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FRACTAL PROPERTIES OF THE MATRIX FOR THE CUPS AND STONES COUNTING PROBLEM

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Abstract: In *Adv. in Appl. Math.*, **21**, 405-423 (1998), Carbonara and Green study a family of integer sequences $S_{k,\sigma}$ (for $\sigma = 1$) that in essence solve a special case of the Cups and Stones problem first posed by Barry Cipra *Math. Mag.*, **65**, No. 1 (1992), 56, which is still open in the general case. Among other things, we show the exact relationship between Cipra's and our version. Carbonara and Green found a recursive formula for $S_{k,1}$ by encoding the problem as a matrix $M_{k,1}$ where each row represents a different configuration (so that $S_{k,1}$ = number of rows of $M_{k,1}$). In addition, they give evidence that Sierpinski's Gasket shows up in the matrix $M_{k,1}$ (note the distribution of zeros in page 421 of the same paper). In our current work, we show that $M_{k,1}$ has a decomposition with fractal characteristics similar to Sierpinski's Gasket. Cipra's problem originated in the field of statistical mechanics (personal communication). The discovery of the fractal properties of $M_{k,1}$ was unexpected. Other fractal matrices have recently

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being used to model different areas of condensed matter research ([5], [6]).

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1. Introduction

In [1] Carbonara and Green study a new family of integer sequences $S_{k,\sigma}$ motivated by a counting problem first posed by Barry Cipra [2] still open in full generality [3], to the best knowledge of the authors. We call this problem the Cups and Stones Counting Problem (CSCP for short).

Combinatorially, the CSCP can be stated as follows.

The Cups and Stones Counting Problem (CSCP). Given an initial configuration and a Transformation rule that modifies the initial configuration, count the number of different configurations possible.

Initial Configuration. Arrange in a circle k cups, each having σ stones with one of the cups marked as a special cup (which we call the root). Obtain the next configuration by applying the following transformation rule.

Transformation Rule (T.r.). Pick up all the stones in the root. Distribute them one cup at a time in a clockwise fashion going around the circle of cups, until all stones that were picked up are reallocated. The last cup to receive a stone on each application of the T.r. (often referred as "the last cup of the T.r.") becomes the root in the resulting configuration.

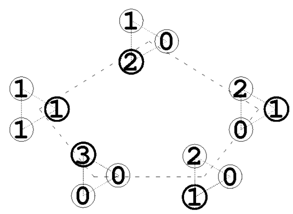
Notation. We denote by $\mathbf{S}_{\mathbf{k},\sigma}$ the number of different configurations in the CSCP where the initial configuration has k cups, each having σ stones.

In this paper we will study the matrix representation $M_{k,\sigma}$ for the CSCP. To obtain $M_{k,\sigma}$, we must first number the cups. Let the root in the first configuration be cup one and number the cups consecutively in a clockwise direction. Then $M_{k,\sigma}(i,j)$ is the number of stones in cup j in the i-th configuration. Thus each column of $M_{k,\sigma}$ corresponds to a particular cup and each row of $M_{k,\sigma}$ corresponds to a particular configuration. Note this means that $M_{k,\sigma}$ has k columns and $S_{k,\sigma}$ rows. To keep track of the location of the root we will underline the entry in each row corresponding to the root.

Example 1. Consider the example with 3 cups (i.e. $k = 3$) and 1 stone per cup (i.e. $\sigma = 1$). We use the transformation rule as a vector function: T.r.$[(\underline{1},1,1)] = (0,\underline{2},1)$. So the first two configurations in this case are $(\underline{1},1,1)$ and $(0,\underline{2},1)$. To get the third configuration we apply again T.r.:

$k = 3$, $\sigma = 1$

$S_{3,1} = 5$



$$\equiv \begin{pmatrix} \underline{1} & 1 & 1 \\ 0 & \underline{2} & 1 \\ \underline{1} & 0 & 2 \\ 0 & \underline{1} & 2 \\ 0 & 0 & \underline{3} \end{pmatrix} = M_{3,1}$$

Darker circle ≡ Special cup ≡ Underlined entry

NOTATION: the special cup will be referred to as the root

Figure 1: Cups and stones counting problem

$\text{T.r.}^2[(\underline{1}, 1, 1)] = \text{T.r.}[(0, \underline{2}, 1)] = (\underline{1}, 0, 2)$. Figure 1 shows all the configurations in the case $k = 3, \sigma = 1$.

Note that since there is a finite number of possible configurations and T.r. has a unique inverse, the configurations T.r. creates will keep changing (never repeating) until they get back to the CSCP initial configuration.

Please note that in our version of the game cyclic permutations of a configuration are considered equal but in Cipra's version (see [2]) they are not. For $\sigma = 1$, every result we obtain can be translated to the original game. We state the result now, and prove it later, after the Fundamental Lemma Part 2.

Corollary 2. (Corollary to Fundamental Lemma Part 2) *Denote by CSCP′ the CSCP with k cups numbered and cyclic permutations considered different (Cipra's version of the game) and let $M'_{k,\sigma}$ be the corresponding matrix. Let $S'_{k,\sigma}$ denote the number of different configurations in the CSCP′. Let 2^p be the largest power of 2 that is a factor of k and let 2^n be the smallest power of 2 greater or equal to $k-1$. Then:*

1. $S'_{k,1} = \frac{k}{2^p} S_{k,1}$.

2. $M'_{k,1}$ is obtained by placing $\frac{k}{2^p}$ copies of the matrix $M_{k,1}$ underneath each other, each copy rotated clockwise $2^n - k$ columns from the previous one (see the example below).

Example 3. The matrix $M'_{3,1}$ is as follows:

$$\begin{pmatrix} \underline{1} & 1 & 1 \\ 0 & \underline{2} & 1 \\ \underline{1} & 0 & 2 \\ 0 & \underline{1} & 2 \\ 0 & 0 & \underline{3} \\ & & \\ 1 & 1 & \underline{1} \\ \underline{2} & 1 & 0 \\ 0 & 2 & \underline{1} \\ \underline{1} & 2 & 0 \\ 0 & \underline{3} & 0 \\ & & \\ 1 & \underline{1} & 1 \\ 1 & 0 & \underline{2} \\ 2 & \underline{1} & 0 \\ 2 & 0 & \underline{1} \\ \underline{3} & 0 & 0 \end{pmatrix}.$$

2. The CSCP Matrix M_k

From now on fix σ to be 1, denote $M_{k,1}$ simply by M_k, and $S_{k,1}$ by S_k. The basic building blocks in this counting problem are the matrices M_k where $k = 2^n + 1$, for all non-negative integers n.

Definition 4. Let M_k be the matrix representation for the CSCP with k cups and 1 stone. Then:

1. The (i, j) entry in M_k is denoted $M_k(i, j)$.
2. Given any matrix M, $||M||$ = number of rows of M. As an example, $||M_k|| = S_k$.
3. $\text{mod}_k(n) = \{x | 1 \le x \le k \; \& \; x = n \text{ Mod } k\}$.
4. $[i]^n = i, i, \cdots, i$ repeated n times (as appropriate within the given context).
5. $C_k(i)$ is the root's position in row i in M_k. In other words, it is the column number of the underlined integer in row i.
6. $R_k(i)$ is the root's value in row i in M_k. Equivalently, $R_k(i) = M_k(i, C_k(i))$ which is the value of the underlined integer in row i.

Lemma 5. *Let M_k be the matrix of the CSCP. Then*

$$\text{mod}_k\Big(\big(\sum_{i=1}^{x-1} R_k(i)\big) + 1\Big) = C_k(x).$$

Proof. Note that $C_k(1) = 1$, and $C_k(x) = \text{mod}_k(C_k(x-1) + R_k(x-1))$. This recursion implies the lemma. □

Definition 6. Let $r_{kl} = \min_{x \in Z^+} \left((\sum_{i=1}^{x-1} R_k(i)) + 1 > (l-1)k \right)$, for all $l \in Z^+$, where the inequality is true for at least one value. For example $r_{31} = 1$ and $r_{32} = 3$ (see Figure 1). Note that r_{kl} denotes the row corresponding to the configuration, where the redistribution of stones has gone beyond the last column of M_k for the $(l-1)^{st}$ time. Call the rows labeled r_{kl} *initiator rows*.

This cellular automaton has many properties. We show many of the simple ones to establish the language as well as a way to build a theory from which beautiful and interesting results follow.

Lemma 7. *Let M_k be the matrix of the CSCP. Then for any row i, $1 \leq i < S_k$, $M_k(i+1, C_k(i)) = 0$. In other words, Lemma 7, row i is empty in row $i+1$.*

Proof. This follows easily from the fact that the root value is less than k except in the last row. □

The transformation rule was defined on configurations. Now we establish a restriction of T.r. to a segment of a configuration.

First note that changes from one configuration to the next only happen at the root cup (say it has n stones in it) and the following n cups. For example, $T.r.(2, 0, \underline{3}, 1, 1, 0, 0) = (2, 0, 0, 2, 2, \underline{1}, 0)$, in which case only the 3-rd, 4-th, 5-th and 6-th cups changed value. From this point of view, we see that the T.r. could be restricted to a section of a configuration, containing the root of value n and at least the following n cups. The resulting segment as well as the original will be in the matrix $M_{k,1}$. If the segment we select is big enough and has the root cup, applying T.r. to it a number of times will produce a block of the original matrix $M_{k,1}$. We call this a *self-contained block*.

We say a self-contained block is *maximal* iff:

1. The first element in the first row is a root, and
2. The value of the root in the last configuration is greater than the number of columns to the right of it.

For example this is a maximal self contained block from $M_{9,1}$:

$$\cdots \begin{matrix} \underline{1} & 0 & 1 & 3 \\ 0 & \underline{1} & 1 & 3 \\ 0 & 0 & \underline{2} & 3 \end{matrix} \cdots$$

and the following is a self-contained block that is not maximal:

$$\cdots \begin{matrix} \underline{1} & 0 & 1 & 3 \\ 0 & \underline{1} & 1 & 3 \end{matrix} \cdots$$

CSCP matrices have structure that lends itself to block matrix notation using self-similar blocks because:

1. All matrices M_k start with a row of 1's (with the first one underlined).

2. Let B be a self-contained block with a rows. If C is a block with a rows located to the right or left of B (occupying the exact same a rows, but different columns) then each row segment in C is an exact copy of the top row segment in C, because each row's root is in B and by definition no stones outside of B are moved.

We will sometimes use matrix block notation and so, if a matrix R has submatrices that encompass all of its elements, instead of writing R as an array of elements, we will write it as an array of submatrices. This can be seen more clearly in the following definition.

Definition 8. Let R be any CSCP matrix. If matrix R can be written in matrix block notation in such a way that the blocks along R's main diagonal are self-contained matrices $T_1, ..., T_r$ (and therefore $||R|| = \sum_i^r ||T_i||$) we say that $R = T_1 \oplus T_2 \cdots \oplus T_r = \oplus_{i=1}^r T_i$. Note that the entries not in T_i are completely determined since all the T_i's are self-contained.

Example 9. Let $R = \begin{pmatrix} T_1 & \oplus & T_2 & \oplus & T_3 & \oplus & T_4 \end{pmatrix}$. Then

$$R = \begin{pmatrix} T_1 & \cdot & \cdot & \cdot \\ \cdot & T2 & \cdot & \cdot \\ \cdot & \cdot & T_3 & \cdot \\ \cdot & \cdot & \cdot & T_4 \end{pmatrix},$$

where the dot is a place holder for a block matrix of the appropriate size.

3. Structure of the Matrices M_k for $k = 2^n + 1$

The key property of these particular matrices is that cup $k = 2^n + 1$ is never the root except in the last configuration, and (except in the last configuration) each time a stone is deposited in cup k, the last stone goes into the next cup, i.e. cup 1. This is stated formally in the following lemma.

Lemma 10. (Fundamental Lemma Part 1) *Let $k = 2^n + 1$ for some non-negative integer n.*

1. *If $i < S_k - 1$ then $C_k(i) + R_k(i) \geq k \rightarrow C_k(i) + R_k(i) = k + 1$.*

2. *If $i = S_k - 1$ then $C_k(i) + R_k(i) = k$.*

In other words, the last stone of any T.r. is never in column k (except in row S_k) and if it drops a stone in column k, then it will drop its last stone in column 1 (i.e. $R_k(r_{kl}) = C_k(r_{kl}) = 1$ for all initiator rows r_{kl}).

Before we prove this lemma, we need to define the sections of M_k.

Definition 11. Let $k = 2^n + 1$ for a non-negative integer n, and let M_k be the CSCP matrix. Assume there are m initiator rows. For $1 \leq i \leq m-1$ define $Section_k(i)$ to be the rows in between r_{ki} and $r_{k(i+1)}$ (including r_{ki} but not $r_{k(i+1)}$). Define $Section_k(m)$ to be the row r_{km} and all the rows following it. For $1 \leq i \leq m-1$ define A_{ki} to be $Section_k(i)$ with the first row and the first entry in each row deleted. It is useful to also give names to the first and last rows of A_{ki}; call the first row α_{ki} and the last row β_{ki}. Define W_k to be the rows (with the first entry deleted) after r_{km}.

For example, in M_3, $A_{31} = (\underline{2}, 1)$ and $W_3 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & \underline{3} \end{smallmatrix} \right)$.

Outline of the Proof of Fundamental Lemma Part 1. We shall use induction on n. It is certainly true for $n = 0$ and 1 (see Figure 1). Assume it is true for $k = 2^n + 1$. We will outline the proof for $k' = 2^{n+1} + 1$ (for details see Section 6).

Consider $M_{k'}$. Divide matrix $M_{k'}$ up into 3 parts as follows (see Figure 2):

— Part 1: column 1;
— Part 2: columns 2 to k;
— Part 3: columns $(k+1)$ to k.

We also define the *divider row* to be the first row where the root has a value of $2^n + 1$ or more (See Figure 2).

We will show that, except for the divider row, whenever a stone lands in the last column of any part, the next root is the *first column* of the cyclically following part.

We prove this by using induction on n and also a nested induction on the sections of $M_{k'}$. This is easy to show for the first section.

By the induction on the sections it is clear that whenever a stone lands in Part 1, it is the root and (by Lemma 7) it is the only stone in the cup. Therefore the following root is in the *first column* of Part 2.

Above the divider row we show that parts 2 and 3 are essentially identical to the A_{ki} parts of M_k, so the above claims are true by induction on n. The lemma is simple to show for the divider row itself. Below the divider row, Part 2 has a very simple structure while Part 3 is again essentially the same as the A_{ki} except for the extra stones in the last column. But since (by induction on n) the root never lands in the last column (until the last row), the root locations and values are not affected. □

The following corollary to Lemma 10 easily follows.

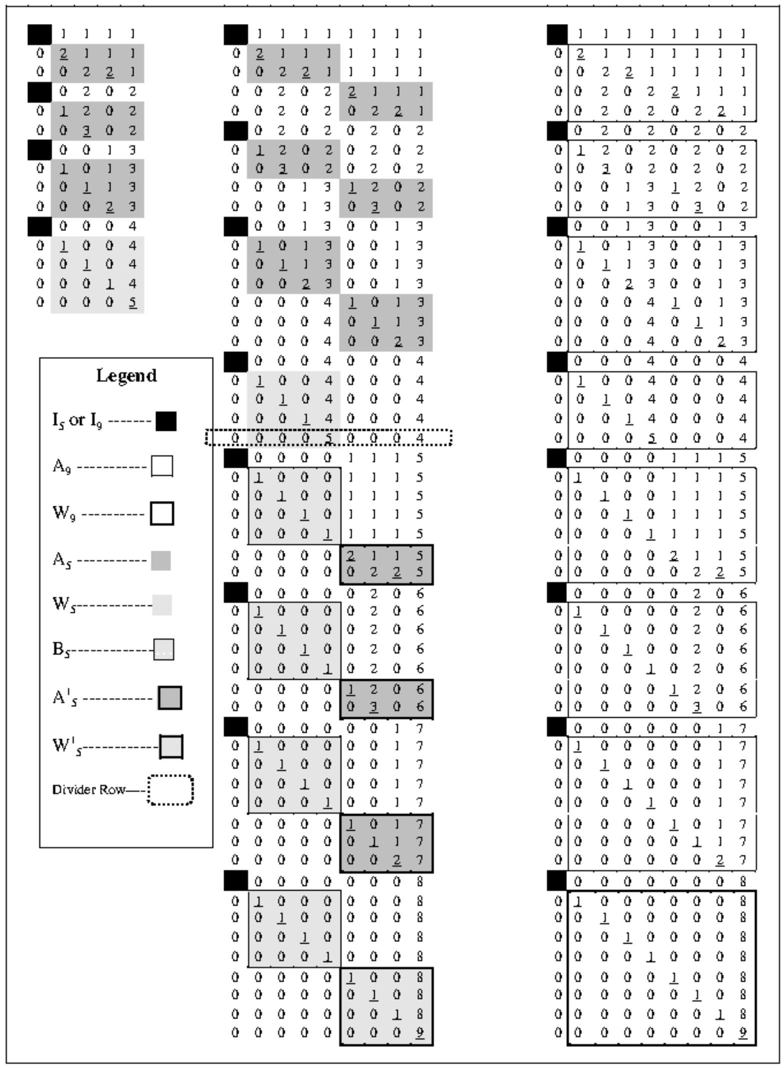


Figure 2: The left and right matrices show the largest $A's$, $I's$ and $W's$ contained in M_5 and M_9. The matrix in the middle shows the start of the fractal decomposition for M_9. For example, note how the first connected component of A_9 in essence is made up of two copies of the first component of A_5, which in turn are each composed of two copies of the first component of A_3 (not shown here).

Corollary 12. *Let $k = 2^n + 1$ for a non-negative integer n, and let M_k be the CSCP matrix. Then:*

1. The row r_{kl} has the root in column 1, i.e. $C_k(r_{kl}) = 1$.

2. The value of the root in row r_{kl} is 1, i.e. $R_k(r_{kl}) = 1$.

3. $M_k(r_{kl}, k) = l$.

4. The root in the last row of M_k is in column k and has value k, i.e. $R_k(S_k) = C_k(S_k) = k$.

5. There are $k - 1 = 2^n$ initiator rows in M_k, which divide M_k into 2^n sections.

6. Row $r_{k(k-1)}$ in M_k is $(\underline{1}, [0]^{k-2}, (k-1))$.

The following lemma could have been included in Corollary 12, but since it plays a significant role later on, we state it by itself, and include a short proof of it.

Lemma 13. *Let $k = 2^n + 1$ for a non-negative integer n, and let M_k be the CSCP matrix. Then the row r_{kl} is $(\underline{1}, [0]^{l-1}, ...)$.*

Proof. We use induction on l. Row $r_{k1} = (\underline{1}, ...)$ which fits the model. Assume it is true for $l - 1$. That is, assume that $r_{k,l-1} = (\underline{1}, [0]^{l-2}, a, ...)$. After applying the T.r. l times we obtain $([0]^{l-1}, \underline{a+1}, ...)$. By Lemma 7, in the next configuration, the l-th cup, where $\underline{a+1}$ was, will be empty. We complete the proof by noting that Lemma 10 implies that $r_{k,l} = (\underline{1}, [0]^{l-1}, ...)$. □

Definition 14. Let $k = 2^n + 1$ for a non-negative integer n, and let M_k be the CSCP matrix. Then:

1. Let I_k be the $(k-1) \times 1$ column matrix of $\underline{1}$'s. This matrix contains all the roots in the initiator rows of M_k.

2. Let A_k be the matrix obtained by placing the matrices $A_{k1}, ..., A_{k(k-2)}$ above each other to form one tall matrix. Call A_k the *atomic matrix*.

Lemma 15. *Let $k = 2^n + 1$ for a non-negative integer n, and let M_k be the CSCP matrix.*

1. The first $l - 1$ columns in A_{kl}, where $1 \leq l \leq k - 2$, have a very simple structure. In fact, the top is a $(l-1) \times (l-1)$ identity matrix (with all 1's underlined), and the rest all zeros.

2. The first $k - 2$ columns in matrix W_k contain a $(k-2) \times (k-2)$ identity matrix (with all 1's underlined) at the top, and the rest all zeros.

Proof. This follows immediately from Lemma 13. □

Lemma 16. *Let $k = 2^n + 1$ for a non-negative integer n. Consider the CSCP with initial configuration $(\underline{1}, [1]^{k-2}, d)$ where d is any positive integer and* *the usual T.r. and let Γ_k^d be the matrix corresponding to this counting problem. Then the first S_k rows in Γ_k^d are related to the matrix M_k as follows:*

$$\begin{aligned} \Gamma_k^d(i,j) &= M_k(i,j) && \text{if } j \neq k\,, \\ \Gamma_k^d(i,j) &= M_k(i,j)+d-1 && \text{if } j = k\,. \end{aligned} \tag{1}$$

Proof. Since by Lemma 10 the last column is never the root (except in row S_k), the value of cup k has no effect on any column other than itself. So the result follows. □

Definition 17. Let $k = 2^n + 1$ for a non-negative integer n, and consider Γ_k^d, the CSCP from Lemma 16, with $d = i2^n + 1$ for a non-negative integer i. Note that if $i = 0$ we are back to the original game.

1. The matrix consisting of the first S_k rows of Γ_k^d is identical to M_k, except that the last column in Γ_k^d is always bigger by the constant $i2^n$. We call it M_k^i.
2. Since M_k^i has an identical structure to M_k, we define in an analogous way the matrices $A_k^i, A_{k1}^i, \ldots, A_{k,(k-2)}^i$, and W_k^i.
3. Let B_k be the $(k-1)\times(k-1)$ identity matrix (with all 1's underlined), 1_k be the $(k-1)\times(k-1)$ matrix with all $1's$ and O_k be the $(k-1)\times(k-1)$ zero matrix. Let B_k^i, 1_k^i and O_k^i be like B_k, 1_k and O_k, except that $i2^n$ is added to each entry in their last column.
4. Let o_k be the $(k-1)\times 1$ column of 0's.

Theorem 18. (Fractal Decomposition; see Figure 2) *Let $k' = 2^{n+1}+1$ and $k = 2^n + 1$ for a non-negative integer n, and let $M_{k'}$ and M_k be the corresponding CSCP matrices. From the definitions we know that each row in $M_{k'}$ contains exactly one row from either $I_{k'}$, $A_{k'}$ or $W_{k'}$. We have already described in detail the structure of $I_{k'}$, and $W_{k'}$. Next we describe the structure of $A_{k'}$ in terms of submatrices of M_k:*

1. for $1 \le i \le k-2$, $A_{k',i} = A_{k,i} \oplus A_{k,i}$.

2. $A_{k',k-1}$ is composed of two contiguous blocks: W_k is the first, and to its right is O_k^1.

3. for $k \le i \le k'-2$, $A_{k',i} = B_k \oplus A^1_{k,i-k+1}$.

Proof. In the full proof of Lemma 10 it is shown that for $1 \le i \le k-2$, $A_{k',i} = A_{k,i} \oplus A_{k,i}$ (in such proof we call it Claim 1). The initiator row after $A_{k',k-2}$ (which is in row $r_{k',k-1}$ in $M_{k'}$) is: $(\underline{1}, [0]^{k-2}, k-1, [0]^{k-2}, k-1)$. Applying the T.r. $k-1$ times to this row will produce configurations which contain an exact copy of W_k next to O_k^1. In particular, $(\text{T.r.})^{k-1}[(\underline{1}, [0]^{k-2}, k-1, [0]^{k-2}, k-1)]$ will equal $(0, [0]^{k-2}, \underline{k}, [0]^{k-2}, k-1)$. The next configuration is an initiator row (with number $r_{k',k}$): $(\underline{1}, [0]^{k-2}, 0, [1]^{k-2}, k)$.

Now apply the T.r. $k-1$ times to the initiator row $(\underline{1}, [0]^{k-2}, 0, [1]^{k-2}, k)$. Rows and columns 2 to k will form an identity matrix of size $(k-1)\times(k-1)$ (which we call B_k) and $(\text{T.r.})^{k-1}[(\underline{1}, [0]^{k-2}, 0, [1]^{k-2}, k)] = (0, [0]^{k-2}, \underline{1}, [1]^{k-2}, k)$. Notice that the last k entries in such configuration, $(\underline{1}, [1]^{k-2}, k)$, is the initial configuration in the matrix M_k^1. Therefore, each time the T.r. goes over the last column in M_k', we get one copy of B_k and a section $A_{k,i}^1$, for $1 \le i \le k-2$. This process repeats exactly $k-2$ times, ending at the initiator row $r_{k',k'-1}$, which looks like $(\underline{1}, [0]^{k-2}, 0, [0]^{k-2}, k'-1)$. This shows that for $k \le i \le k'-2$, $A_{k',i} = B_k \oplus A_{k,i-k+1}^1$. □

4. Structure of the Matrices M_k for a General k

The case where k is any integer is significantly more involved than the case where k is a power of two plus one because Lemma 10 no longer holds. There is an unexpected new parameter that must be introduced. We do that in the next definition.

Definition 19. Consider the CSCP with k cups. Given a configuration, in addition to having a root, we now define a home cup as follows. In the initial configuration, the home cup is the root. Now, assume in configuration g the home cup is cup number n. The home cup will remain the same, until one of the following happens:

Case 1. If after distributing the stones from the root cup to produce configuration $g+1$, the last stone lands in cup n (and so the root cup is cup n), then in configuration $g+1$ the home cup is still cup n, but in configuration $g+2$ the home cup is cup $\text{mod}_k(n+1)$.

Case 2. If after distributing the stones from the root cup to produce configuration $g+1$, cup n receives a stone but does not become the root, then in configuration $g+1$ the home cup is cup $\text{mod}_k(n+1)$.

The home cup will provide a frame of reference in addition to the original numbering of the cups (which correspond to the column numbers of M_k). In fact, for some developments, we may want to shift M_k so that the home cup is first (but stays in the same column); in such cases, the matrix would look like a staircase with steps of different size. The following definition will be useful dealing with this change in the frame of reference.

Definition 20. Consider the CSCP problem with k cups. Recall that we label the cups by their position (cup 1 to cup k), which correspond to columns 1 to k in M_k.

$$M_4 = \begin{pmatrix} \boxed{\underline{1}} & 1 & 1 & 1 \\ \boxed{0} & \underline{2} & 1 & 1 \\ \boxed{0} & 0 & 2 & \underline{2} \\ 1 & \boxed{\underline{1}} & 2 & 0 \\ 1 & \boxed{0} & \underline{3} & 0 \\ 2 & \boxed{\underline{1}} & 0 & 1 \\ 2 & 0 & \boxed{\underline{1}} & 1 \\ 2 & 0 & \boxed{0} & \underline{2} \\ 3 & \underline{1} & \boxed{0} & 0 \\ 3 & 0 & \boxed{\underline{1}} & 0 \\ 3 & 0 & 0 & \boxed{\underline{1}} \\ \underline{4} & 0 & 0 & \boxed{0} \end{pmatrix} \quad M_4^* = \begin{pmatrix} \underline{1} & 1 & 1 & 1 & & & \\ 0 & \underline{2} & 1 & 1 & & & \\ 0 & 0 & 2 & \underline{2} & & & \\ & \underline{1} & 2 & 0 & 1 & & \\ & 0 & \underline{3} & 0 & 1 & & \\ & \underline{1} & 0 & 1 & 2 & & \\ & & \underline{1} & 1 & 2 & 0 & \\ & & 0 & \underline{2} & 2 & 0 & \\ & & 0 & 0 & 3 & \underline{1} & \\ & & \underline{1} & 0 & 3 & 0 & \\ & & & \underline{1} & 3 & 0 & 0 \\ & & & 0 & \underline{4} & 0 & 0 \end{pmatrix}$$

Framed box ≡ Home cup
Underlined ≡ Root cup

Figure 3: The home cup shift

1. We define the *staircase matrix* M_k^* as the matrix of configurations where the first cup in each row is always the *home* cup. To obtain M_k^* from M_k, we proceed as follows: Say in configuration g the home cup is cup n; then move cups 1 through $n-1$ by displacing them to the end of row g, so that cups 1 through $n-1$ in fact would occupy *virtual* columns $k+1$ to $k+n-1$.

2. The home cup determines a partition of the matrix M_k^* into sections (call them *step sections*). Each step section consists of the set of consecutive rows where the home cup is in the same column.

3. Let r_{kl}^* be the row number of the first row in section l of M_k^*.

4. In this setting, we also define $C_k^*(x) \equiv \mathrm{Mod}_k(C_k(x)-n+1)$ and $R_k^*(x) \equiv R_k(x)$.

Note that the picture of M_k^* is not a matrix in the traditional sense. M_k^* looks like a staircase, where each indentation occurs at the places where the home cup changes.

Example 21. Figure 3 illustrates the home cup shift for $k=4$. Note that there are 4 sections.

Definition 22. Let $(M_k^*)_1, \ldots, (M_k^*)_m$ be the step sections of M_k^*. Each of the sections is a rectangular submatrix in M_k^*. We define the *core* of each section, $A_{k,i}^*$ $(1 \le i \le m)$ to be the submatrix of $(M_k^*)_i$ obtained from $(M_k^*)_i$ by removing the rows where the root is at the home cup, and by eliminating as well the entries corresponding to the home cup.

Before we state and prove a lemma for general k, we need to consider the special case $k = 2^n + 1$ in the staircase matrix.

Corollary 23. (Corollary to Fundamental Lemma, Part 1)

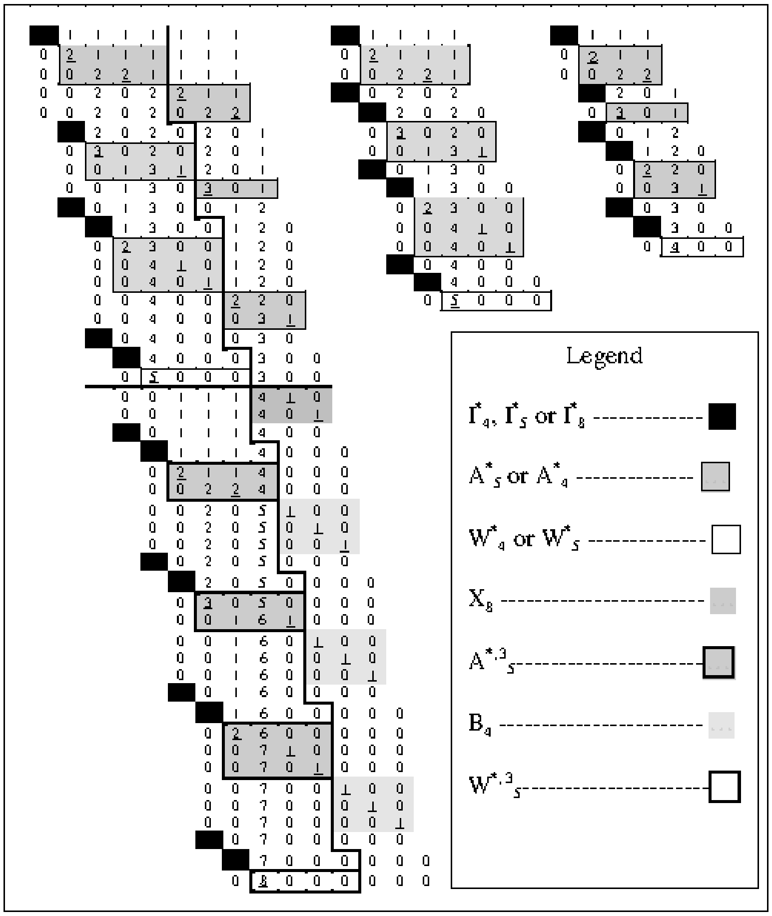


Figure 4: Matrices M_8^* and M_7^* (note T_1, T_2, T_3, T_4), with M_5^*, M_4^* and M_3^* (note the sections' cores).

Let $k = 2^n + 1$ for any positive integer n. Then:

1. If $1 \leq i < S_k$, then $C_k^*(i) + R_k^*(i) > k \rightarrow C_k^*(i) + R_k^*(i) = k + 1$.

2. There are $k - 1 = 2^n$ step sections in M_k^*

3. For $1 \leq l \leq k - 2$, section l consists of all the rows with row number in the interval $[r_{kl} + l - 1, r_{k,l+1} + l - 1]$ (inclusive). Section $(k - 1)$ includes row $r_{k,k-1} + k - 2$ and all subsequent rows.

Proof. We know from Lemma 13 that the initiator row $r_{kl} = (\underline{1}, [0]^{l-1}, \dots)$ in M_k. This implies that the step matrix M_k^* is obtained from the matrix M_k

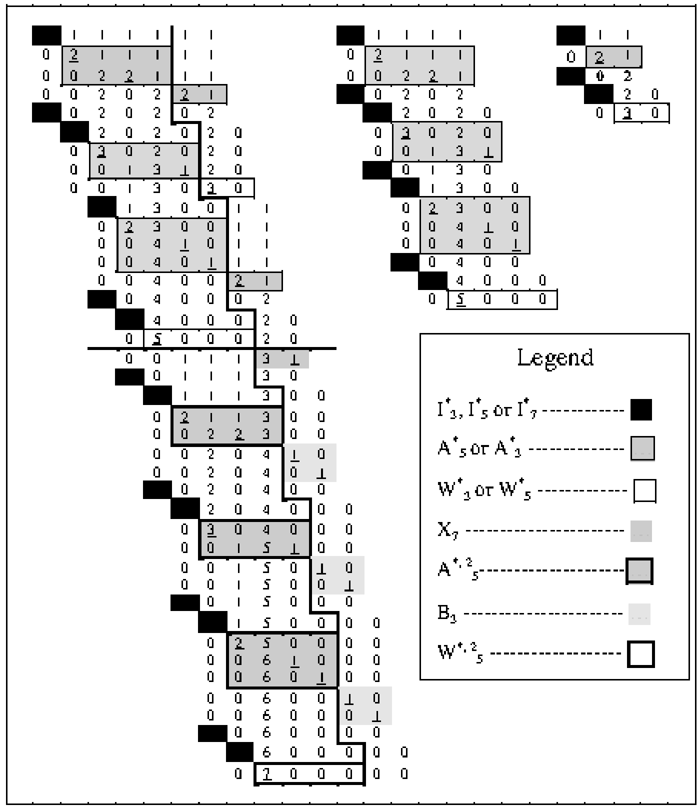


Figure 4: Continuation: Matrices M_8^* and M_7^* (note T_1, T_2, T_3, T_4), with M_5^*, M_4^* and M_3^* (note the sections' cores).

by shifting the rows as follows:

1. Rows i, for $1 \leq i \leq r_{k2}$ are the same for both M_k and M_k^*.

2. Row $r_{k2} = (\underline{1}, 0, \cdots)$ in M_k and M_k^*. Clearly in M_k, row $r_{k2} + 1 = (0, \underline{1}, \cdots)$.

From the definition of home cup shift, row $r_{k2} + 1 = (\underline{1}, \ldots, 0)$ in M_k^*. Therefore row $r_{k2}+1$ and every row below it gets shifted one unit (i.e. the first entry gets relocated to virtual column $k+1$) in M_k to build M_k^*.

3. We continue in this fashion, where in general for all l (where $1 \leq l \leq k-1$) each row i (where $r_{kl} + l - 1 \leq i \leq r_{k,l+1} + l - 1$) gets shifted $l-1$ units.

Since there are 2^n initiator rows, the construction above shows parts 2 and 3 of the lemma.

We defined r^*_{kl}, where $1 \le l \le k-1$, to be the row number of the first row of section l in M^*_k. It is easy to see from the construction of M^*_k that if $C^*_k(i) + R^*_k(i) > k$, then $i = r^*_{kl} - 2$ and row $r^*_{kl} - 1 = (\underline{1}, 0, \dots)$, which means that $C^*_k(i) + R^*_k(i) = k+1$. □

Lemma 24. *Consider M^*_k, for $k = 2^n + 1$ and let Σ be any section in M^*_k. Then:*

1. The home cup has value 0 or $\underline{1}$.

2. The rows after the second row in Σ and not containing the root in the home cup are (if they exist) of the form $(0, 0, ...)$.

Proof. The result follows easily from Lemma 7 and the construction in the proof of Corollary 23. □

Definition 25. Let 2^n be the biggest power of two less than or equal to $k-2$ (i.e. $n = \lfloor \log_2(k-2) \rfloor$). Next, divide M^*_k into four parts, the top left (denote it T_1) the top right (T_2) the bottom left (T_3) and the bottom right (T_4), as follows:

1. Let the first row i such that $R_k(i) \ge 2^n + 1$ be defined to be the *divider row*. Then the top parts (T_1 and T_2) will contain all the rows above and including the divider row while the bottom parts (T_3 and T_4) will contain the rows below the divider row (see Figure 4).

2. The left parts (T_1 and T_3) will contain the first $2^n + 1$ entries in each row of M^*_k. The right parts (T_2 and T_4) will contain the last $k - 2^n - 1$ entries in each row of M^*_k.

Define the sections of T_1 to be the intersection of T_1 with the step sections of M^*_k. Similarly we define the sections of T_2, T_3, and T_4. Call them $(T_1)_1, (T_1)_2 ...,$ $(T_2)_1, (T_2)_2, ...,$ $(T_3)_1, (T_3)_2, ...,$ and $(T_4)_1, (T_4)_2, ...$

Finally we define the *core* of each section of T_1, T_2, T_3 and T_4 to be the submatrix of them such that:

1. Each row has a root in it.
2. The root is not in the home cup.
3. The elements corresponding to the home cup are eliminated.

Call them

$$(T_1)'_1, (T_1)'_2 ..., \quad (T_2)'_1, (T_2)'_2, ..., \quad (T_3)'_1, (T_3)'_2, ...,$$

and $(T_4)'_1, (T_4)'_2, ...$

We are now ready to state the fundamental lemma for a general k.

Lemma 26. (Fundamental Lemma Part 2) *Let M^*_k be the staircase matrix for the CSCP, where k is any integer and let $n = \lfloor \log_2(k-2) \rfloor$. Then*

1. $C_k^*(i) + R_k^*(i) > k \to [(C_k^*(i) + R_k^*(i) = k+1)$ *or* $(C_k^*(i) + R_k^*(i) = k+2)]$.
2. M_k^* *has* $2(2^n)$ *sections.*
3. The last row of M_k^* *is* $(0, \underline{k}, [0]^{k-2})$.

Outline of proof of Fundamental Lemma Part 2. The proof of Fundamental Lemma Part 2 is similar to the proof of Part 1, although it is much more complicated. The proof is outlined below (for a more detailed proof, see Section 6).

We use induction on k, assuming that the lemma is true for all $M_{k'}^*$, where $k' < k$, and then show it is true for M_k^*. We also use a nested induction on the step sections of M_k^*.

We show that the cores of T_1 and T_3 behave just like the cores of M_{2^n+1} where $n = \lfloor \log_2(k-2) \rfloor$ while the cores of T_2 behave like the cores of $M_{k'}$, where $k' = k - (2^n + 1)$. We show this by proving the following points:

Except for the divider row, whenever the stone lands in the last column of T_1 (or T_3), the next root is in the first column of T_2 (or T_4) by the Corollary to Fundamental Lemma Part 1.

Whenever a stone lands in the last column of T_2 or T_4, the next root is either one or two columns past this last column (by induction on k). It is easy to show this root will be the home cup and that it will only have one stone in it (by Lemma 7). Then it is simple to show that when a stone lands in the second column of T_1 or T_3 (the beginning of a core section), it will always be a root.

Finally, the divider row itself can be shown to be $(0, \underline{2^n + 1}, [0]^{2^n - 1} \mid k - 2^n - 1, [0]^{k-2^n-2})$ which clearly obeys the lemma. □

We shall now prove Corollary 2, which was stated in the introduction.

Proof. From parts (2) and (3) of the Fundamental Lemma Part 2, we see that all the stones will land in column $\text{mod}_k(2^{n+1} + 1)$. The number of times this shift must occur until the stones pile up back in column 1, is τ, where τ is the smallest integer such that $\text{mod}_k(\tau 2^{n+1} + 1) = 1$. This means that $\text{mod}_k(\tau 2^{n+1}) = 0$, and therefore, if 2^p is the largest power of 2 that divides k, $\tau = k/2^p$. □

Definition 27. Let 2^n be the biggest power of two less than or equal to $(k-2)$, i.e. $n = \lfloor \log_2(k-2) \rfloor$. We introduce the notation $p_2(k) \equiv 2^n$.

Corollary 28. *Consider* M_k^* *with its four parts* $T_1, T_2, T_3,$ *and* T_4*. Let* $n = \lfloor \log_2(k-2) \rfloor$*, let* $m = 2(p_2(2^n + 1)) = 2(2^{n-1}) = 2^n$ *and* $\mu = 2(p_2(k - 2^n))$ *(see Definition 27). Then* T_1 *has* m *sections,* T_2 *has* m-1 *sections and:*
1. $(T_1)_j' = (M_{2^n+1}^*)_j', 1 \le j \le m$.
2. $(T_2)_j' = (M_{k-2^n}^*)_{Mod_\mu(j)}', 1 \le j \le m-1$.
3. The home cup in Part T_1 *has value* 0 *or* $\underline{1}$.

4. The divider row is $(0, \underline{2^n+1}, [0]^{2^n-1} \mid k-2^n-1, [0]^{k-2^n-2})$*, where the vertical line ("|") divides* T_1 *and* T_2*.*

Proof. Now that Lemma 26 has been established, the same reasoning as in such lemma can be used to show this corollary. □

Corollary 29. *Consider* M_k^* *with its four parts* $T_1, T_2, T_3,$ *and* T_4*. Let* $n = \lfloor \log_2(k-2) \rfloor$*. Then:*

1. The home cup in Part T_3 *has value* 0 *or* $\underline{1}$*.*

2. T_3 *has* 2^n *core sections, and they are identical to the corresponding core sections from* $M_{2^n+1}^*$ *except for the cup that is in position* 2^n+1 *in the first row of* $M_{2^n+1}^*$ *which is always* $k-2^n-1$ *smaller.*

3. T_4 *has* 2^n *core sections.* $(T_4)_1'$ *is Part of section number* 2^n *in* M_k^* *and looks like a modified* $(k-2^n-1)\times(k-2^n-1)$ *identity matrix, where the first row has been eliminated and the value* $k-2^n$ *has been added to each entry in the first column. The other core sections of* T_4 *all look like the* $(k-2^n-1)\times(k-2^n-1)$ *identity matrix (i.e.* B_{k-2^n}*).*

Proof. This result follows easily from the constructions in Lemma 26 . □

Definition 30. (see Figure 4) Let k be an arbitrary positive integer, let M_k^* be the staircase matrix of the CSCP and let $n = \lfloor \log_2(k-2) \rfloor$. Then

1. Let I_k^* be the column matrix of $\underline{1}$'s, of size equal to the number of rows in M_k^* where the root is at the home cup.

2. Let A_k^* be the matrix obtained by placing all but the last of the core sections of M_k^* (i.e. $A_{k,1}^*, \ldots, A_{k,2(2^n)-1}^*$) above each other to form one shifted matrix (keeping the same positional relation as in M_k^*).

3. Let $W_k^* = A_{2(2^n)}^*$.

Definition 31. Let $k = 2^n + 1$ for a non-negative integer n, and consider the CSCP from Lemma 16 with initial configuration $(\underline{1}, [1]^{k-2}, d)$, where d is any positive integer.

1. Let $\Gamma_k^{*,d}$ be the shifted version of Γ_k^d (see Definition 17).

2. Let $M_k^{*,d}$ be shifted matrix consisting of the first S_k rows of Γ_k^d.

3. Since $M_k^{*,d}$ has an identical structure to M_k^*, we define in an analogous way the matrices $A_k^{*,d}, A_{k1}^{*,d}, ..., A_{k,(k-1)}^{*,d}$, and $W_k^{*,d} = A_{k,(k-1)}^{*,d}$.

We now summarize the results for M_k^*, where k is an arbitrary non-zero integer.

Theorem 32. (Fractal Decomposition for Arbitrary k) *Let* M_k^* *be the staircase matrix for an arbitrary positive integer* k *and let* $n = \lfloor \log_2(k-2) \rfloor$*.*

Define X_k to be a modified $(k-2^n-1)\times(k-2^n-1)$ identity matrix, where the first row has been eliminated and the value $k-2^n$ has been added to each entry in the first column. Then:

1. $A^*_{k,l} = A^*_{2^n+1,l} \oplus A^*_{k-2^n,Mod_\mu(l)}$, *where* $1 \le l \le 2^n-1$ *and* $\mu = 2(2^{\lfloor \log_2(k-2^n-2)\rfloor})$.
2. $A^*_{k,2^n} = W^*_{2^n+1} \oplus X_k$.
3. $A^*_{k,l} = A^{*,k-2^n-1}_{2^n+1,Mod_{2^n}(l)} \oplus B_{k-2^n}$, *where* $2^n+1 \le l \le 2^{n+1}-1$.
4. $A^*_{k,2^{n+1}} = (\underline{k}, [0]^{k-1})$.

Proof. This result follows from the statement and proof of Lemma 26 (Corollarie 28 and Corollarie 29 describe this result in the language used in the lemma). $\square$

5. The Fractal Structure of M_k

5.1. The Fractal Structure of M_k for $k = 2^n + 1$

Define α_k to be matrix obtained by combining A_k and W_k and define α^i_k to be matrix obtained by combining A^i_k and W^i_k. Then, in the example we presented, α_9 and A_9 are of the form:

$\alpha_9 =$

<table><tr><td>α_5</td><td>A_5</td></tr><tr><td>B^4_5</td><td>α^1_5</td></tr></table>

$A_9 =$

<table><tr><td>α_5</td><td>A_5</td></tr><tr><td>B^3_5</td><td>A^1_5</td></tr></table>

where B^n_k is B_k repeated n times. Each α and A can then be decomposed in a recursive manner as follows:

$\alpha_{2^n+1} =$

<table><tr><td>$\alpha_{2^{n-1}+1}$</td><td>$A_{2^{n-1}+1}$</td></tr><tr><td>$B^{2^{n-1}}_{2^{n-1}+1}$</td><td>$\alpha^1_{2^{n-1}+1}$</td></tr></table>

$A_{2^n+1} =$

<table><tr><td>$\alpha_{2^{n-1}+1}$</td><td>$A_{2^{n-1}+1}$</td></tr><tr><td>$B^{2^{n-1}-1}_{2^{n-1}+1}$</td><td>$A^1_{2^{n-1}+1}$</td></tr></table>

Clearly this creates a fractal like object.

5.2. The fractal nature of M^*_k for any integer k

Let α^*_k be A^*_k and W^*_k combined and define α^{i*}_k similarly. Let β^n_k be X_k and B^n_k combined. Then in the example for $k = 7$, α^*_7 and A^*_7 can be decomposed into the following form:

$\alpha^*_7 =$

<table><tr><td>α^*_5</td><td>α^*_3
A^*_3</td></tr><tr><td>α^{1*}_5</td><td>β^3_3</td></tr></table>

$A^*_7 =$

<table><tr><td>α^*_5</td><td>α^*_3
A^*_3</td></tr><tr><td>A^{1*}_5</td><td>β^3_3</td></tr></table>

Each α^* and A^* can then be decomposed just as before. The general form of the recursion is:

$$\alpha_k^* = \begin{array}{|c|c|}\hline \alpha^*_{2^n+1} & \begin{array}{c}\alpha^*_{k-2^n}\\ \vdots \\ A^*_{k-2^n}\end{array} \\ \hline \alpha^{1*}_{2^n+1} & \beta^{2^n-1}_{k-2^n} \\ \hline \end{array} \qquad A_k^* = \begin{array}{|c|c|}\hline \alpha^*_{2^n+1} & \begin{array}{c}\alpha^*_{k-2^n}\\ \vdots \\ A^*_{k-2^n}\end{array} \\ \hline A^{1*}_{2^n+1} & \beta^{2^n-1}_{k-2^n} \\ \hline \end{array}$$

where n is the largest power of 2 that is strictly less than $k-1$. In general the number of terms in the top right section will be a power of 2, the last being an A^* and the rest α^*'s.

Clearly this creates a fractal like object.

6. Proofs of the Lemmas

6.1. Proof of Fundamental Lemma 1 Part 1

Proof of Fundamental Lemma 1 (Induction on n). It is certainly true for $n = 0$ and 1 (see Figure 1). Assume it is true for $k = 2^n + 1$, and we shall prove it for $k' = 2^{n+1} + 1$.

Our assumption implies that in M_k the following holds: since the root never lands in cup k until the very end, this cup gains a stone every time the T.r. passes over it until it has all k stones.

We can summarize the structure of M_k as follows:

1. The row r_{kl} has the root in column 1, i.e. $C_k(r_{kl}) = 1$.
2. The value of the root in row r_{kl} is 1, i.e. $R_k(r_{kl}) = 1$.
3. $M_k(r_{kl}, k) = l$.
4. The root in the last row of M_k is in column k and has value k, i.e. $R_k(S_k) = C_k(S_k) = k$.
5. There are $k - 1 = 2^n$ initiator rows in M_k, which divide M_k into 2^n sections.
6. Row $r_{k(k-1)}$ in M_k is $(\underline{1}, [0]^{k-2}, (k-1))$.

Now consider the matrix $M_{k'}$. Divide it into three parts, the first with one column and the next two with $k-1$ columns each (see the second matrix in figure 2):

Part 1. Column 1.

Part 2. Columns 2 to k.

Part 3. Columns $(k+1)$ to k'.

We first look at rows $r_{k'i}$, where $1 \leq i \leq 2^n$ (because M_k has 2^n sections).

Claim 1. *$A_{k'i} = A_{ki} \oplus A_{ki}$ and $R_{k'}(r_{k'i}) = C_{k'}(r_{k'i}) = 1$, where $1 \leq i < 2^n$ and $R_{k'}(r_{k'2^n}) = C_{k'}(r_{k'2^n}) = 1$ (see Figure 2).*

We use induction on i to prove Claim 1.

For $i = 1$, $R_{k'}(r_{k'i}) = C_{k'}(r_{k'i}) = 1$ is true since the initial row is $(\underline{1}, [1]^{k'-1})$. Applying T.r. to the initial row produces $(0, \underline{2}, [1]^{k'-1})$. The segment of this row which is in Part 2 (i.e. $(\underline{2}, [1]^{k-2})$) equals α_{k1} (see Definition 11). The maximal self-contained block generated by α_{k1} will equal A_{k1}.

The last row in this self-contained block is β_{k1}. Say it is in row ρ in $M_{k'}$. By the induction hypothesis on n applied to A_{k1}, the last stone of $T.r.(\beta_{k1})$ will land on column $k+1$. Therefore the last $k-1$ entries of $M_{k'}$ on row $\rho+1$ look like $(\underline{2}, [1]^{k-2})$, which equals α_{k1} and will therefore generate another maximal self-contained block identical to A_{k1}. The last row of such block is Part of row $r_{k'2} - 1$ in $M_{k'}$, so that in fact $A_{k'1} = A_{k1} \oplus A_{k1}$.

Now assume Claim 1 is true for $l < (k-1)$. We will show it is true for $l+1$. Basically we are assuming that rows $r_{k'l}$ to $r_{k',l+1}$-1 look like $(\underline{1}) \oplus A_{kl} \oplus A_{kl}$.

Now, row $r_{k',l+1} - 1$ equals $(0, T.r.(\beta_{kl}), \beta_{kl}) = (0, \beta_{k'l})$. By induction on n applied to A_{kl} (whose last row is β_{kl}), the last cup of $T.r.(\beta_{k'l})$ is $k' + 1$. Therefore row $r_{k',l+1}$ will look like $(\underline{1}, T.r.(\beta_{kl}), T.r.(\beta_{kl}))$.

By the lemma's induction hypothesis, the initiator row r_{kl} is $(\underline{1}, T.r.(\beta_{kl}))$ and $T.r.(\underline{1}, T.r.(\beta_{kl})) = (0, \alpha_{k,l+1})$. Applying this to the matrix $M_{k'}$ we get

$$T.r.(\underline{1}, T.r.(\beta_{kl}), T.r.(\beta_{kl})) = (0, \alpha_{k,l+1}, T.r.(\beta_{kl}))$$

in which configuration the root is in the second column. The maximal self-contained matrix generated by $\alpha_{k,l+1}$ is identical to $A_{k,l+1}$.

The last row of this self-contained matrix is identical to $\beta_{k,l+1}$. The row containing such segment in $M_{k'}$ looks like $(0, \beta_{k,l+1}, T.r.(\beta_{kl}))$. Then by induction on n, $T.r.(0, \beta_{k,l+1})$ will place its last stone on column $k+1$, and $T.r.(0, \beta_{k,l+1}, T.r.(\beta_{kl})) = (0, T.r.(\beta_{k,l+1}), \alpha_{k,l+1})$. Now $\alpha_{k,l+1}$ will generate another self-contained maximal matrix equal to $A_{k,l+1}$ in the last $k-1$ columns of $M_{k'}$. We are done now proving Claim 1.

This implies that in $M_{k'}$, for $1 \leq l \leq k-1$, rows $r_{k',l}$ have the root in column 1 and such roots have value 1. Note that the section in between rows $r_{k'l}$ and $r_{k',l+1}$ has two identical copies of A_{kl}. In fact $A_{k',l} = A_{kl} \oplus A_{kl}$. So far we have shown the inductive step in the first $r_{k',k-1}$ rows of $M_{k'}$.

Now, row $r_{k',(k-1)}$ in $M_{k'}$ is: $(\underline{1}, [0]^{k-2}, k-1, [0]^{k-2}, k-1)$. After applying T.r. $k-2$ times, we get the row $([0]^{k-1}, \underline{k}, [0]^{k-2}, k-1)$, which immediately leads to $(\underline{1}, [0]^{k-1}, [1]^{k-2}, k)$. This shows the inductive step up to row $r_{k',k}$.

Applying the T.r. $k-1$ times to $(\underline{1},[0]^{k-1},[1]^{k-2},k)$ gives $([0]^{k-1},\underline{1},[1]^{k-2},k)$, i.e. $\text{T.r.}^{k-1}(\underline{1},[0]^{k-1},[1]^{k-2},k) = ([0]^{k-1},\underline{1},[1]^{k-2},k)$ and the latter will behave under the T.r. almost exactly like $(\underline{1},[1]^{k-2},1)$, which is the initial configuration of M_k. In other words, the zeros on the left in $([0]^{k-1},\underline{1},[1]^{k-2},k)$ will not affect what happens when the T.r. gets to or beyond the last column (which is what we are addressing in this lemma); the zeros will just carry the single stone that lands in the first column of the initiator rows to column $k+1$ to continue generating identical segments to the ones in M_k (except for the extra stones in the last column). Because of our inductive assumption on n, the k in the last column (i.e. the extra stones as compared to M_k) will not affect our hypothesis either (the stones just continue to accumulate there, but do not affect other cups). This implies that the inductive step in this proof is true up to the configuration $(\underline{1},[0]^{k'-2},k'-1)$, which is row $r_{k',k'-1}$. Applying T.r. $k'-1$ times gives $([0]^{k'-1},\underline{k'})$. This shows the inductive step holds, and we are therefore done with the induction proof. □

6.2. Proof of Fundamental Lemma Part 2

Proof of Fundamental Lemma Part 2. We use induction on n, where $n = \lfloor \log_2(k-2) \rfloor$ (if $k<3$ the result follows easily, so we only consider the cases where $k \geq 3$).

If $n=0$ then $k=3$; in this case the result can be checked directly. Assume the lemma holds for all values $M^*_{k'}$, where $\lfloor \log_2(k'-2) \rfloor < n$; we'll show it is true for any M^*_k, where $\lfloor \log_2(k-2) \rfloor = n$.

Claim 1. *Consider M^*_k with its parts T_1, T_2, T_3, T_4. Let $m = 2(p_2(2^n+1)) = 2(2^{n-1}) = 2^n$ and $\mu = 2(p_2(k-2^n))$ (see Definition 25). Then T_1 has* m *sections, T_2 has* m−1 *sections and*

1. $(T_1)'_j = (M^*_{2^n+1})'_j, 1 \leq j \leq m$.

2. $(T_2)'_j = (M^*_{k-2^n})'_{Mod_\mu(j)}, 1 \leq j \leq m-1$.

3. The home cup in Part T_1 has value 0 *or* $\underline{1}$ *(see Figure 4).*

Proof of Claim 1. We shall use induction on j, the core index. For the base case, the first row in $(T_1)'_1$ equals the first row in $(M^*_{2^n+1})'_1$ and the root is there and in the same position, so $(T_1)'_1 = (M^*_{2^n+1})'_1$. By Corollary 23 and using a similar argument as in the base case for the Fundamental Lemma Part 1, similar conditions hold for $(T_2)'_1$ and $(M^*_{k-2^n})'_1$, therefore $(T_2)'_1 = (M^*_{k-2^n})'_1$. Part (3) of the claim follows since the initial configuration is $(\underline{1},[1]^{k-1})$, and the home cup as well as the root cup is cup 1. This takes care of the base step in the induction. We next deal with the inductive step.

Before addressing the inductive step, it is useful to note that:

1. The core sections of T_1 will be shown to be identical to the core sections of $M^*_{2^n+1}$. By the induction hypothesis Part 2 of this lemma (note that $\lfloor \log_2(2^n + 1 - 2) \rfloor < \lfloor \log_2(k-2) \rfloor$) $M^*_{2^n+1}$ has m sections, as described in Corollary 23.

2. The only row in $M^*_{2^n+1}$ that contains an element of value 2^n+1 is its last row (see Corollary 12), and such row looks like $(0, \underline{2^n+1}, [0]^{2^n-1})$ (by the induction hypothesis from this lemma Part 3).

3. We argue that the divider row is the last row of the last core section of T_1.

By definition, the divider row contains the first root cup with at least 2^n+1 stones. The rows that are not in the core sections of either T_1 or T_2 have the root in the home cup which has a value of $\underline{1}$ by the induction hypothesis of Claim 1. So the divider row must either be in a core section of T_1 or T_2.

The core sections of T_2 will be shown to be equal to the core sections of $M^*_{k-2^n}$. If $k - 2^n < 2^n + 1$ the root cup with value 2^n+1 cannot be in $M^*_{k-2^n}$, or equivalently T_2; on the other hand if $k - 2^n = 2^n + 1$, for each l, the core section $(T_1)'_l$ will be identical to and come before the core section $(T_2)'_l$. The divider row will therefore be a row in T_1.

4. From the facts above, we see that T_1 will have m sections, and T_2 will have $m-1$ sections.

5. From the way sections were defined, for $1 \leq l \leq m-1$ the core section $(T_2)'_l$ comes immediately after the core section $(T_1)'_l$. The divider row is the last row of $(T_1)'_m$, so there is no section $(T_2)'_m$. This is why T_2 has $m-1$ sections.

6. Since $\lfloor \log_2(k-2^n-2) \rfloor$ and $\lfloor \log_2(2^n+1-2) \rfloor$ are both less than $\lfloor \log_2(k-2) \rfloor$ we can use induction on n to say that the number of core sections in both $M^*_{k-2^n}$ and $M^*_{2^n+1}$ is a power of 2. Since $\lfloor \log_2(k-2^n-2) \rfloor \leq \lfloor \log_2(2^n+1-2) \rfloor$ then the number of core sections of $M^*_{k-2^n}$ divides the number of core sections of $M^*_{2^n+1}$.

7. By induction on n if a section of T_2 equals the last section of $M^*_{k-2^n}$, the last row of such section will look like $(\underline{k-2^n-1}, [0]^{k-2^n-2})$, the following section of T_2 will start with a row of 1's, and its core section below will equal the first core section of $M^*_{k-2^n}$.

8. If (number of core sections of $M^*_{2^n+1}$)/(number of core sections of $M^*_{k-2^n}$) $= c$, then Claim 1 implies that T_2 will have $c-1$ complete sets of core sections (identical to the set of core sections in $M^*_{k-2^n}$), and 1 incomplete set of core sections where the last section is missing. Note that when comparing core sections of T_2 to core sections of $M^*_{k-2^n}$ we need to index the latter using Modulo notation to describe how the sections cycle.

First we show the inductive step for Claim 1 Part 1: Assume $(T_1)'_{j-1} = (M^*_{2^n+1})'_{j-1}$ and $(T_2)'_{j-1} = (M^*_{k-2^n})'_{Mod_\mu(j-1)}$. We shall show $(T_1)'_j =$

$(M^*_{2^n+1})'_j$.

Note that we are assuming the lemma is true for any $M^*_{k'}$ where $\lfloor \log_2(k'-2)\rfloor < \lfloor \log_2(k-2)\rfloor$. We will show it is true for M^*_k. In particular, the lemma is true for $M^*_{k-2^n}$. We are also using induction of the section number, and so $(T_2)'_{j-1} = (M^*_{k-2^n})'_{Mod_\mu j-1}$. Therefore, the last line of $(T_2)'_{j-1}$ (say it is in row ν_2 of M^*_k) will have one of the two following cases (since the equivalent is true in $M^*_{k-2^n}$):

1. $C^*_k(\nu_2) + R^*_k(\nu_2) = k+1$.
2. $C^*_k(\nu_2) + R^*_k(\nu_2) = k+2$.

Consider case 1. We know the last row of $(T_1)'_{j-1}$ equals the last row of $(M^*_{2^n+1})'_{j-1}$, call that segment γ_1, and say it is contained in row ν_1 in M^*_k. Say $T.r.(\gamma_1) = \gamma'_1$. In M^*_k, let row $\nu_1 = (..., \gamma_1|x, ...)$, then by Corollary 23 row ν_1+1 looks like $(\dots, \gamma'_1|\underline{x+1}, \dots)$, where "|" indicates the place where T_1 ends and T_2 starts. We will show that (for different reasons) both the first row in $(T_1)'_j$ and the first row in $(M^*_{2^n+1})'_j$ look like γ'_1 with the first entry removed, an increase by 1 to the second entry (which is the root), and a 0 added at the end. Indeed, on the last row of $(T_2)'_{j-1}$ (which has row number ν_2), $C^*_k(\nu_2) + R^*_k(\nu_2) = k+1$. Because of our assumption on Claim 1 Part 3, the home cup in row ν_2 has value zero. Therefore $C^*_k(\nu_2+1) = R^*_k(\nu_2+1) = 1$. Row ν_2+2 is where the home cup shift occurs in M^*_k. At this point, $C^*_k(\nu_2+2)$ is in the same column where the root in the first row of $(T_1)'_{j-1}$ was, which necessarily became empty (as every cup that is a root does after T.r. acts on it). Therefore $C^*_k(\nu_2+2) = R^*(\nu_2+2) = 1$ (needed to show Part 3 of Claim 1). The first row of $(T_1)'_j$ is in row ν_2+3, and we can now see that indeed, the first entry in $(T_1)'_j$ is obtained from γ'_1 by removing its first entry (because of the home cup shift) and increasing the second entry by 1, which becomes the root (because $C^*_k(\nu_2+2) = R^*(\nu_2+2) = 1$ and the home cup shift occurs in row ν_2+3).

Because of the home cup shift, the last entry in the first row of $(T_1)'_j$ is in the same column as the first entry of the rows from $(T_2)_{j-1}$. We mentioned above that the row containing the first row from $(T_2)_{j-1}$ looks like $(\dots, \gamma'_1|\underline{x+1}, \dots)$. But the root cup becomes empty (until the T.r. goes by it again). This implies that the last entry in the first row of $(T_1)'_j$ is 0 as we wanted to show.

Finally, in $M^*_{2^n+1}$, an easy application of Corollary 23 shows that the first row of $(M^*_{2^n+1})'_j$ is obtained from γ'_1 exactly as we showed for γ'_1 in $(T_1)'_j$. Therefore the first line from $(T_1)'_j$ is identical to the first line from $(M^*_{2^n+1})'_j$, and since they are maximal self contained blocks, $(T_1)'_j = (M^*_{2^n+1})'_j$.

Now consider case 2 and use the same notation as in the discussion of case 1. In case 2, the home cup shift happens in row ν_2+1 (instead of ν_2+2). It is also the case that $C_k^*(\nu_2+1)$ will be on the home cup. The column where the root is in row ν_2+1 is the same where the root was in $(T_1)'_{j-1}$, which became empty after the T.r. acted on that configuration. Therefore $R_k^*(\nu_2+1)=1$. The exact same situation is in place now as in case 1, and we can conclude that $(T_1)'_j = (M^*_{2^n+1})'_j$.

We are done now showing Claim 1 Part 1.

Claim 1 Part 3 readily follows from the constructions to prove Part 1. In fact the home cup had value $\underline{1}$ when it was root cup, and then had value 0 until the T.r. distributed its last stone on it (to again have value $\underline{1}$) or passed by it (in which case the next cup became the home cup with value $\underline{1}$).

Next we show the inductive step for Claim 1 Part 2: Assume $(T_1)'_{j-1} = (M^*_{2^n+1})'_{j-1}$ and $(T_2)'_{j-1} = (M^*_{k-2^n})'_{j-1}$. We shall show $(T_2)'_j = (M^*_{k-2^n})'_j$. Say the last row in $(T_2)'_{j-1}$ and $(M^*_{k-2^n})'_{j-1}$ is γ_2 (say it happens in rows ν_2 and ν'_2 respectively), and let $T.r(\gamma_2)=\gamma'_2$. We will show that the first row in both $(T_2)'_j$ and $(M^*_{k-2^n})'_j$ is obtained from γ'_2 by removing the first entry, increasing by 1 the second entry (which becomes the root), and adding a 1 or a 0 at the end. Removing the first entry follows since there is a home cup shift in both of them before the beginning of a new core section. Increasing the second entry by 1 (and making it a root) follows in T_2 by Corollary 23 applied to the core section $(T_1)'_j$. In $M^*_{k-2^n}$ the increase by 1 of the second entry of γ'_2 is true by the lemma's induction hypothesis Part 1 (the reasoning is slightly different depending on whether we have $C_k^*(\nu'_2)+R_k^*(\nu'_2)=k+1$ or $k+2$, where ν'_2 is the row number of the last row of $(M^*_{k-2^n})'_{j-1}$). If $C_k^*(\nu_2)+R_k^*(\nu_2)=k+1$ then $C_k^*(\nu_2+1)=R^*(\nu_2+1)=1$, and the element shifted to the right will be a 0. Similarly, if in $M^*_{k-2^n}$ it is the case that $C^*_{k-2^n}(\nu'_2)+R^*_{k-2^n}(\nu'_2)=k+1$, then $C^*_{k-2^n}(\nu'_2+1)$ is the home cup, and so it was empty and became 1. The first row of $(M^*_{k-2^n})'_j$ is in row ν_2+2, and the element shifted to the right is a 0 (as we wanted to show). Now, if in T_2 $C_k^*(\nu_2)+R_k^*(\nu_2)=k+2$, the T.r drops a stone in the column where the home cup of section $j-1$ was, and the last stone in the home cup of section j. In this case, the shifted element will be a 1. The same applies to $M^*_{k-2^n}$. Therefore the first row in both are the same, and since they are self contained maximal blocks, $(T_2)'_j = (M^*_{k-2^n})'_{Mod_\mu j}$ as we wanted to show. We are now done proving Claim 1. □

Claim 1 implies that Part 1 of this Lemma is true for all the rows up to and including the divider row. Next, we show the structure of the divider row:

Claim 2. *The divider row looks like* $(0, \underline{2^n+1}, [0]^{2^n-1} \mid k-2^n-1, [0]^{k-2^n-2})$, *where the vertical line ("|") divides* T_1 *and* T_2.

Proof of Claim 2. From Claim 1 we know that the core of $M^*_{2^n+1}$ is identical to the core of T_1. By the induction hypothesis Part 3 of this lemma, we know

now that the divider row in M_k^* looks like $(0, \underline{2^n+1}, [0]^{2^n-1} \mid ...)$, where the horizontal bar "|" shows the boundary between T_1 and T_2.

Note that the number of sections of $M_{k-2^n}^*$ divides the number of sections of $M_{2^n+1}^*$ (they are both a power of 2), and the last core section of T_2 looks like the next to last core section in $M_{k-2^n}^*$. Say the last row in the last section of T_2 as well as the last row of the next to last core section of $M_{k-2^n}^*$ is γ. Let $T.r(\gamma) = \gamma'$. By the induction hypothesis Part 3, the last row of $M_{k-2^n}^*$ looks like $(0, \underline{k-2^n}, [0]^{k-2^n-2})$. Since the first entry in such row is the home cup, the first (and last) row in the last core section of $M_{k-2^n}^*$ is $(\underline{k-2^n}, [0]^{k-2^n-2})$. Using an argument similar to the one used in the proof of Claim 1, $(\underline{k-2^n}, [0]^{k-2^n-2})$ is obtained from γ' by removing the first entry, increasing by 1 the second entry, and adding a 1 or a 0 at the end. Therefore γ' must look like $(x, k-2^n-1, [0]^{k-2^n-3})$. After the home shift occurs, γ' becomes $(k-2^n-1, [0]^{k-2^n-2})$, which are the last $k-2^n-1$ elements of the divider row, as claimed. We are now done proving Claim 2. □

It is now easy to see that the row immediately below the divider row will be like

$$(0, 0, [1]^{2^n-1} | k-2^n, \underline{1}, [0]^{k-2^n-3}) .$$

Note that the divider row, as well as the row below it are in section 2^n of M_k^*. The last row of this section will be $k-2^n-1$ rows after the divider row, when the transformation rule "cycles over" and produces the row

$$(\underline{1}, 0, [1]^{2^n-1} | k-2^n, 0, [0]^{k-2^n-3}) .$$

The first row of section 2^n+1 looks like $(\underline{1}, [1]^{2^n-1}, k-2^n | 0, [0]^{k-2^n-3}, 0)$.

But Part of this row in T_3 behaves like

$$(\underline{1}, [1]^{2^n}),$$

that is, the fact that there are zeros at the end and the fact that the value of the cup in position 2^n+1 is not 1 does not affect in essence the core (see Lemma 16). In addition, the zeros on the right will not affect the fact that $C_{2^n+1}^*(i) + R_{2^n+1}^* > 2^n+1 \rightarrow C_{2^n+1}^*(i) + R_{2^n+1}^*(i) = 2^n+2$, and it easily follows that $C_k^*(i) + R_k^*(i) > k \rightarrow C_k^*(i) + R_k^*(i) = k+1$ for all rows i below the divider row. Also it immediately shows that there are 2^n new sections below the divider row (for a total of $2(2^n)$ in M_k^*). The only thing left to do is to show that the last row in fact looks like $(0, \underline{k}, [0]^{k-2})$. But that easily follows by counting the number of home cup shifts, and noting that in M_{2^n+1} the last row looks like $([0]^{2^n}, \underline{2^n+1})$. □

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