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ROGERS-HÖLDER'S INEQUALITY ON TIME SCALES

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Dedicated to Professor Haruo Murakami
on his 75-th Birthday.

Abstract: Some inequalities of Rogers-Hölder's type are established on time scales cersion by using elementary method. Time scales versions of some discrete inequalities in the book "Inequalities" by Hardy et al are also given.

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1. Introduction

To unify the theory of continuous and discrete dynamic systems, in 1990, Hilger [8] proposed the study of dynamic systems on a time scale and developed the calculus for functions on a time scale (that is, any closed subset of reals).

Recently, a time scales version of the well known Rogers-Hölder's inequality has been established, see for example, Agarwal, Bohner and Peterson [1].

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The purpose of this paper is to generalize the above results on time scales versions. Some related results are also given. For other related resuls, we refer to [2-7, 9-10, 12-15].

2. Preliminaries and Lemmas

We first briefly introduce the time scales calculus.

By a times scale $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$ with order and topological structure in a canonical way. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

Definition. Let $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale, then two mappings

$$\sigma,\ \rho : \mathbb{T} \to \mathbb{R}$$

satisfying

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} | s > t\}$$

are called the jump operators.

The point $t \in \mathbb{T}$ is called right-scattled, left-scaltted, right-dense or left dense according to $\sigma(t) > t$, $\rho(t) < t$, $\sigma(t) = t$, or $\rho(t) = t$, respectively.

Definition. Let $\mathbb{T}$ be a time scale. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if the following two conditions hold:

(i) f is continuous at each right-dense point or maximal point of $\mathbb{T}$;

(ii) $\lim\limits_{s \to t^-} g(s) = g(t^-)$ exists for each left-dense point t of $\mathbb{T}$.

Throughout this paper, we suppose that:

(a) $\mathbb{R} = (-\infty, \infty)$;

(b) $\mathbb{T}$ is a time scale;

(c) an interval means the intersection of a real interval with the given time scale;

(d) The set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}[\mathbb{T}, \mathbb{R}]$;

(e) $||f||_r := \big(\int_a^b h(x)|f(x)|^r \Delta x \big)^{\frac{1}{r}}$ for $r > 0$, where $h,\ f \in C_{rd}([a, b], \mathbb{R})$;

(f) $L^r[a, b] := \{f \in C_{rd}([a, b], \mathbb{R}) |\ ||f||_r < \infty\}$ if $r > 0$;

(g) $\mathbb{T}^k := \begin{cases} \mathbb{T} - \{m\}, \text{ if } \mathbb{T} \text{ has a left-scattered maximal point } m, \\ \mathbb{T}, \text{ otherwise.} \end{cases}$

Definition. Assume that $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, then we define $f^{\Delta}(t)$ to be the number (if it exists) with property that, for any given $\epsilon > 0$, there exists a neighborhood U of t such that

$$\Big| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \Big| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. In this case, $f^{\Delta}(t)$ is called the *delta*-derivative of $f(t)$ at t. If f is differentiable at each $t \in \mathbb{T}$, then f is called *delta*-differentiable on $\mathbb{T}$.

Definition. A function $g : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. In this case, we define the integral of f by

$$\int_s^t f(u)\Delta u = g(t) - g(s)$$

for all $s, t \in \mathbb{T}$, and we say that f is integrable on $\mathbb{T}$.

It follows from Theorem 1.74 of [2] that every rd-continuous function has an antiderivative.

For further concerning the time scale, we refer to [2, 8, 11].

To discuss our main results, we need the following two lemmas.

The first one is an extension of Schlömilch's inequality, see [7]. The second one is Radon's inequality.

Lemma A. (Schlömilch's Inequality) *Let* $h,\ f \in C_{rd}([a,b],[0,\infty))$ *with* $\int_a^b h(x)\Delta x > 0$. *If* $r < s$, *then*

$$M_r(h,f) \leq M_s(h,f),$$

where $M_r(h,f)$ *is defined by*

$$M_r(h,f) := \begin{cases} \left(\dfrac{\int_a^b h(x) f^r(x)\Delta x}{\int_a^b h(x)\Delta x} \right)^{\frac{1}{r}}, & r \neq 0; \\[2ex] \exp\left(\dfrac{\int_a^b h(x) ln f(x)\Delta x}{\int_a^b h(x)\Delta x} \right), & r = 0. \end{cases}$$

Proof. In order to prove this lemma, we need Bernoulli's inequality, that is, if $y > 0$, then

$$y^p > p\,y + 1 - p \quad \text{if} \quad p > 1 \quad \text{or} \quad p < 0.$$

Without loss of generality, we may assume that $\int_a^b h(x)\Delta x = 1$. If $r > s > 0$ or $r < s < 0$, then $\frac{r}{s} > 1$. Thus, by Bernoulli's inequality,

$$\int_a^b h(x)\Big(\frac{f(x)}{\displaystyle\int_a^b h(x)f(x)\Delta x}\Big)^{\frac{r}{s}}\Delta x \geq \int_a^b h(x)\Big[\frac{\frac{r}{s}f(x)}{\displaystyle\int_a^b h(x)f(x)\Delta x} + 1 - \frac{r}{s}\Big]\Delta x = 1,$$

that is

$$\int_a^b h(x)f^{\frac{r}{s}}(x)\Delta x \geq \Big(\int_a^b h(x)f(x)\Delta x\Big)^{\frac{r}{s}}.$$

Letting f be replaced by f^s in the above inequality, we get

$$\int_a^b h(x)f^r(x)\Delta x \geq \Big(\int_a^b h(x)f^s(x)\Delta x\Big)^{\frac{r}{s}}.$$

Hence

$$\Big(\int_a^b h(x)f^r(x)\Delta x\Big)^{\frac{1}{r}} \geq \Big(\int_a^b h(x)f^s(x)\Delta x\Big)^{\frac{1}{s}} \quad \text{if} \quad r > s > 0;$$

$$\Big(\int_a^b h(x)f^r(x)\Delta x\Big)^{\frac{1}{r}} \leq \Big(\int_a^b h(x)f^s(x)\Delta x\Big)^{\frac{1}{s}} \quad \text{if} \quad r < s < 0.$$

Clearly, if $s \to 0$, then the above two inequalities are reduced to

$$M_r(h,f) \geq M_0(h,f) \quad \text{if} \quad r > 0;$$

$$M_r(h,f) \leq M_0(h,f) \quad \text{if} \quad r < 0.$$

This completes our proof. □

Remark A. Let $\mathbb{T} = \mathbb{Z}$. If x_i, α_i are real positive numbers with $\sum_{i=1}^n \alpha_i = 1$ for $i = 1, 2, \cdots, n$. Then it follows from Lemma A that for $r < s$,

$$\Big(\sum_{i=1}^n \alpha_i x_i^r\Big)^{\frac{1}{r}} \leq \Big(\sum_{i=1}^n \alpha_i x_i^s\Big)^{\frac{1}{s}}.$$

Lemma B. (Radon's Inequality, see page 61 in Hardy, Littlewood and Polya [7]) *Let* $a_i > 0$ *and* $b_i > 0$ *for* $i = 1, 2, \cdots, n$, *then*

$$(a) \quad \frac{\left(\sum_{i=1}^{n} a_i\right)^{\lambda+1}}{\left(\sum_{i=1}^{n} b_i\right)^{\lambda}} \le \frac{a_1^{\lambda+1}}{b_1^{\lambda}} + \frac{a_2^{\lambda+1}}{b_2^{\lambda}} + \cdots + \frac{a_n^{\lambda+1}}{b_n^{\lambda}}, \quad if \quad \lambda > 0 \quad or \quad \lambda < -1;$$

(b) *The inequality of* (a) *is reversed if* $-1 < \lambda < 0$.

3. Rogers-Hölder's Inequality

We are in a position to establish the time scales version of Rogers-Hölder's inequality which extends some results of [1-4, 6-7, 9-10, 12-15].

Theorem 1. *Let* $h, f, g \in C_{rd}([a, b], [0, \infty))$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p, q \in \mathbb{R} - \{0, 1\}$. *Then the following two statements hold:*

(a) If $p > 1$, *then*

$$\int_a^b h(x)f(x)g(x)\Delta x \le \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}} \Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}},$$

and hence $M_1(h, fg) \le M_p(h, f)M_q(h, g)$;

(b) If $p < 1$, $p \ne 0$ *(for* $p < 0$, *we assume that* $\int_a^b h(x)f^p(x)\Delta x > 0$*), then*

$$\int_a^b h(x)f(x)g(x)\Delta x \ge \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}} \Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}},$$

and hence $M_1(h, fg) \ge M_p(h, f)M_q(h, g)$.

Proof. (a) It follows from the arithematic-geometric inequality that

$$\begin{aligned}
&\frac{\displaystyle\int_a^b h(t)f(t)g(t)\Delta t}{\Big(\displaystyle\int_a^b h(t)f^p(t)\Delta t\Big)^{\frac{1}{p}} \Big(\displaystyle\int_a^b h(t)g^q(t)\Delta t\Big)^{\frac{1}{q}}} \\
&\qquad = \int_a^b \left(\frac{h(t)f^p(t)}{\int_a^b h(t)f^p(t)\Delta t}\right)^{\frac{1}{p}} \left(\frac{h(t)g^q(t)}{\int_a^b h(t)g^q(t)\Delta t}\right)^{\frac{1}{q}} \Delta t \\
&\qquad \le \int_a^b \Big[\frac{1}{p}\frac{h(t)f^p(t)}{\int_a^b h(t)f^p(t)\Delta t} + \frac{1}{q}\frac{h(t)g^q(t)}{\int_a^b h(t)g^q(t)\Delta t}\Big]\Delta t
\end{aligned}$$

$$= \frac{1}{p} + \frac{1}{q} = 1.$$

(b) Without loss of generality, we prove only the case that $p < 0$ (and hence $q > 0$). Let

$$\lambda = -\frac{p}{q} (> 0), \quad \mu = \frac{1}{q} (> 0).$$

Then

$$\frac{1}{\lambda} + \frac{1}{\mu} = -\frac{q}{p} + q = 1.$$

It follows from (a) that, for $F, G \in C_{rd}([a,b],[0,\infty))$,

$$\int_a^b h(x)F(x)G(x)\Delta x \le \Big(\int_a^b h(x)F^{\lambda}(x)\Delta x\Big)^{\frac{1}{\lambda}} \Big(\int_a^b h(x)G^{\mu}(x)\Delta x\Big)^{\frac{1}{\mu}}.$$

Taking $F(x) = \frac{1}{f^q(x)}$ and $G(x) = f^q(x)g^q(x)$, we get the desired result. □

Remark 1. (a) Let $h, f, g \in C_{rd}([a,b],[0,\infty))$. Then by (a) of Theorem 1,

$$\int_a^b h(x)f^{\lambda}(x)g^{\mu}(x)\Delta x \le \Big(\int_a^b h(x)f(x)\Delta x\Big)^{\lambda} \Big(\int_a^b h(x)g(x)\Delta x\Big)^{\mu}, \qquad (R)$$

where $\lambda, \mu \in (0,1)$ with $\lambda + \mu = 1$.

Let $\lambda = \dfrac{1}{n}$, $n \in \{2,3,4,\cdots\}$, $\mu = 1 - \dfrac{1}{n}$, then (R) is reduced to

$$\int_a^b h(x)f^{\frac{1}{n}}(x)g^{\frac{n-1}{n}}(x)\Delta x \le \left(\int_a^b h(x)f(x)\Delta x\right)^{\frac{1}{n}} \left(\int_a^b h(x)g(x)\Delta x\right)^{\frac{n-1}{n}}.$$

(b) If $p, q \in (0,\infty)$ or $p,\ q \in (-\infty,0)$, then by (a) of Theorem 1,

$$\int_a^b h(x)f^{\frac{p}{p+q}}(x)g^{\frac{q}{p+q}}(x)\Delta x \le \left(\int_a^b h(x)f(x)\Delta x\right)^{\frac{p}{p+q}} \left(\int_a^b h(x)g(x)\Delta x\right)^{\frac{q}{p+q}}.$$

Theorem 2. *Let $f \in C_{rd}([a,b],[0,\infty))$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If A and B are positive constants, then*

$$\int_a^b f^p(x)\Delta x \le A$$

if and only if, for any $g \in C_{rd}([a,b],[0,\infty))$ with $\int_a^b g^q(x)\Delta x \le B$,

$$\int_a^b f(x)g(x)\Delta x \le A^{\frac{1}{p}} B^{\frac{1}{q}}.$$

Proof. (⇒) It follows from Hölder's inequality.

(⇐) Suppose not. That is, if $\int_a^b f^p(s)\Delta x > A$, then there exists $g \in C_{rd}([a,b],[0,\infty))$ such that

$$\frac{g^q(x)}{f^p(x)} = \text{constant} \quad \text{and} \quad \int_a^b g^q(x)\Delta x = B.$$

It follows from $p > 1$ and Hölder inequality that

$$\int_a^b f(x)g(x)\Delta x \geq \Big(\int_a^b f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)\Delta x\Big)^{\frac{1}{q}} > A^{\frac{1}{p}}B^{\frac{1}{q}},$$

which is a contradiction. This contradiction completes the proof. □

Theorem 3. *Let $h, f, g \in C_{rd}([a,b],[0,\infty))$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ with $p,q,r \in \mathbb{R}-\{0\}$. Then:*

(a) $\Big(\int_a^b h(x)f^r(x)g^r(x)\Delta x\Big)^{\frac{1}{r}} \leq \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}}$
if $p > 0$ and $q > 0$ or $p > 0$, $q < 0$ and $r < 0$;

(b) $\Big(\int_a^b h(x)f^r(x)g^r(x)\Delta x\Big)^{\frac{1}{r}} \geq \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}}$
if $p > 0$, $q < 0$ and $r > 0$ or $p < 0$ and $q < 0$.

Proof. (a) By $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, where $p > 0, q > 0$, we get

$$\frac{p}{r} > 1, \quad \frac{1}{p/r}+\frac{1}{q/r}=1.$$

Thus, it follows from (a) of Theorem 1 that

$$\int_a^b h(x)f^r(x)g^r(x)\Delta x \leq \Big(\int_a^b h(x)\big(f^r(x)\big)^{\frac{p}{r}}\Delta x\Big)^{\frac{r}{p}}\Big(\int_a^b h(x)\big(g^r(x)\big)^{\frac{q}{r}}\Delta x\Big)^{\frac{r}{q}},$$

which implies

$$\Big(\int_a^b h(x)f^r(x)g^r(x)\Delta x\Big)^{\frac{1}{r}} \leq \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}}.$$

Similarly, we can prove the other case.

(b) If $p > 0$, $q < 0$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ with $r > 0$, then $\frac{r}{p} > 1$ and

$$\frac{1}{r/p}+\frac{1}{-q/p}=\frac{p}{r}-\frac{p}{q}=p\Big(\frac{1}{r}-\frac{1}{q}\Big)=1.$$

Hence, it follows from (a) of Theorem 1 that

$$\int_a^b h(x)f^p(x)\Delta x = \int_a^b h(x)f^p(x)g^p(x)g^{-p}(x)\Delta x$$
$$\leq \Big(\int_a^b h(x)\big(f^p(x)g^p(x)\big)^{\frac{r}{p}}\Delta x\Big)^{\frac{p}{r}}\Big(\int_a^b h(x)\big(g^{-p}(x)\big)^{-\frac{q}{p}}\Delta x\Big)^{-\frac{p}{q}}$$
$$= \Big(\int_a^b h(x)f^r(x)g^r(x)\Delta x\Big)^{\frac{p}{r}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{-\frac{p}{q}},$$

which implies

$$\Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}} \leq \Big(\int_a^b h(x)f^r(x)g^r(x)\Delta x\Big)^{\frac{1}{r}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{-\frac{1}{q}}.$$

Thus

$$\Big(\int_a^b h(x)f^r(x)g^r(x)\Delta x\Big)^{\frac{1}{r}} \geq \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}},$$

and hence, the proof is complete.

Similarly, we can prove the other case. □

Remark 2. Let $\mathbb{T} = \mathbb{R}$. If $h, f, g \in C([a, b], [0, \infty))$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then the results of (a), (b) in Theorem 3 are reduced to the following results, respectively,

$$(a^*)\quad \Big(\int_a^b h(x)f^r(x)g^r(x)dx\Big)^{\frac{1}{r}} \leq \Big(\int_a^b h(x)f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)dx\Big)^{\frac{1}{q}}$$

if $p > 0$ and $q > 0$ or $p > 0$, $q < 0$ and $r < 0$;

$$(b^*)\quad \Big(\int_a^b h(x)f^r(x)g^r(x)dx\Big)^{\frac{1}{r}} \geq \Big(\int_a^b h(x)f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)dx\Big)^{\frac{1}{q}}$$

if $p > 0$, $q < 0$ and $r > 0$ or $p < 0$ and $q < 0$.

Remark 3. Let $\mathbb{T} = \mathbb{Z}$. If $a_k, b_k, c_k > 0$ for $k = 1, 2, \cdots, n$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then (a), (b) in Theorem 3 are reduced to the following results, respectively,

$$(a^*)\quad \Big(\sum_{k=1}^n a_k b_k^r c_k^r\Big)^{\frac{1}{r}} \leq \Big(\sum_{k=1}^n a_k b_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n a_k c_k^q\Big)^{\frac{1}{q}}, \quad \text{if } p > 0 \text{ and } q > 0$$

or $p > 0$, $q < 0$ and $r < 0$;

$$(b^*)\quad \Big(\sum_{k=1}^n a_k b_k^r c_k^r\Big)^{\frac{1}{r}} \geq \Big(\sum_{k=1}^n a_k b_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n a_k c_k^q\Big)^{\frac{1}{q}}, \quad \text{if } p > 0,\ q < 0 \text{ and } r > 0$$

or $p < 0$ and $q < 0$.

Theorem 4. *Let $h, f, g \in C_{rd}([a,b],[0,\infty))$. If $p > 0, q > 0$ with $\frac{1}{p} + \frac{1}{q} < 1$, then*

$$\int_a^b h(x)f(x)g(x)\Delta x \le \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}}.$$

That is,

$$M_1(h, fg) \le M_p(h, f)M_q(h, g).$$

Proof. Let $\lambda := \frac{1}{p} + \frac{1}{q}$, $\alpha = p\lambda, \beta = q\lambda$. Then $\alpha > 1$, $\beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Hence, it follows from Lemma A with $\alpha < p$, $\beta < q$ and (a) of Theorem 1 that

$$\begin{aligned}\int_a^b h(x)f(x)g(x)\Delta x &\le \Big(\int_a^b h(x)f^\alpha(x)\Delta x\Big)^{\frac{1}{\alpha}}\Big(\int_a^b h(x)g^\beta(x)\Delta x\Big)^{\frac{1}{\beta}}\\ &\le \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}}. \quad \square\end{aligned}$$

Theorem 5. *Let $h, f_1, f_2, \cdots, f_n \in C_{rd}([a,b],[0,\infty))$. If $\lambda_1, \lambda_2, \cdots, \lambda_n \in (0,1)$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, then*

$$\begin{aligned}&\int_a^b h(t)f_1^{\lambda_1}(t)f_2^{\lambda_2}(t)\cdots f_n^{\lambda_n}(t)\Delta t\\ &\quad\le \Big(\int_a^b h(t)f_1(t)\Delta t\Big)^{\lambda_1}\Big(\int_a^b h(t)f_2(t)\Delta t\Big)^{\lambda_2}\cdots\Big(\int_a^b h(t)f_n(t)\Delta t\Big)^{\lambda_n} \qquad (R_1)\end{aligned}$$

and hence

$$M_1(h,\ f_1^{\lambda_1}\ f_2^{\lambda_2}\ \cdots\ f_n^{\lambda_n}) \le M_1^{\lambda_1}(h,\ f_1)M_1^{\lambda_2}(h,\ f_2)\cdots M_1^{\lambda_n}(h,\ f_n).$$

Proof. If there exists a $k \in \{1, 2, \cdots, n\}$ such that $\int_a^b h(t)f_k(t)\Delta t = 0$, then $h(t)f_k(t) = 0$ almost everywhere (a.e.) on $[a,b]$. Thus without loss of generality, we may assume that $\int_a^b h(t)f_i(t)\Delta t > 0$ for $i = 1, 2, \cdots, n$. Hence, it follows from the arithematic-geometric means inequality that

$$\begin{aligned}
&\frac{\displaystyle\int_a^b h(t)f_1^{\lambda_1}(t)f_2^{\lambda_2}(t)\cdots f_n^{\lambda_n}(t)\Delta t}{\Big(\displaystyle\int_a^b h(t)f_1(t)\Delta t\Big)^{\lambda_1}\cdots\Big(\displaystyle\int_a^b h(t)f_n(t)\Delta t\Big)^{\lambda_n}}\\
&=\int_a^b\left(\frac{h(t)f_1(t)}{\int_a^b h(t)f_1(t)\Delta t}\right)^{\lambda_1}\cdots\left(\frac{h(t)f_n(t)}{\int_a^b h(t)f_n(t)\Delta t}\right)^{\lambda_n}\Delta t\\
&\le\int_a^b\left(\frac{\lambda_1 h(t)f_1(t)}{\int_a^b h(t)f_1(t)\Delta t}+\frac{\lambda_2 h(t)f_2(t)}{\int_a^b h(t)f_2(t)\Delta t}+\cdots+\frac{\lambda_n h(t)f_n(t)}{\int_a^b h(t)f_n(t)\Delta t}\right)\Delta t\\
&=\lambda_1+\lambda_2+\cdots+\lambda_n=1
\end{aligned}$$

with inequality unless

$$\frac{h(t)f_1(t)}{\displaystyle\int_a^b h(t)f_1(t)\Delta t}=\frac{h(t)f_2(t)}{\displaystyle\int_a^b h(t)f_2(t)\Delta t}=\cdots=\frac{h(t)f_n(t)}{\displaystyle\int_a^b h(t)f_n(t)\Delta t}\quad\text{a.e. on}\quad[a,b].$$

This completes our proof. □

Remark 4. Under the assumptions of Theorem 5, if $f_i^{\lambda_i}(t)=g_i(t)$, i.e., $f_i(t)=g_i^{\frac{1}{\lambda_i}}(t)$ for $i=1,2,\cdots,n$, then Theorem 5 becomes the following inequality:

$$\begin{aligned}
&\int_a^b h(t)g_1(t)g_2(t)\cdots g_n(t)\Delta t\\
&\quad\le\Big(\int_a^b h(t)g_1^{\frac{1}{\lambda_1}}(t)\Delta t\Big)^{\lambda_1}\Big(\int_a^b h(t)g_2^{\frac{1}{\lambda_2}}(t)\Delta t\Big)^{\lambda_2}\cdots\Big(\int_a^b h(t)g_n^{\frac{1}{\lambda_n}}(t)\Delta t\Big)^{\lambda_n},
\end{aligned}$$

and hence

$$M_1(h,\ g_1\ g_2\ \cdots g_n)\le M_{\frac{1}{\lambda_1}}(h,\ g_1)\ M_{\frac{1}{\lambda_2}}(h,\ g_2)\cdots M_{\frac{1}{\lambda_n}}(h,\ g_n).$$

Theorem 6. *Let $h,f_1,\cdots,f_n,\lambda_1,\cdots,\lambda_n$ be defined as in Theorem 5 with $\int_a^b h(x)\Delta x=1$. If $\lambda_1+\lambda_2+\cdots+\lambda_n=k<1$, then (R_1) holds.*

Proof. Let $\mu_i=\frac{\lambda_i}{k}$ for $i=1,2,\cdots,n$. Then

$$\mu_1+\mu_2+\cdots+\mu_n=1.$$

It follows from Theorem 5, Lemma A and $k<1$ that, for $f_i^k(t)=g_i(t)$,

$$\begin{aligned}\int_a^b h(t)f_1^{\lambda_1}(t)\cdots f_n^{\lambda_n}(t)\Delta t &= \int_a^b h(t)g_1^{\mu_1}(t)\cdots g_n^{\mu_n}(t)\Delta t \\ &\le \Big(\int_a^b h(t)g_1(t)\Delta t\Big)^{\mu_1}\cdots\Big(\int_a^b h(t)g_n(t)\Delta t\Big)^{\mu_n} \\ &= \Big(\int_a^b h(t)f_1^k(t)\Delta t\Big)^{\frac{\lambda_1}{k}}\cdots\Big(\int_a^b h(t)f_n^k(t)\Delta t\Big)^{\frac{\lambda_n}{k}} \\ &\le \Big(\int_a^b h(t)f_1(t)\Delta t\Big)^{\lambda_1}\cdots\Big(\int_a^b h(t)f_n(t)\Delta t\Big)^{\lambda_n}.\end{aligned}$$

Thus, the proof is complete. □

The following theorem extends Theorem 3 of Maligranda [10]. However, our proof is simpler than that of Maligranda [10].

Theorem 7. *Let $f,g,h\in C_{rd}([a,b],[0,\infty))$. Then the following three statements are equivalent:*

(A) Rogers-Hölder's inequality:

$$\int_a^b h(x)f(x)g(x)\Delta x \le \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}},$$

if $p,q\in(1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$;

(B) Rogers' inequality:

$$\Big(\int_a^b h(x)f(x)g^s(x)\Delta x\Big)^{t-r} \le \Big(\int_a^b h(x)f(x)g^r(x)\Delta x\Big)^{t-s}\Big(\int_a^b h(x)f(x)g^t(x)\Delta x\Big)^{s-r},$$

if $r<s<t$;

(C) Hölder's inequality:

$$\Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^{t} \le \Big(\int_a^b h(x)f(x)\Delta x\Big)^{t-1}\Big(\int_a^b h(x)f(x)g^t(x)\Delta x\Big),$$

if $1<t$.

Proof. $(A)\Rightarrow(B)$ Taking $p=\frac{t-r}{t-s}$, $q=\frac{t-r}{s-r}$ and replacing h,f,g by $hf, g^{\frac{r}{p}}, g^{\frac{t}{q}}$, repectively, (R_1) is reduced to (R_2).

$(B)\Rightarrow(C)$ Letting $r\to 0^+$ and $s=1$, (B) is reduced to (C).

$(C) \Rightarrow (A)$ Let $p = \frac{t}{t-1}$ and $q = t$. Then $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\begin{aligned}
&\int_a^b h(x)f(x)g(x)\Delta x = \int_a^b h(x)f^{\frac{t}{t-1}}(x)g(x)f^{-\frac{1}{t-1}}(x)\Delta x \\
&\leq \Big(\int_a^b h(x)f^{\frac{t}{t-1}}(x)\Delta x\Big)^{1-\frac{1}{t}}\Big(\int_a^b h(x)f^{\frac{t}{t-1}}(x)\big(g(x)f^{-\frac{1}{t-1}}(x)\Delta x\big)^t\Big)^{\frac{1}{t}} \\
&= \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^t(x)\Delta x\Big)^{\frac{1}{t}} \\
&= \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}}.
\end{aligned}$$

Remark 5. If $\mathbb{T} = \mathbb{R}$, then Theorem 7 is reduced to Theorem 3 in Maligranda [10]. Maligranda [10] proved this Theorem by using "$(A) \Leftrightarrow (B)$" and "$(C) \Leftrightarrow (A)$".

Similarly, we can prove the following theorem.

Theorem 8. *The following three statements are equivalent:*

$$(A^*) \quad \int_a^b h(x)f(x)g(x)\Delta x \geq \Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}},$$

if $\frac{1}{p} + \frac{1}{q} = 1$, $p < 1$, $p \neq 0$;

$$(B^*) \quad \Big(\int_a^b h(x)f(x)g^s(x)\Delta x\Big)^{t-r} \geq \Big(\int_a^b h(x)f(x)g^r(x)\Delta x\Big)^{t-s}\Big(\int_a^b h(x)f(x)g^t(x)\Delta x\Big)^{s-r},$$

if $r < t < s$;

$$(C^*) \quad \Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^t \geq \Big(\int_a^b h(x)f(x)\Delta x\Big)^{t-1}\Big(\int_a^b h(x)g^t(x)\Delta x\Big),$$

if $0 < t < 1$.

4. Applications

Theorem 9. (Minkowski's Inequality) *Suppose that* $h,\ f_i \in C_{rd}([a,b],[0,\infty))$ *for* $i = 1, 2, \cdots, n$. *Then the following two statements hold:*

(a) $\left(\int_a^b h(t)\Big(\sum_{i=1}^n f_i(t)\Big)^p\Delta t\right)^{\frac{1}{p}} \leq \sum_{i=1}^n \left(\int_a^b h(t)f_i^p(t)\Delta t\right)^{\frac{1}{p}}$, *if* $p > 1$;

(b) $\left(\int_a^b h(t)\Big(\sum_{i=1}^n f_i(t)\Big)^p \Delta t\right)^{\frac{1}{p}} \geq \sum_{i=1}^n \left(\int_a^b h(t) f_i^p(t)\Delta t\right)^{\frac{1}{p}}$, *if* $p<1$ *and* $p \neq 0$.

Proof. We only consider the case (a). For $p>1$, it follows from Jenson's inequality that

$$\Big(f_1(t)+f_2(t)+\cdots+f_n(t)\Big)^p \geq f_1^p(t)+f_2^p(t)+\cdots+f_n^p(t).$$

Let $A(t)=f_1(t)+f_2(t)+\cdots+f_n(t)$. Without loss of generality, we may assume that $\int_a^b h(t)A^p(t)\Delta t>0$. It follows from (a) of Theorem 1 that

$$\int_a^b h(t)f_i(t)A^{p-1}(t)\Delta t \leq \Big(\int_a^b h(t)f_i^p(t)\Delta t\Big)^{\frac{1}{p}}\Big(\int_a^b h(t)A^{(p-1)q}(t)\Delta t\Big)^{\frac{1}{q}},$$

where $q\in\mathbb{R}$ satisfies $\frac{1}{p}+\frac{1}{q}=1$. Thus,

$$\begin{aligned}\int_a^b h(t)A^p(t)\Delta t &= \int_a^b h(t)f_1(t)A^{p-1}(t)\Delta t+\cdots+\int_a^b h(t)f_n(t)A^{p-1}(t)\Delta t\\ &\leq \left(\int_a^b h(t)A^p(t)\Delta t\right)^{\frac{1}{q}}\left(\sum_{i=1}^n\Big(\int_a^b h(t)f_i^p(t)\Delta t\Big)^{\frac{1}{p}}\right).\end{aligned}$$

Hence, we obtain the desired result. □

Corollary 10. *Let* h, f_i *be defined as in Theorem 9 and* $\int_a^b h(t)\Delta t>0$. *Then:*

(a^*) *if* $p>1$, *then*

$$\left(\frac{\int_a^b h(t)\Big(\sum_{i=1}^n f_i(t)\Big)^p\Delta t}{\int_a^b h(t)\Delta t}\right)^{\frac{1}{p}} \leq \sum_{i=1}^n\left(\frac{\int_a^b h(t)f_i^p(t)\Delta t}{\int_a^b h(t)\Delta t}\right)^{\frac{1}{p}}$$

(b^*) *if* $p<1$ *and* $p\neq 0$, *then*

$$\left(\frac{\int_a^b h(t)\Big(\sum_{i=1}^n f_i(t)\Big)^p\Delta t}{\int_a^b h(t)\Delta t}\right)^{\frac{1}{p}} \geq \sum_{i=1}^n\left(\frac{\int_a^b h(t)f_i^p(t)\Delta t}{\int_a^b h(t)\Delta t}\right)^{\frac{1}{p}};$$

that is:

(a^{**}) if $p > 1$, then $M_p(h, f_1 + f_2 + \cdots + f_n) \le M_p(h, f_1) + M_p(h, f_2) + \cdots + M_p(h, f_n)$;
(b^{**}) if $p < 1$ and $p \neq 0$, then $M_p(h, f_1 + f_2 + \cdots + f_n) \ge M_p(h, f_1) + M_p(h, f_2) + \cdots + M_p(h, f_n)$.

Using Lemma B, we can extend Beckenbach-Dresher's inequality [3, 5] as follows.

Theorem 11. *Let $h, f_1, f_2, \cdots, f_n \in C_{rd}([a,b],[0,\infty))$ with $\int_a^b h(x) f_i^r(x)\Delta x > 0$ for $i = 1, 2, \cdots, n$. Then:*

$$(a) \quad \left(\frac{\displaystyle\int_a^b h(x)\Big(\sum_{i=1}^n f_i(x)\Big)^p \Delta x}{\displaystyle\int_a^b h(x)\Big(\sum_{i=1}^n f_i(x)\Big)^r \Delta x} \right)^{\frac{1}{p-r}} \le \sum_{i=1}^n \left(\frac{\displaystyle\int_a^b h(x)[f_i(x)]^p \Delta x}{\displaystyle\int_a^b h(x)[f_i(x)]^r \Delta x} \right)^{\frac{1}{p-r}},$$

if $p \ge 1 \ge r \ge 0$;

$$(b) \quad \left(\frac{\displaystyle\int_a^b h(x)\Big(\sum_{i=1}^n f_i(x)\Big)^p \Delta x}{\displaystyle\int_a^b h(x)\sum_{i=1}^n f_i(x) \Delta x} \right)^{\frac{1}{p-r}} \ge \sum_{i=1}^n \left(\frac{\displaystyle\int_a^b h(x)[f_i(x)]^p \Delta x}{\displaystyle\int_a^b h(x)[f_i(x)]^r \Delta x} \right)^{\frac{1}{p-r}},$$

if $0 < p < r$ or $0 > p > r$.

Proof. Assume initially that $p > r > 0$. Let

$$a_i = \Big(\int_a^b h(x) f_i^p(x)\Delta x\Big)^{\frac{1}{p}}, \quad b_i = \Big(\int_a^b h(x) f_i^r(x)\Delta x\Big)^{\frac{1}{r}}, \quad \lambda = \frac{r}{p-r}.$$

Then $\lambda + 1 = \frac{p}{p-r}$ and

$$\frac{\Big(\sum_{i=1}^n a_i\Big)^{\lambda+1}}{\Big(\sum_{i=1}^n b_i\Big)^{\lambda}} = \frac{\left[\sum_{i=1}^n \Big(\int_a^b h(x) f_i^p(x)\Delta x\Big)^{\frac{1}{p}}\right]^{\frac{p}{p-r}}}{\left[\sum_{i=1}^n \Big(\int_a^b h(x) f_i^r(x)\Delta x\Big)^{\frac{1}{r}}\right]^{\frac{r}{p-r}}}.$$

It follows from $p \ge 1 \ge r \ge 0$ and Theorem 9 that

$$\left[\sum_{i=1}^n \Big(\int_a^b h(x) f_i^p(x)\Delta x\Big)^{\frac{1}{p}}\right]^p \ge \int_a^b h(x)\Big(f_1(x) + f_2(x) + \cdots + f_n(x)\Big)^p \Delta x$$

and

$$\Big[\sum_{i=1}^{n}\Big(\int_a^b h(x)f_i^r(x)\Delta x\Big)^{\frac{1}{r}}\Big]^r \le \int_a^b h(x)\Big(f_1(x)+f_2(x)+\cdots+f_n(x)\Big)^r\Delta x.$$

These imply the desired result holds for $p \ge 1$ and $p > r > 0$. Clearly, the desired result also holds for $p = r$ or $r = 0$. This completes the proof of case (a).

Similarly, we can prove the case (b). □

As an application of Minkowski's inequality, we have the following theorem.

Theorem 12. *Let $h, f_1, f_2, \cdots, f_n \in C_{rd}([a,b],[0,\infty))$ with $\int_a^b h(t)\Delta t > 0$ and $\lambda_1+\lambda_2+\cdots+\lambda_n = 1$ with $\lambda_i \in (0,1)$ for each $i = 1,2,\cdots,n$. Then:*

$$(a)\quad M_r(h, f_1f_2\cdots f_n) \le M_{\frac{r}{\lambda_1}}(h,f_1)\ M_{\frac{r}{\lambda_2}}(h,f_2)\cdots M_{\frac{r}{\lambda_n}}(h,f_n)$$

for all $r > 0$;

$$(b)\quad M_r(h, f_1f_2\cdots f_n) \ge M_{\frac{r}{\lambda_1}}(h,f_1)\ M_{\frac{r}{\lambda_2}}(h,f_2)\cdots M_{\frac{r}{\lambda_n}}(h,f_n)$$

for all $r < 0$.

Proof. Case (a). If follows from Theorem 5 that

$$\begin{aligned}
M_r^r(h,\ f_1\ f_2\ \cdots\ f_n) &= \Big(\frac{\int_a^b h(x)f_1^r(x)f_2^r(x)\cdots f_n^r(x)\Delta x}{\int_a^b h(x)\Delta x}\Big)\\
&= \Big(\frac{\int_a^b (h(x)f_1^{\frac{r}{\lambda_1}}(x))^{\lambda_1}(h(x)f_2^{\frac{r}{\lambda_2}}(x))^{\lambda_2}\cdots(h(x)f_n^{\frac{r}{\lambda_n}}(x))^{\lambda_n}\Delta x}{\int_a^b h(x)\Delta x}\Big)\\
&= \Big(\int_a^b h(x)\Delta x\Big)^{-1}\Big(\int_a^b \prod_{i=1}^{n}(h(x)f_i^{\frac{r}{\lambda_i}}(x))^{\lambda_i}\Delta x\Big)\\
&\le \Big(\int_a^b h(x)\Delta x\Big)^{-1}\Big\{\prod_{i=1}^{n}\Big(\int_a^b h(x)f_i^{\frac{r}{\lambda_i}}(x)\Delta x\Big)^{\frac{\lambda_i}{r}}\Big\}^r,
\end{aligned}$$

which completes the proof of (a).

(b) Next, if $r < 0$, then $-r > 0$. Thus

$$M_r(h, f_1 f_2 \cdots f_n) = \frac{1}{M_{-r}\Big(h, \frac{1}{f_1 f_2 \cdots f_n}\Big)}$$
$$\geq \frac{1}{M_{-\frac{r}{\lambda_1}}\big(h, \frac{1}{f_1}\big) \cdots M_{-\frac{r}{\lambda_n}}\big(h, \frac{1}{f_n}\big)} = M_{\frac{r}{\lambda_1}}\big(h, f_1\big) \cdots M_{\frac{r}{\lambda_n}}\big(h, f_n\big). \quad \square$$

Theorem 13. *Let $\lambda, \lambda_1, \lambda_2, \cdots, \lambda_n$ be positive real numbers and $h, f_1, f_2, \cdots, f_n \in C_{rd}([a,b],[0,\infty))$ with $\int_a^b h(t)\Delta t > 0$. If $\lambda \geq \lambda_1 + \lambda_2 + \cdots + \lambda_n$, then*

$$M_{\frac{1}{\lambda}}(h, f_1 f_2 \cdots f_n) \leq M_{\frac{1}{\lambda_1}}(h, f_1)\ M_{\frac{1}{\lambda_2}}(h, f_2) \cdots M_{\frac{1}{\lambda_n}}(h, f_n).$$

Proof. (a) If $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, then $\frac{\lambda_1}{\lambda} + \frac{\lambda_2}{\lambda} + \cdots + \frac{\lambda_n}{\lambda} = 1$. It then follows from (a) of Theorem 12 with $r = \frac{1}{\lambda}$ that

$$M_{\frac{1}{\lambda}}(h, f_1\ f_2 \cdots f_n) \leq M_{\frac{1}{\lambda_1}}(h, f_1)\ M_{\frac{1}{\lambda_2}}(h, f_2) \cdots M_{\frac{1}{\lambda_n}}(h, f_n).$$

(b) If $\lambda > \lambda_1 + \lambda_2 + \cdots + \lambda_n := \mu$, then $\frac{1}{\mu} > \frac{1}{\lambda}$. Hence, it follows from (a) and Lemma A that

$$M_{\frac{1}{\lambda}}(h, f_1\ f_2 \cdots f_n) \leq M_{\frac{1}{\mu}}(h, f_1\ f_2 \cdots f_n) \leq M_{\frac{1}{\lambda_1}}(h, f_1) \cdots M_{\frac{1}{\lambda_n}}(h, f_n). \quad \square$$

Theorem 14. *Let $r > 0$ and $h, f, g \in C_{rd}([a,b],[0,\infty))$ with $\int_a^b h(x)\Delta x > 0$. If f and g are similarly (or oppositely) ordered on $[a,b]$ (i.e., for $x, t \in [a,b]$, $[f(x) - f(t)][g(x) - g(t)] \geq$ (or $\leq$)0). Then:*

$$M_r(h, f) M_r(h, g) \leq (or \geq) M_r(h, fg).$$

In general, if $f_1, f_2, \cdots, f_n \in C_{rd}([a,b],[0,\infty))$ are similarly (or oppositely) ordered, then

$$M_r(h, f_1) M_r(h, f_2) \cdots M_r(h, f_n) \leq (or \geq) M_r(h, f_1 f_2 \cdots f_n).$$

Proof. Since

$$M_r(h, f) = \left(\frac{\displaystyle\int_a^b h(x) f^r(x) \Delta x}{\displaystyle\int_a^b h(x)\Delta x} \right)^{\frac{1}{r}} := \Big(M_1(h, f^r)\Big)^{\frac{1}{r}} = \Big(A(h, f^r)\Big)^{\frac{1}{r}},$$

it is enough to consider the case $r = 1$ only. It follows from

$$\begin{aligned}T[f,g] := \frac{1}{2}\int_a^b\int_a^b h(x)h(t)\Big[f(x)-f(t)\Big]\Big[g(x)-g(t)\Big]\Delta x\Delta t\\ = \int_a^b h(x)\Delta x\int_a^b h(x)f(x)g(x)\Delta x\\ - \int_a^b h(x)f(x)\Delta x\int_a^b h(x)g(x)\Delta x \geq 0(\leq 0)\end{aligned}$$

that

$$A(h,fg) \geq (\leq) A(h,f)A(h,g).$$

Thus

$$M_r(h,fg) \geq (\leq) M_r(h,f)M_r(h,g).$$

This completes the proof. □

The following two theorems extend two results of [9].

Theorem 15. *Let $f \in L^p[a,b] \cap L^q[a,b]$ with $0 < p < q$. If $r \in (p,q)$, then*

$$f \in L^r[a,b] \quad \text{and} \quad ||f||_r \leq ||f||_p^{1-\alpha}||f||_q^{\alpha},$$

where $\alpha \in (0,1)$ is defined by $\frac{1}{r} = \frac{1-\alpha}{p} + \frac{\alpha}{q}$ and

$$||f||_r := \Big(\int_a^b h(x)|f(x)|^r\Delta x\Big)^{\frac{1}{r}}.$$

Proof. Let $\lambda = \frac{p}{(1-\alpha)r}$, $\mu = \frac{q}{\alpha r}$. Then

$$\frac{1}{\lambda} + \frac{1}{\mu} = \frac{(1-\alpha)r}{p} + \frac{\alpha r}{q} = 1$$

and

$$\mu = \frac{q}{\alpha r} = \frac{(1-\alpha)q}{p\alpha} + \frac{\alpha}{\alpha} > 1.$$

Thus, by (a) of Theorem 1,

$$\begin{aligned}||f||_r^r = \int_a^b h(x)|f(x)|^r\Delta x = \int_a^b h(x)|f(x)|^{r\alpha}|f(x)|^{(1-\alpha)r}\Delta x\\ \leq \Big(\int_a^b h(x)|f(x)|^{r\alpha\mu}\Delta x\Big)^{\frac{1}{\mu}}\Big(\int_a^b h(x)|f(x)|^{(1-\alpha)r\lambda}\Delta x\Big)^{\frac{1}{\lambda}}\\ = \Big[\Big(\int_a^b h(x)|f(x)|^q\Delta x\Big)^{\frac{1}{q}}\Big]^{\alpha r}\Big[\Big(\int_a^b h(x)|f(x)|^p\Delta x\Big)^{\frac{1}{p}}\Big]^{(1-\alpha)r}\end{aligned}$$

$$= ||f||_q^{\alpha r}||f||_p^{(1-\alpha)r}. \quad \square$$

Theorem 16. *Let* $1 \le p \le r \le q < \infty$ *and* $f \in L^p[a,b] \cap L^q[a,b]$. *Then:*
(a) $||f||_r^r \le ||f||_p^p + ||f||_q^q$;
(b) For $\alpha \in [0,1]$,

$$||f||_r^r \le ||f||_p^{\alpha p}||f||_q^{(1-\alpha)q},$$

where $r = \alpha p + (1-\alpha)q$.

Proof. (a) Let

$$A = \{x \in [a,b] |\ |f(x)| \le 1\} \quad \text{and} \quad B = \{x \in [a,b] |\ |f(x)| > 1\}.$$

Then $[a,b] = A \cup B$ and

$$\begin{aligned} ||f||_r^r &= \int_A h(x)|f(x)|^r \Delta x + \int_B h(x)|f(x)|^r \Delta x \\ &\le \int_A h(x)|f(x)|^p \Delta x + \int_B h(x)|f(x)|^q \Delta x \\ &\le \int_a^b h(x)|f(x)|^p \Delta x + \int_a^b h(x)|f(x)|^q \Delta x = ||f||_p^p + ||f||_q^q. \end{aligned}$$

(b) We have

$$\begin{aligned} ||f||_r^r &= \int_a^b h(x)|f(x)|^r \Delta x = \int_a^b h(x)|f(x)|^{\alpha p + (1-\alpha)q} \Delta x \\ &= \int_a^b h(x)|f(x)|^{\alpha p}|f(x)|^{(1-\alpha)q} \Delta x \\ &\le \Big(\int_a^b h(x)|f(x)|^p \Delta x\Big)^{\alpha}\Big(\int_a^b h(x)|f(x)|^q \Delta x\Big)^{(1-\alpha)} = ||f||_p^{\alpha p}||f||_q^{(1-\alpha)q}. \quad \square \end{aligned}$$

Theorem 17. *Let* $h \in C_{rd}([a,b],[0,\infty))$ *and* $0 < \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$.
(a) If $f,\ g \in C_{rd}([a,b],(0,1])$, *then*

$$\begin{aligned} &\Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^{\frac{1}{r}} \\ &\qquad \le \Big(\int_a^b h(x)\frac{f^2(x)g^2(x)}{f^p(x)}\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)\frac{f^2(x)g^2(x)}{f^q(x)}\Delta x\Big)^{\frac{1}{q}}. \end{aligned}$$

(b) If $f,\ g \in C_{rd}([a,b],[1,\infty))$, *then*

$$\Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^{\frac{1}{r}} \le \Big(\int_a^b h(x)f^2(x)g^2(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)f^2(x)g^2(x)\Delta x\Big)^{\frac{1}{q}}.$$

Proof. (a) It follows from $0 < f(x)g(x) \le 1$ on (a,b) and $\frac{1}{r} < 1$ that

$$f(x)g(x) \le f(x)g(x)(f(x)g(x))^{1-r} = \frac{f^2(x)g^2(x)}{f^r(x)g^r(x)}$$

for $x \in [a,b]$. Thus, by (a) of Theorem 1,

$$\begin{aligned}\int_a^b h(x)f(x)g(x)\Delta x &\le \int_a^b h(x)\frac{f^2(x)g^2(x)}{f^r(x)g^r(x)}\Delta x\\ &= \int_a^b h^{\frac{r}{p}}(x)\frac{f^{\frac{2r}{p}}(x)g^{\frac{2r}{p}}(x)}{f^r(x)}h^{\frac{r}{q}}(x)\frac{f^{\frac{2r}{q}}(x)g^{\frac{2r}{q}}(x)}{g^r(x)}\Delta x\\ &\le \Big(\int_a^b h(x)\frac{f^2(x)g^2(x)}{f^p(x)}\Delta x\Big)^{\frac{r}{p}}\Big(\int_a^b h(x)\frac{f^2(x)g^2(x)}{g^q(x)}\Delta x\Big)^{\frac{r}{q}}.\end{aligned}$$

That is,

$$\Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^{\frac{1}{r}} \le \Big(\int_a^b h(x)\frac{f^2(x)g^2(x)}{f^p(x)}\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)\frac{f^2(x)g^2(x)}{g^q(x)}\Delta x\Big)^{\frac{1}{q}}.$$

(b) It follows from that $fg \ge 1$ on $[a,b]$ that

$$fg \le f^2g^2$$

on $[a,b]$ and

$$\begin{aligned}\int_a^b f(x)g(x)\Delta x &\le \int_a^b f^2(x)g^2(x)\Delta x\\ &= \int_a^b h^{\frac{r}{p}}(x)f^{\frac{2r}{p}}(x)g^{\frac{2r}{p}}(x)h^{\frac{r}{q}}(x)f^{\frac{2r}{q}}(x)g^{\frac{2r}{q}}(x)\Delta x\\ &\le \Big(\int_a^b h(x)f^2(x)g^2(x)\Delta x\Big)^{\frac{r}{p}}\Big(\int_a^b h(x)f^2(x)g^2(x)\Delta x\Big)^{\frac{r}{q}}.\end{aligned}$$

Thus

$$\Big(\int_a^b h(x)f(x)g(x)\Delta x\Big)^{\frac{1}{r}}$$
$$\leq \Big(\int_a^b h(x)f^2(x)g^2(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)f^2(x)g^2(x)\Delta x\Big)^{\frac{1}{q}}. \quad \square$$

Theorem 18. *Let $f, g \in C_{rd}([a,b],[0,\infty))$ with $g > 0$ on $[a,b]$ and $q > p > 0$. If*

$$\int_a^b h(x)f^q(x)\Delta x \leq \int_a^b h(x)g^q(x)\Delta x, \tag{C_1}$$

Then

$$\int_a^b h(x)f^p(x)\Delta x \leq \int_a^b h(x)g^p(x)\Delta x. \tag{R_1}$$

Proof. Let $r = \frac{q}{p}$ and $s = \frac{q}{q-p}$. Then $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. Thus it folllows from Roger-Hölder's ineguality ((a) of Theorem 1) that

$$\Big(\int_a^b h(x)f^p(x)\Delta x\Big)^q = \Big(\int_a^b \frac{h^{\frac{1}{r}}(x)f^p(x)}{g^{\frac{p}{s}}(x)} g^{\frac{p}{s}}(x)h^{\frac{1}{s}}(x)\Delta x\Big)^q$$
$$\leq \Big(\int_a^b \frac{h(x)f^{pr}(x)}{g^{\frac{pr}{s}}(s)}\Delta x\Big)^{\frac{q}{r}}\Big(\int_a^b h(x)g^p(x)\Delta x\Big)^{\frac{q}{s}}$$
$$= \Big(\int_a^b \frac{h(x)f^q(x)}{g^{q-p}(x)}\Big)^p\Big(\int_a^b h(x)g^p(x)\Delta x\Big)^{q-p}$$
$$= \left(\int_a^b h(x)g^p(x)\Delta x - \int_a^b \frac{h(x)[g^q(x) - f^q(x)]}{g^{q-p}(x)}\Delta x\right)^p\Big(\int_a^b h(x)g^p(x)\Delta x\Big)^{q-p}$$
$$\leq \Big(\int_a^b h(x)g^p(x)\Delta x\Big)^q. \quad \square$$

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