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WEAK MONOTONICITY OF HILBERT SPACE OPERATORS

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Abstract: We study the notion of weak monotone operators between Hilbert spaces. Weak monotonicity ensures positive solvability of the linear system $Ax = b,\ b \geq 0$. We provide settings for operators over infinite dimensional Hilbert spaces which enable suitable generalizations of the results in the finite dimensional case. It is known that if A is weak monotone and has full-rank, then A has a nonnegative full-rank $\{1\}$-inverse. In this article, among others we generalize this result to weak monotone operators between Hilbert spaces. We also investigate the connections between weak monotonicity and other types of operator monotonicity and their relationships with generalized inverses.

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1. Introduction

A matrix $A \in \mathbb{R}^{m\times n}$ is called weak monotone if $Ax \in \mathbb{R}^m_+ \Rightarrow x \in \mathbb{R}^n_+ + N(A)$. Here $\mathbb{R}^k_+$ denotes the nonnegative orthant of $\mathbb{R}^k$. Weak monotonicity is the most general of the several types of monotonicity of matrices (see for instance Berman et al [6] or [7]). There is an obvious generalization of this concept to bounded operators between Hilbert spaces namely, $Ax \in P_2 \Rightarrow x \in P_1 + N(A)$, where P_1, P_2 are cones in Hilbert spaces H_1, H_2 respectively and, A is a bounded operator between H_1 and H_2. Equivalenty, a bounded operator A between

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Hilbert spaces H_1 and H_2 is weak monotone if $Ax \in P_2 \Rightarrow Ax = Ay$ for some $y \in P_1$. A good source of reference for various types of matrix monotonicities is the paper by Berman et al [6] (see also Berman et al [7]). A collection of results concerning weak monotonicity of matrices and their relationship with generalized B-splittings can be found in Peris et al [14]. A short preliminary investigation of weak monotonicity of bounded operators between Hilbert spaces was also done by Sivakumar [16].

In this article we make a detailed study of weak monotonicity of operators between Hilbert spaces. It is of intrinsic interest to know when an operator equation $Ax = b$ (with a nonnegative requirement vector b) has a nonnegative solution. The operator A being weak monotone guarantees the possibility of a such solution. One of our main objectives is to study the convergence of iterative schemes for infinite dimensional systems, similar to the ones already available in the finite dimensional setting. It is our intention to explore the applicability of weak monotonicity for such problems. Some work on iterative methods for linear systems has already been done by Marek et al [11] and Kammerer et al [9]. However, weak monotone operators were not considered in either of these works. Our endeavor is to extend many of the results available in the setting of finite dimensional spaces (with the nonnegative orthant as the cone) to infinite dimensional spaces (with a general closed cone). This is an improvement on the previous works which hold good only for polyhedral cones. We provide sufficient conditions under which such generalizations are possible. Wherever possible, we have presented examples (Remark 3.7 and Remark 3.21, Examples 3.10, 3.11, 3.14, and 3.19) to show that some of these sufficient conditions are indispensable. This results in sharpening the existing results even in the finite dimensional case.

After stating the required definitions in the second section, we move to Section 3 where we prove our results. We first show that any bounded linear operator with closed range has a full-rank factorization (Lemma 3.3). We then present a sufficient condition under which weak monotonicity of A implies the weak monotonicity of its adjoint A^* (Theorem 3.12). We illustrate that this is not always the case (Example 3.14). We next extend a result of Jeter and Pye (Theorem 1, Jeter et al [8]) on the structure of a weak monotone matrix with a nonnegative full-rank factorization (Theorem 3.16). We obtain a corollary of this, a result that appears to be new even in the finite dimensional setting. We then study an extension of range monotonicity which we call generalized range monotonicity. We show how this concept is useful in obtaining a relationship with weak monotonicity more suitable for operators over infinite dimensional spaces (Theorem 3.24). Finally we study the connections between weak monotonicity of A and A being MP. Here again, we give sufficient conditions under which the finite dimensional results can be extended (Theorem 3.26 and Corollary 3.27). We mention that our result makes use of boundedness of positive operators between spaces with certain cones (Theorem 3.25) and an infinite dimensional Farkas Lemma (Theorem 3.4).

2. Notations and Basic Definitions

Throughout, H_1 and H_2 will denote complex Hilbert spaces, equipped with cones P_1 and P_2, respectiely. We shall denote the set of all bounded operators between H_1 and H_2, by $\mathcal{B}(H_1, H_2)$. For $A \in \mathcal{B}(H_1, H_2)$ denote the range and null spaces of A by $R(A)$ and $N(A)$, respectively. Throughout, it will be assumed that the range of any bounded linear operator is closed, unless stated otherwise.

$A \in \mathcal{B}(H_1, H_2)$ is said to be monotone if $Ax \in P_2 \Rightarrow x \in P_1$.

$A \in \mathcal{B}(H_1, H_2)$ is said to be weak monotone if $Ax \in P_2 \Rightarrow x \in P_1 + N(A)$.

We shall give a few examples of weak monotone operators (both in finite as well as in infinite dimensions).

Example 2.1. Consider the Jordan block J_λ corresponding to the eigenvalue 0. J_λ (for $\lambda > 0$) is given by $J_\lambda = \begin{pmatrix} 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots & \lambda \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$. Then $J_\lambda x = \lambda(x_2, ..., x_n, 0)^t$. It then follows that $J_\lambda x \geq 0 \Rightarrow x_j \geq 0 \ \forall j \geq 2$. It is then clear that the system $J_\lambda x \geq 0$ has a nonnegative solution y (for instance $y = (1, x_2, ..., x_n)^t$). Thus the matrix J_λ is weak monotone.

Example 2.2. Let E be the square matrix all of whose entries are 1. We show that E is weak monotone (with the usual cone). Observe that $Ex \geq 0 \Leftrightarrow Ax \geq 0$, where $A = (1, ..., 1)$. Then, proving that E is weak monotone boils down to the following question: Does the system $Ax = b$ $(b \geq 0)$ have a nonnegative solution? Consider the system $Ax = b,\ b \geq 0$. Note that b is a nonnegative real number. Now consider the dual system give by $A^t y \geq 0$. This forces y to be nonnegative. Therefore $\langle y, b \rangle < 0$ is impossible. By Farkas Lemma, the system $Ax = b$ has a nonnegative solution, thereby proving that E is weak monotone.

Remark. The matrix E is also weak monotone with respect to the cone $P_0 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 \geq 0, \ x_3 \geq 0, \ x_2^2 \leq 2x_1x_3\}$, as we show next.

Example 2.3. Consider the square matrix E (of size 3) as in the previous example. Then $Ex \in P_0 \Rightarrow x_1 + x_2 + x_3 \geq 0$. Also, $N(A)$ is a two dimensional subspace spanned by the vectors $(0, 1, -1)^t$ and $(1, -1, 0)^t$. Then, we can write $(x_1, x_2, x_3)^t = -x_3(0, 1, -1)^t - (x_2 + x_3)(1, -1, 0)^t + (x_1 + x_2 + x_3(1, 0, 0)^t$. This proves that E is weak monotone with respect to the cone P_0.

Example 2.4. Consider the matrix A with only finitely many -1's on the diagonal, 1's in the super-diagonal and 0's elsewhere. We claim that A is weak monotone. Without loss of generality, we assume that all the rows containing the -1's appear as the first finitely many rows. Let $Ax^0 = b$ be consistent, where $b = (b_1, b_2, ..., b_n)^t \in \mathbb{R}^n_+$. Let k be the number of -1's. Then, define $y^0 := (y^0_1, y^0_2, ..., y_n)^t$ as follows: $y^0_1 = 0$, $y^0_j = b_1 + ... + b_{j-1}$ for $j = 2, ..., k$ and $y^0_j = b_j \ \forall j \geq k$. Then, we see that $y^0 \in \mathbb{R}^n_+$ and that $Ay^0 = Ax^0$. This shows that J is weak monotone.

Remark. The above example can be extended to infinite dimensions. Again, we assume that there are only finitely many -1's on the diagonal, 1's in the super-diagonal and 0's elsewhere. The same argument proves the weak monotonicity of A.

Our next example shows that any bounded projection on a Hilbert space with respect to any cone P.

Example 2.5. Let $A \in \mathcal{B}(H)$ be a projection operator. Then $A^2 = A$. Suppose $Ax \in P$. Since A is a projection, we can write $x = u + v$, where $u \in N(A)$ and $v \in R(A)$. Now, $Ax = Av$. Since A is a projection, $v = Av = Ax \in P$. Thus A is weak monotone.

Remark. Note that the above example is independent of the dimension of the Hilbert space.

We now give two examples in the infinite dimensional setting.

Example 2.6. Let A be a weighted left shift operator, where the weights are given by a square summable sequence of positive real numbers. Then A is defined by $x \mapsto (a_1x_2, a_2x_3, ..., a_jx_{j+1}, ...)$. Then $Ax \in l^2_+ \Rightarrow a_ix_{i+1} \geq 0, i = 1, 2, ...$ This implies that $x_{i+1} \geq 0, i = 1, 2, ...$ Let $y = (0, x_2, x_3, ...)$. Then it is easily verified that $Ax = Ay$. This shows that A is weak monotone.

As a final example, we consider a Toeplitz matrix that is weak monotone.

Example 2.7. Consider the infinite matrix

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 0 & 1 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

It is easily seen that A represents the bounded operator on l^2 defined by $x \mapsto (x_2 - x_3, x_3 - x_4, x_4 - x_5, ...)$. Let $y = (y_1, y_2, ...)$ be any decreasing sequence of square summable numbers. Square summability forces these numbers to be nonnegative. Then $Ax = Ay$. Thus A is weak monotone.

An element $X \in \mathcal{B}(H_2, H_1)$ is called the Moore-Penrose inverse of A, if it satisfies the following four equations.

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA. \tag{4}$$

Such an X is denoted by $A^\dagger$. It is well known that $A^\dagger$ is bounded iff $R(A)$ is closed (Nashed [13]).

$A \in \mathcal{B}(H_1, H_2)$ is said to be nonnegative, if $AP_1 \subseteq P_2$, where P_1 and P_2 are cones in H_1 and H_2, respectively.

For a non-empty subset λ of $\{1, 2, 3, 4\}$, $T \in \mathcal{B}(H_2, H_1)$ is called a λ-inverse of A, if T satisfies those equations corresponding to each element of λ. For example, $T \in \mathcal{B}(H_2, H_1)$ is called a $\{1\}$-inverse of $A \in \mathcal{B}(H_1, H_2)$, if T satisfies the equation $ATA = A$.

For a non-empty subset of $\{1, 2, 3, 4\}$, $A \in \mathcal{B}(H_1, H_2)$ is called λ-monotone, if it has a nonnegative λ-inverse. For example, we say that A is $\{1\}$-monotone, if A has a nonnegative $\{1\}$-inverse.

$A \in \mathcal{B}(H)$ is said to be range monotone if $Ax \in P_1$ and $x \in R(A) \Rightarrow x \in P_1$. We next propose a notion extending range monotonicity. An element $A \in \mathcal{B}(H_1, H_2)$ is said to be generalized range monotone if for some $\{1\}$-inverse T of A, $Ax \in P_2, x \in R(TA) \Rightarrow x \in P_1$.

$A \in \mathcal{B}(H_1, H_2)$ is said to be MP if A has a nonnegative $\{1\}$-inverse which is either injective or surjective. MP matrices were introduced by Meyer et al [12].

A Hilbert space H has positive property or $\mathcal{P}$ for short, if it has an orthonormal basis $\{u^\alpha\}$ such that $u^\alpha \in P$, $\forall\ \alpha \in J$, J being an index set and P is the underlying cone in H. $\mathbb{R}^n$ with the usual cone $\mathbb{R}^n_+$ has property $\mathcal{P}$. The standard basis $\{e^1, ..., e^n\}$ satisfies $e^i \in \mathbb{R}^n_+ \ \forall\ i$. l^2, the Hilbert space of all absolutely square summable sequences, with the usual cone $l^2_+ := \{x = (x_i) \in l^2 : x_i \geq 0 \ \forall\ i\}$ has property $\mathcal{P}$.

For a cone P in a complex Hilbert space H, the dual cone is defined by

$$P^* := \{y \in H\colon \text{Re}\ \langle x, y\rangle \ \geq\ 0 \ \forall\ x \in P\},$$

where Re $\langle x, y\rangle$ denotes the real part of $\langle x, y\rangle$. A cone P is said to be self-dual if $P^* = P$. A cone P in a real Hilbert space is said to be generating if $H = P - P$. The nonnegative orthant $\mathbb{R}^n_+$ of the Euclidean space $\mathbb{R}^n$ is a self-dual (closed) generating cone. The cone l^2_+ is a self-dual (closed) generating cone. Another example of a self-dual cone is the P_0 cone considered in Example 2.3.

A cone P in $\mathbb{R}^m$ is said to be polyhedral, if it is the image of $\mathbb{R}^n_+$, under a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$. It is a well known result in the finite dimensional case that, $N(A) + P$ is closed, where A is linear and P is polyhedral (see A. Ben-Israel [2] for instance).

3. Main Results

In this section, we prove our main results concerning weak monotone operators. We shall first prove some preliminary results.

Definition 3.1. $A \in \mathcal{B}(H_1, H_2)$ is said to have a full-rank factorization if there exists a Hilbert space H, and linear maps $B \in \mathcal{B}(H, H_2)$, $C \in \mathcal{B}(H_1, H)$ such that $A = BC$, with $N(B) = \{0\}$, $R(C) = H$, $R(B) = R(A)$ and $N(C) = N(A)$.

Definition 3.2. $A \in \mathcal{B}(H_1, H_2)$ is said to have a nonnegative full-rank factorization if in addition to the above, P_1, P_2 and P are cones in H_1, H_2 and H, respectively, such that $BP \subseteq P_2$ and $CP_1 \subseteq P$.

Note that if $A = BC$ is a full-rank factorization, then $R(B)$ and $R(C)$ are closed by definition. It is well known that any finite matrix over $\mathbb{R}$ or $\mathbb{C}$ always has a full-rank factorization (A. Ben-Israel and T.N.E. Greville [4]). We next show that any bounded linear map with closed range has a full-rank factorization.

Lemma 3.3. *Every $A \in \mathcal{B}(H_1, H_2)$ has a full-rank factoriation $A = BC$.*

Proof. Since we are assuming that $R(A)$ is closed, $R(A)$ is itself a Hilbert space. Let us denote this space by H. Let $C : H_1 \to H$ be defined by $C = A$ and let $B : H \to H_2$ be the inclusion operator. Then B is one-one, C is onto, $A = BC, R(B) = R(A)$, $N(C) = N(A)$ and $R(C) = H$. □

We need the following result subsequently.

Theorem 3.4. (A. Ben-Israel et al [3] Corollary 2) *Let H_1 and H_2 be real Hilbert spaces with H_1 partially ordered by the self-dual cone P_1. Let $A \in \mathcal{B}(H_1, H_2)$ with $N(A) + P_1$ closed. Then either:*
(1) $Ax = b, x \in P_1$ has a solution;
or (exclusive)
(2) $A^*y \in P_1$, $\langle y, b\rangle < 0$ has a solution.

Remark 3.5. The above lemma gives an extension of Farkas' Lemma to infinite dimensional spaces. We have stated it in a slightly different form than the one that appears in the literature (A. Ben-Israel et al [3]). The above result holds in a more general setting of topological vector spaces.

Lemma 3.6. *Let A be $\{1\}$-monotone. Then A is weak monotone.*

Proof. Let A be $\{1\}$-monotone. Let T be a nonnegative $\{1\}$-inverse of A. Let $Ax = y \in P_2$. Set $z := Ty$. Then by definition, it follows that $z \in P_1$ and clearly, $Az = y = Ax$. Thus A is weak monotone. □

Remark 3.7. The converse of Lemma 3.6 is not true even in the finite dimensional case. If $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$, then A is weak monotone, but not $\{1\}$-monotone [8].

We next show that if A is a nonnegative operator between two real Hilbert spaces equipped with self-dual cones, then its adjoint A^* is also nonnegative.

Lemma 3.8. *Let P_1 and P_2 be self-dual cones in the Hilbert spaces H_1 and H_2, respectively. Let $A \in \mathcal{B}(H_1, H_2)$. Then A is nonnegative if and only if A^* is nonnegative.*

Proof. It suffices to prove that if A is nonnegative, then A^* is nonnegative. So, let us assume that A is nonnegative. We want to prove that $A^*P_2 \subseteq P_1$. Let $y \in A^*P_2$. Then $\exists x \in P_2$ such that $y = A^*x$. Let $u \in P_1$. Then, $\langle u, A^*x\rangle = \langle Au, x\rangle$. Now, Re $\langle Au, x\rangle \geq 0$ as $Au \in AP_1 \subseteq P_2$, $x \in P_2$ and $P_2^* = P_2$, which implies that $A^*x \in P_1^* = P_1$. This shows that A^* is nonnegative. □

Remark 3.9. The condition that P_1 is self-dual cannot be dispensed with, even in the finite dimensional case, as the following examples illustrate.

Example 3.10. Consider $\mathbb{R}^2$ equipped with the cone $P = \{(x_1, x_2) : x_1 \geq 0\}$. Then the dual cone of P is given by $P^* = \{(x_1, x_2) : x_1 \geq 0, x_2 = 0\}$. This shows that P is not self-dual. Let H_1 and H_2 be $\mathbb{R}^2$ both being partially ordered by the cone P. Consider the linear operator $T : H_1 \to H_2$ given by $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$. Then T is clearly nonnegative. However T^t is not nonnegative. If $x^0 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then $T^t x^0$ is given by $T^t x^0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin P$, although $x^0 \in P$.

Example 3.11. Consider $\mathbb{R}^{2\times 2}$ equipped with the inner product $\langle X, Y\rangle = tr(Y^t X)$. Consider the cone $P_1 = \{kI : k \geq 0\}$. Then the dual of P_1 is the cone $P_1^* = \{X : trX \geq 0\}$. This shows that P_1 is not self-dual. Let H_1 and H_2 be the Hilbert spaces $\mathbb{R}^{2\times 2}$ partially ordered by the cones P_1 and $P_2 = P_1^*$, respectively. Let B be a diagonal matrix with strictly positive diagonal entries b_{11} and b_{22}. Let T be the linear operator from H_1 to H_2 given by $T(X) = XB$. It is easily seen that T is a nonnegative operator. However T^t is not nonnegative. For, if $X^0 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, then $X^0 \in P_2$; however, $T^t(X^0) = \begin{pmatrix} kb_{11} & 0 \\ 0 & -kb_{22} \end{pmatrix} \notin P_1$.

If A is a finite matrix with real entries, then the weak monotonicity of A imples that of A^t, where the underlying cones are the usual cones (see for instance Theorem 2, Peris et al [14]). We now prove an analogous result for weak monotone operators between Hilbert spaces under certain additional assumptions.

Theorem 3.12. *Let $A \in \mathcal{B}(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces equipped with cones P_1 and P_2, respectively. Further assume that P_1 is self-dual and that $N(A^*) + P_2$ is closed. If A is weak monotone, then A^* is also weak monotone.*

Proof. Let A be weak monotone. Suppose A^* is not weak monotone. Then $\exists\ b = A^* y \in P_1$ such that for no $x \in P_1$, we have $A^* y = A^* x$. In otherwords, $A^* x = b, x \in P_1$ has no solution. By Theorem 3.4, $\exists\ v \in H_1$ such that $Av \in P_2$ and $\langle b, v\rangle < 0$ (using the condition that $N(A^*) + P_2$ is closed). Now, since A is weak monotone, $Av \in P_2 \Rightarrow v = r + s, r \in N(A), s \in P_1$. Consider $0 > \langle b, v\rangle = \langle b, r + s\rangle = \langle A^* y, r + s\rangle = \langle y, As\rangle = \langle A^* y, s\rangle \geq 0$, since $A^* y \in P_1$, $s \in P_1$ and $P_1^* = P_1$. This contradiction establishes our result. □

Remark 3.13. 1. If we assume that P_2 is self-dual and $N(A) + P_1$ is closed, then the weak monotonicity of A^* implies that of A. Here we are assuming that

A^* has closed range.

2. The condition that $N(A^*) + P_2$ is closed cannot be dropped even in the finite dimensional case, as the following example shows.

Example 3.14. Consider $\mathbb{R}^2$ with the usual cone $\mathbb{R}^2_+$ and $\mathbb{R}^3$ with the cone P_0, defined earlier. Then, $\mathbb{R}^2_+$ is self-dual. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is clear that A is monotone and hence, weak monotone. Also the vector $(1,0,0)^t$ spans $N(A^t)$. Now, if $x^0 = (0,1,0)^t$, then $A^t(x^0) \in \mathbb{R}^2_+$. However, $x^0 \notin P_0 + N(A^t)$. For, if $\exists\, y \in P_0$ such that $x = y + \beta(1,0,0)^t$, then we can conclude that $1 = 0$, an absurdity. This establishes our result that A^t is not weak monotone. Now, it remains to show that $N(A^t) + P_0$ is not closed.

If we set $u^k = \begin{pmatrix} -k \\ 0 \\ 0 \end{pmatrix}$ and $v^k = \begin{pmatrix} k \\ 1 \\ 1/k \end{pmatrix}$, then for all k, $A^t(u^k) = 0$ and $v^k \in P_0$. Setting $z^k = u^k + v^k = \begin{pmatrix} 0 \\ 1 \\ 1/k \end{pmatrix}$, we see that z^k converges to $z^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Let $z^0 = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$, where $\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \in P_0, r \in \mathbb{R}$. As $t_3 = 0$, we have $t_2 = 0$ and so 1=0, an absurdity. Thus $N(A^t) + P_0$ is not closed. Note that P_0 is not polyhedral.

We now state a result which will be used in the theorem that follows.

Theorem 3.15. (Theorem 2.16, Kulkarni et al [10]) *Let H_1 and H_2 be Hilbert spaces with conesP_1 and P_2, respectively with P_1 generating and P_2 self-dual. Assume that H_1 has property $\mathcal{P}$. Let $A \in \mathcal{B}(H_1, H_2)$ with $N(A^*) + P_2$ closed. Then A is monotone if and only if $\exists$ a positive operator $Y \in \mathcal{B}(H_2, H_1)$ such that $YA = I$.*

Theorem 3.16. *Let $A : H_1 \to H_2$ be a weak monotone operator, with a nonnegative full-rank factorization $A = BC$ with $B \in \mathcal{B}(H, H_2)$ and $C \in \mathcal{B}(H_1, H)$. Assume that H_1 and H_2 have property $\mathcal{P}$, and that $N(A) + P_1$ and $N(A^*) + P_2$ are closed. Further, suppose that P_2 is generating, P_1 is self-dual, and P is self-dual and generating. Then $\exists\, X \in \mathcal{B}(H_2, H)$, $Y \in \mathcal{B}(H, H_1)$ such that X and Y are nonnegative, and $XAY = I$.*

Proof. We first show that B is monotone. Suppose $Bz \in P_2$. Since $R(A) = R(B)$, $\exists\, x \in H_1$ such that $Bz = Ax$. Now, $Ax \in P_2 \Rightarrow\ x =$

$u+v, u \in N(A), v \in P_1$, as A is weak monotone. So $Bz = Ax = A(u+v) = Av = BCv$. Since B is injective, $z = Cv \in P$, the positive cone in H. Thus $Bz \in P_2 \Rightarrow z \in P$, so that B is monotone. Using Theorem 3.15, wee see that B has a nonnegative left-inverse which we denote by X (note that $N(B^*) + P_2 = R(B)^\perp + P_2 = R(A)^\perp + P_2 = N(A^*) + P_2$, so that $N(B^*) + P_2$ is closed). Since A is weak monotone, it follows that A^* is weak monotone, by Theorem 3.12. So, $A^* = C^*B^*$ is a nonnegative full-rank factorization of A^*. Proceeding as before, C^* is monotone and so, $\exists\, Y^*$, such that $Y^*C^* = I$ (note that $N((C^*)^*) + P_1 = N(C) + P_1 = N(A) + P_1$, so that $N((C^*)^*) + P_1$ is closed). Thus $XB = I, CY = I$ and so $XAY = X(BC)Y = I$. □

Corollary 3.17. *Let the assumptions of Theorem 3.16 hold. Then A is $\{1,2\}$-monotone.*

Proof. Let X and Y be as in previous Theorem 3.16. Set $Z = YX$. Then $AZA = B(CY)(XB)C = BC = A$. Similarly, $ZAZ = Z$. □

Remark 3.18. 1. The conclusion in Corollary 3.17 appears to be new even in the finite dimensional case.

2. Theorem 3.16 can be considered as a generalization of Theorem 1, pp. 172 of Jeter and Pye [8]. We observe that in spite of the assumptions of Theorem 3.16 holding, $A^\dagger$ can fail to be nonnegative, as the following example illustrates.

Example 3.19. Consider $\mathbb{R}^2$ and $\mathbb{R}^3$ with their usual cones. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$. Clearly, A is weak monotone (in fact A is monotone). Also, A is one-one and nonnegative. Hence, A has the trivial nonnegative full-rank factorization $A = AI$. However, $A^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ is not nonnegative.

It was proved in Kulkarni et al [10] that if A is a real square matrix that is monotone, then A^t is also monotone (see Proposition 2.12, Kulkarni et al [10]). An analogous result holds good for weak monotone operators between two Hilbert spaces under some additional assumptions, as we have shown in Theorem 3.12. Again, for real square matrices A and B that are monotone, their product AB and BA are also monotone. However, in the case of weak monotonicity, this result fails; but with the added assumption that one of them is monotone, their product turns out to be monotone.

Theorem 3.20. *Let $A \in \mathcal{B}(H_1, H_2)$ be weak monotone and, $B \in \mathcal{B}(H_2, H_3)$ be monotone. Then $BA \in \mathcal{B}(H_1, H_3)$ is weak monotone.*

Proof. $B(Ax) \in P_3 \Rightarrow Ax \in P_2$, as B is monotone. Thus $x \in N(A) + P_1$, as A is weak monotone. But then we have $x \in N(BA) + P_1$, as $N(A) \subseteq N(BA)$. □

Remark 3.21. The following example shows that if the operators A and B are just weak monotone, then their product need not be weak monotone. Consider $\mathbb{R}^2$ with its usual cone $\mathbb{R}^2_+$. $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. It is not hard to show that A and B are weak monotone. Consider the product $C = AB = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$. We have $N(C) = \{x \in \mathbb{R}^2 : x_2 = -x_1\}$. Consider $x^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $Cx^0 \geq 0$. However, if $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$, with β_1 and $\beta_2 \geq 0$, then $\alpha = \beta_2 \geq 0$. This contradicts $\alpha + \beta_1 = -1 < 0$. Thus, C is not weak monotone.

We next study generalized range monotone operators. We first characterize such operators: Note that (b), below is equivalent to saying that T is positive on $R(A)$.

Theorem 3.22. *For a bounded $T \in A(\{1\})$, the following are equivalent:*
(a) $Ax \in P_2$, $x \in R(TA) \Rightarrow x \in P_1$;
(b) $y \in R(A) \cap P_2 \Rightarrow Ty \in P_1$.

Proof. Suppose (a) holds. Now, $y \in R(A) \cap P_2 \Rightarrow \exists\ x \in H_1$ such that $y = Ax$. Consider Ty. It is clear that $Ty \in R(TA)$ and that $A(Ty) \in P_2$. It now follows from (a) that $Ty \in P_1$. Thus (b) holds.

Conversely, suppose (b) holds. Let $Ax \in P_2$ and $x \in R(TA)$. Set $y := Ax$. Then $y \in P_2$ and $Ty = T(Ax) = TA(x) = x \in R(TA)$ (Recall that TA is an idempotent). Now it follows from (b) that $Ty = x \in P_1$. Thus (a) holds. □

Remark 3.23. Theorem 3.22 resembles Theorem 4 in Peris et al [14], which characterizes the existence of the group inverse of A and it being non-negative on its range. It can be seen that this theorem of Peris and Subiza does not hold in infinite dimensional spaces, as $R(A) = R(A^2)$ and $N(A) = N(A^2)$ are not equivalent (for instance the right shift operator on l^2). This is one of the reasons for proposing the notion of generalized range monotonicity.

We next study the relationship between weak monotonicity and generalized range monotonicity.

Theorem 3.24. *If $A \in \mathcal{B}(H_1, H_2)$ is generalized range monotone, then A is weak monotone. Conversely, suppose that A is weak monotone and that the other conditions of Theorem 3.16 hold. Then, A is generalized range monotone.*

Proof. Suppose $A : H_1 \to H_2$ is generalized range monotone. Let $Ax \in P_2$. Then $x = u + v$, where $u = TAx \in R(TA)$ and $v = (I - TA)x \in N(A)$. Now $Au = Ax \in P_2$ with $u \in P_1$. By the generalized range monotonicity of A, we have $u \in P_1$. Thus A is weak monotone.

Conversely, suppose that A satisfies the conditions of Theorem 3.16. By Corollary 3.17, $Z = YX$ is a nonnegative {1,2}-inverse of A. Let $Ax \in P_2$ and $x \in R(ZA)$. Then $x = ZAx \in P_1$, as Z and A are nonnegative. Thus A is generalized range monotone. □

We next present a generalization of a theorem of Peris and Subiza [14], where they prove that if $A \in \mathbb{R}^{m \times n}$ is weak monotone and of full rank, then A is MP. The following theorem due to Lozanovsky will be used in the proof of the aforesaid result.

Theorem 3.25. (Lozanovsky, Corollary 2.5 in Abramovich et al [1]) *Let X_1, X_2 be partially ordered Banach spaces with closed cones. If the cone of X_1 is also generating, then every positive operator $A \in \mathcal{B}(X_1, X_2)$ is continuous.*

Theorem 3.26. *Let H_1 and H_2 be Hilbert spaces where H_2 has property $\mathcal{P}$. Let $A \in \mathcal{B}(H_1, H_2)$ be surjective. Further assume that P_2 is generating. Then A is MP if and only if A is weak monotone.*

Proof. If A is MP, then A is {1}-monotone and so A is weak monotone.

Conversely, suppose that A is onto and weak monotone. We shall prove that A has a nonnegative {1}-inverse that is injective. Let $\{u^\alpha : \alpha \in J\}$ be an orthonormal basis for H_2 with $u^\alpha \in P_2 \ \forall \ \alpha \in J$ (such a basis exists as H_2 has property $\mathcal{P}$). Since A is surjective and weak monotone, the system $Ay = u^\alpha$ has a solution, for each α say, v^α. Now, define a map $X : H_2 \to H_1$ by, $X(u^\alpha) = v^\alpha$, and extend X linearly to the whole of H_2. Then by the definition of X, we see that it is nonnegative. By Theorem 3.25, X is bounded, as P is generating. Also, $AX(u^\alpha) = A(v^\alpha) = u^\alpha$, so that $AX = I$ (and hence X is a {1,2,3}-inverse of A). Since X has A as a left inverse, it follows that X is injective. Hence A is MP. □

Corollary 3.27. *Let H_1 and H_2 be Hilbert spaces, where H_1 has property $\mathcal{P}$. Assume that the cones P_1 and P_2 are self-dual and generating and that $N(A^*) + P_2$ is closed. Let $A \in \mathcal{B}(H_1, H_2)$ be injective. Then A is MP if and only if A is weak monotone.*

Proof. The proof of the "only if" part is as in the previous theorem. Conversely, suppose that A is injective and weak monotone. Then A^* is surjective and by Theorem 3.12 it is weak monotone. By Theorem 3.27, A^* is MP. Let Y be a nonnegative injective right inverse of A^*. Then Y^* is nonnegative and surjective. Hence A is MP. □

4. Concluding Remarks

It has been shown in Peris et al [14] (Theorem 4) that, for a square matrix A, range monotonicity of A is equivalent to the existence of the group inverse of A and its nonegativity on $\mathrm{R}(A)$. Both of these imply weak monotonicity of A. However, the above conditions are only sufficient conditions (refer to the example after Theorem 4 in Peris et al [14]). In the infinite dimensional setting, a relationship between weak montonicty and group monotonicity was given in Sivakumar [16]. However, the operators were assumed to belong to a class of operators $P_{\#}$. The class $P_{\#}$ contains those operators for which the group inverses exists and $AA^{\#}$ being nonnegative. It was then proved that if a weak monotone operator $A \in P_{\#}$, then A is group monotone. It would be interesting to see if this assumption can be replaced by some other weaker assumption. Also, in finite dimensions, if A has the full-rank factorization $A = BC$, then the group inverse exists if and only if CB is invertible. Consequently, we see that if A admits a nonnegative full-rank factorization, then the group inverse is also nonnegative. It would be interesting to study conditions (sufficient as well as necessary) that will ensure the nonnegativity of CB as well as its inverse.

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