{(1)}$ in the clear fluid as the oscillating disk is given by

$$\begin{aligned}
[\tau_{z\theta}^{(1)}]_{z=d} \;=\; r\Omega\mu \Bigg[& \frac{\alpha+(1-\beta\sqrt{2})\sigma}{1+\alpha+(1-\beta\sqrt{2})\sigma} \\
& + \frac{R_e^2}{4\sigma^3(1+\alpha+(1-\beta\sqrt{2})\sigma)}\{4+8(1-\beta)\sigma^3\beta^2 \\
& - 2\sqrt{2}\sigma^2\beta+\alpha(-\sqrt{2}-\frac{3}{4\sqrt{2}}\sigma-\sigma\beta-\frac{3}{\sqrt{2}}\sigma^2\beta-\sigma^2\beta^2-\sigma^2)\cos nt\} \\
& + \frac{R_e}{2\sigma(1-\beta\sqrt{2})^2}\{2+2\sigma^2+\sqrt{2}\sigma-3\sigma\beta-\frac{3}{\sqrt{2}}\sigma^2\beta
\end{aligned}$$

$$+\alpha(-1+\frac{1}{2}\sigma+(\frac{3}{8}+\frac{3}{\sqrt{2}})\sigma^2\beta-(\frac{3}{4}+\frac{1}{2\sqrt{2}})\sigma\beta+(\frac{1}{2\sqrt{2}}-1)\sigma^2\beta)\sin nt\}]\,. \tag{81}$$

The amplitude of oscillations of the shearing stress is

$$\begin{aligned}|[\tau_{z\theta}^{(1)}]_{z=d}| \\ &= \frac{r(\alpha+(1-\beta\sqrt{2})\sigma)}{1+\alpha+(1-\beta\sqrt{2})\sigma}\Bigg[1+\frac{R_e^2}{8\sigma^4((1-\beta\sqrt{2})^2(1+\alpha+(1-\beta\sqrt{2})\sigma)} \\ &\quad \{8+16\sigma(1-\beta)+2\sigma^2\beta^2+\alpha(7-7\sigma^2\beta^2-13\sqrt{2}\sigma^3\beta \\ &\quad -(9+\frac{9}{\sqrt{2}})\sigma^2\beta+8\sigma^2+4\sigma^4-12\sigma^3\beta-\frac{12}{\sqrt{2}}\sigma^4\beta-\frac{9}{2}\sigma^4\beta^2)\}\Bigg]\,. \end{aligned} \tag{82}$$

This shows that when R_e^2 is negligible,

$$|[\tau_{z\theta}^{(1)}]_{z=d}| \;\cong\; r\Omega\left[1-\frac{1}{(1+\alpha+(1-\beta\sqrt{2})\sigma)}\right]. \tag{83}$$

Hence we conclude that for small values of R_e the shearing stress on the oscillating disk increases with the increse of σ as well as non-Newtonian parameter α. In zone II which is filled by porous medium we observe that for large values of R_e there is a boundary layer formation near the oscillating disk and the shearing stress $\tau_{z\theta}^{(2)}$ on it behaves as:

$$[\tau_{z\theta}^{(2)}]_{z=d} \;=\; r\Omega A_1(\cos nt-\sin nt)e^{-A_2}\,. \tag{84}$$

The amplitude of shearing stress is

$$[\tau_{z\theta}]_{z=d} \;\cong\; r\Omega\sqrt{2A_1^2}e^{-A_2}\,, \tag{85}$$

which is unaffected by the presence of porous medium but increases with the increase of non-Newtonian parameter α. This result may be observed from the Table 1. In human joints the motion of bone may be oscillatory as well as rotatory, the stresses in the synovial fluid which is bounded above by the bone and below by porous cartilage may be predicated by this analysis.

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Appendix

$$\begin{aligned}E_1 = \frac{1}{2}R_e[&(16A_1^3)^{-1}(A_1^2+A_2^2+K_1^2+K_2^2)\sinh 2A_1\\
&-(16A_2^3)^{-1}(A_1^2+A_2^2-K_1^2-K_2^2)\sin 2A_2\\
&+(8A_1^3)^{-1}(A_1K_1+A_2K_2)\cosh 2A_1+(8A_2^3)^{-1}(A_2K_1-A_1K_2)\cos 2A_2]\\
&-\frac{\alpha}{2}[(16A_1^3)^{-1}\{(A_1^2+A_2^2)^2\\
&+(A_1K_1-A_2K_2)^2\}\sinh 2A_1+(16A_2^3)^{-1}\{A_1^2+A_2^2)^2+(A_1K_2+A_2K_1)^2\}\sin 2A_1\\
&+(8A_2^3)^{-1}\{2(A_1^2-A_2^2)(A_1K_2+A_2K_1)-2A_1A_2(A_1K_1-A_2K_2)\}\cos 2A_2\\
&+(8A_1^3)^{-1}\{2(A_1^2-A_2^2)(A_1K_1-A_2K_2)-2A_1A_2(A_1K_2+A_2K_1)\}\cosh 2A_1]\,,\end{aligned}$$

$$\begin{aligned}E_2 = \frac{1}{2}R_e[&(8A_1^2)^{-1}(A_1^2+A_2^2+K_1^2+K_2^2)\cosh 2A_1\\
&-(8A_2^2)^{-1}(A_1^2+A_2^2-K_1^2-K_2^2)\cos 2A_2\\
&+(4A_1^2)^{-1}(A_1K_1+A_2K_2)\sinh 2A_1-(4A_2^2)^{-1}(A_2K_1-A_1K_2)\cos 2A_2]\\
&-\frac{\alpha}{2}[(8A_1^2)^{-1}\{(A_1^2+A_2^2)^2\\
&+(A_1K_1-A_2K_2)^2\}\cosh 2A_1+(8A_2^2)^{-1}\{A_1^2+A_2^2)^2+(A_1K_2+A_2K_1)^2\}\cos 2A_2\\
&-(4A_2^2)^{-1}\{2(A_1^2-A_2^2)(A_1K_2+A_2K_1)-2A_1A_2(A_1K_1-A_2K_2)\}\sin 2A_2\\
&+(4A_1^2)^{-1}\{2(A_1^2-A_2^2)(A_1K_1-A_2K_2)+2A_1A_2(A_1K_2+A_2K_1)\}\sinh 2A_1]\,,\end{aligned}$$

$$\begin{aligned}E_3 = -\frac{1}{2}R_e[&(8A_1^3)^{-1}(A_1K_1+A_2K_2)\\
&+(8A_2^3)^{-1}(A_2K_1-A_1K_2)]+\alpha[(8A_1^3)^{-1}\{(A_1^2-A_2^2)(A_1K_1-A_2K_2)\\
&+2A_1A_2(A_1K_2+A_2K_1)\}+(8A_2^3)^{-1}\{(A_1^2-A_2^2)(A_1K_2+A_2K_1)\\
&-2A_1A_2(A_1K_1-A_2K_2)\}]\,,\end{aligned}$$

$$E_4 = -\frac{1}{2}R_e[(8A_1^2)^{-1}(A_1^2 + A_2^2 + K_1^2 + K_2^2) - (8A_2^2)^{-1}(A_1^2 + A_2^2 - K_1^2 - K_2^2)] + \frac{\alpha}{2}[(8A_1^2)^{-1}\{(A_1^2 + A_2^2)^2 + (A_1^2 + A_2^2)(K_1^2 + K_2^2)\} + (8A_2^2)^{-1}\{(A_1^2 + A_2^2)^2 - (A_1^2 + A_2^2)(K_1^2 + K_2^2)\}],$$

$$E_5 = -\frac{1}{2}R_e[(2A_1)^{-1}(A_1K_1 + A_2K_2) + (2A_2)^{-1}(A_2K_1 - A_1K_2)] + \frac{\alpha}{2}[(A_1)^{-1}\{(A_1^2 - A_2^2) + (A_1K_1 - A_2K_2) + 2A_1A_2(A_1K_2 + A_2K_1)\} + (A_2)^{-1}\{(A_1^2 + A_2^2)(A_1K_2 + A_2K_1) - 2A_1A_2(A_1K_1 - A_2K_2)\}],$$

$$Q_1 = -\frac{R_e}{2}\frac{(A_1^2 + A_2^2)}{B_1N(4B_1^2 - \sigma^2)} + \frac{\alpha(A_1^2 + A_2^2)B_1^2}{2B_1N(4B_1^2 - \sigma^2)}.$$

	A_1			A_2		
R_e / α	1.0	2.0	3.0	1.0	2.0	3.0
0	0.70712	1.00000	1.22472	0.70712	1.00000	1.22472
0.05	0.72451	1.01233	1.23481	0.68923	0.98749	1.21451
0.10	0.74123	1.02445	1.24472	0.67182	0.97492	1.20439
0.15	0.75721	1.03606	1.25435	0.65321	0.96269	1.19447
0.20	0.77214	1.04697	1.26336	0.63544	0.95075	1.18482
0.25	0.78592	1.05712	1.27189	0.61835	0.93934	1.17575
0.30	0.79853	1.06654	1.27961	0.60196	0.92861	1.16726
0.35	0.80985	1.07513	1.28684	0.58653	0.91873	1.15932
0.40	0.82006	1.08272	1.29325	0.57237	0.90972	1.15228

Table 1: The values of constants A_1 and A_2 for $R_e = 1.0, 2.0, 3.0$ and various values of α

R_e / α	$\sigma = 3$			$\sigma = 5$		
	1.0	2.0	3.0	1.0	2.0	3.0
0	0.32353	0.30322	0.27732	0.21552	0.20805	0.19725
0.10	0.31542	0.29771	0.27221	0.21156	0.20501	0.19346
0.20	0.30905	0.29194	0.26482	0.20624	0.20009	0.18992
0.30	0.30072	0.28585	0.26309	0.20081	0.19492	0.18584
0.40	0.29443	0.27849	0.25835	0.19538	0.18956	0.18156

Table 2: The amplitude of oscillations of transverse component of velocity

α / σ	$\beta = 0.2$			$\beta = -0.2$		
	3.0	4.0	5.0	3.0	4.0	5.0
0	0.86451	0.78523	0.63942	0.65933	0.53342	0.43269
0.10	0.83083	0.75495	0.61925	0.64526	0.52464	0.41095
0.20	0.79965	0.71309	0.58639	0.61394	0.50826	0.39929
0.30	0.79852	0.68524	0.54463	0.59948	0.49328	0.34565
0.40	0.79434	0.64316	0.51245	0.54399	0.45361	0.33924

Table 3: Phase lag of the oscillation of the transverse component of velocity with the oscillations of the plate

α / σ	3.0	4.0	5.0
0	0.09502	0.60223	0.3321
0.05	0.09623	0.06412	0.03563
0.10	0.10245	0.06823	0.03812
0.15	0.10326	0.07024	0.04215
0.20	0.11324	0.07214	0.04501
0.25	0.16234	0.08151	0.05234
0.30	0.17329	0.8923	0.06231
0.35	0.17845	0.9125	0.07525
0.40	0.18623	0.10252	0.09314

Table 4: The value of constant D_2/R_e taking $\gamma = 1.0$, $\beta = 0.2$ for various values of α and σ