International Journal of Pure and Applied Mathematics

Volume 29 No. 3 2006, 311-316

COMPACT NORMAL COMPLEX SURFACES WITH LOCALLY FREE AND NON-NEGATIVE TANGENT SHEAF

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we exclude the existence of compact and normal complex analytic surfaces whose tangent sheaf is locally free and rather positive (except, of course, $\mathbf{P}^2$).

AMS Subject Classification: 32C38, 32J15, 14F10, 14J99
Key Words: tangent sheaf, compact analytic surface, automorphism, spanned sheaf, ample tangent sheaf, spanned tangent sheaf

1. The Statements

For any algebraic scheme X defined over an algebraically closed field $\mathbb{K}$ (resp. reduced complex analytic space X) let TX denote its tangent sheaf, i.e. the dual of the sheaf Ω^1_X of germs of regular (resp. holomorphic) 1-forms on X. It is well-known that Ω^1_X is locally free if and only X is smooth. We are interested in the case in which TX is locally free, but X is singular. Here, following [3], we study the case $\dim(X) = 2$, X projective or compact and TX is rather positive (e.g. it is spanned).

Definition 1. Let X be an integral normal projective surface and E a vector bundle on X. We will say that E is weakly ample if there is a very ample line bundle L on X such that for a general $C \in |L|$ there is no non-zero morphism $E|C \to R$ with R a rank one torsion free on C with degree at most zero, i.e. there is no injective map $A \to E^*|C$ with A a rank one torsion free sheaf on C and $\deg(A) \geq 0$.

Received: May 10, 2006

Theorem 1. *Assume* $\mathrm{char}(\mathbb{K}) \neq 2$. *There is no integral normal projective surface X such that X is singular and TX is locally free and weakly ample.*

Theorem 2. *Let X be a compact and connected complex normal surface and $f : S \to X$ its minimal desingularization. Assume* $\mathrm{Sing}(X) \neq \emptyset$ *and* $TX \cong \mathcal{O}_X^{\oplus 2}$. *Then $a(S) = 2$, S is rational and X has a unique singular point. This singular point is a minimally elliptic singularity which is neither a simple elliptic singularity nor a hypersurface singularity.*

Let X be either an n-dimensional compact complex manifold or an n-dimensional smooth projective variety defined over an algebraically closed field with $\mathrm{char}(\mathbb{K}) = 0$. If TX is spanned, then it is a homogeneous space, i.e. $X \cong G/H$ for some complex Lie group (resp. algebraic group) G and some closed complex Lie subgroup (resp. closed algebraic subgroup) H of G. This is not true in positive characteristic (see [10]). Furthermore, in characteristic zero if we also assume that ω_X^* is ample, then G is an affine algebraic group and hence X is unirational. What happens when X is singular?

Theorem 3. *Let X be an integral normal projective variety defined over an algebraically closed field with* $\mathrm{char}(\mathbb{K}) = 0$. *Assume that the sheaf TX is spanned, that ω_X is locally free and that ω_X^* is ample. Then X is smooth and hence it is homogeneous.*

2. The Proofs

Remark 1. Let X be either an integral projective variety defined over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$ or a compact reduced and irreducible complex space. Let $\mathrm{Aut}^0(X)$ denote the connected component of the identity of the automorphism group of X. Hence $\mathrm{Aut}^0(X)$ is either a smooth algebraic group or a complex Lie group. By Hironaka's equivariant desingularization theorem there is a $\mathrm{Aut}^0(X)$-equivariant desingularization $f : Y \to X$. Since $H^0(X, TX)$ (resp. $H^0(Y, TY)$) is the tangent space at the identity of $\mathrm{Aut}^0(X)$ (resp. $\mathrm{Aut}^0(Y)$), we get $h^0(Y, TY) \geq h^0(X, TX)$. If $\dim(X) = 2$ and X is normal we may (and always will) take as $f : Y \to X$ the relatively minimal desingularization of X; in this case we will write S instead of Y.

Remark 2. Let X be either an integral normal variety over an algebraically closed field or a normal and connected complex analytic space. Assume that TX is locally free. Then ω_X is locally free and $\omega_X^* \cong \det(TX)$ (see [3], Lemma 2.3).

Proof of Theorem 1. Since $\omega_X \cong \det(TX)^*$ (see [3], Lemma 2.3), X is Gorenstein and ω_X^* is nef. Since $\mathrm{char}(\mathbb{K}) \neq 2$, X has no rational double point (see [12]). Since X is assumed to be singular, this is implies $\omega_S \cong f^*(\omega_X)(-D)$ with D non-zero effective divisor whose support contains all curves contracted by f. Thus $\kappa(S) = -\infty$. To show that S is rational, it is sufficient to prove that $h^1(S, \mathcal{O}_S) = 0$. Assume $q := h^1(S, \mathcal{O}_S) > 0$. Thus there is a morphism $h : S \to C$ with C a smooth curve of genus q. Let $A \subset X$ be a smooth curve such that $A \cap \mathrm{Sing}(X) = \emptyset$. Thus $B := f^{-1}(A) \cong A$ and $TS|B = f^*(TX)|B$. The morphism h induces a non-zero morphism $\beta : TS|B \to h^*(TC)|B$. Since $q > 0$, $\deg(h^*(TC))|B \leq 2 - 2q \leq 0$, contradicting the weak ampleness of TX. Indeed, we may take as A a general member of a very ample linear system on X. Hence from now on we may assume S rational. Thus $h^1(S, \mathcal{O}_S) = 0$. The Leray spectral sequence of f gives that X has only rational singularities. Since X is Gorenstein, we get that X has only rational double points. Since $\mathrm{Sing}(X) \neq 0$, X has at least one quotient singularity, contradicting the local freeness of TX and [12]. □

Proof of Theorem 2. By [12], X has no quotient singularity and in particular no rational double point. By [15] or [16] no singular point of X is a hypersurface singularity. By [7] no singularity of X is a graded singularity. By [14], Corollary. 1.4, the last assertion shows that X has no simple elliptic singularity. In the case $a(S) = 2$, the proof of [3], Proposition 2.2. works verbatim (except the added informations on the minimal elliptic singularity just checked); we only point out that the proof of the case "S ruled and $q = 1$" may be simplified, because the quasi-homogeneity of S easily implies that $S = S'$ is a projectivization of a rank two semistable vector bundle on an elliptic curve and no such surface has a contractible curve (see [2]). Hence we may assume $a(S) \leq 1$. Since X is assumed to be singular and it has no rational singularity, this is implies $\omega_S \cong f^*(\omega_X)(-D)$ with D non-zero effective divisor whose support contains all curves contracted by f. Thus $\kappa(S) = -\infty$. Since $a(S) \geq 1$, S is neither rational nor a ruled surface. Notice that $\mathrm{Aut}^0(X)$ is a connected complex Lie group acticting complex analytically on X and whose tangent space at the identity is isomorphic to $H^0(X, TX)$ (see [1], p. 40). Since $TX \cong \mathcal{O}_X^{\oplus 2}$ we get $\dim(\mathrm{Aut}^0(X)) = 2$. Furthermore, for every $P \in X_{reg}$ the vector space $H^0(X, \mathcal{I}_P \otimes TX) \cong \{0\}$ is the tangent space at the identity of the stabilizer of P in $\mathrm{Aut}^0(X)$. Thus the action of $\mathrm{Aut}^0(X)$ on X has an open orbit. Since $f : S \to X$ is the minimal desingularization, $\mathrm{Aut}^0(X)$ acts on X. Thus the action of $\mathrm{Aut}^0(S)$ on S has an open orbit, i.e. S is quasi-homogeneous. Let S' be the minimal model of S. Hence $a(S) = a(S')$ and $\kappa(S) = \kappa(S')$. The universal property of the minimal model implies $\mathrm{Aut}(S) \subseteq \mathrm{Aut}(S')$. Thus

S' is quasi-homogeneous. Since X is singular, S' must contain at least one compact curve. We may apply the classification of all smooth complex quasi-homogeneous surfaces (see [12], [6]). Hence S' is a surface of class VII_0. First assume $b_2(S') = 0$. All such surfaces (even without the quasi-homogeneity condition) are classified (see [17]) We have $S = S'$. Since S' contains curves, it must be a Hopf surface (see [17], Section 2). Hence each contacted curve is a smooth elliptic curve (see [4], Proposition 18.2). Hence each singular point of X is a simple elliptic singularity. However, each such singularity is a graded singularity (see [14], Corollary 1.4) and hence TX cannot be locally free (see [7]). Now assume $b_2(S') > 0$. By [11], Théorème at p. 1245, the quasi-homogeneity of S' easily implies that S' is an Inoue-Hirzebruch singularity. Again, the contraction of any of its cycles gives a graded singularity of X (use their construction given in [5], Theorem 2.1 and 3 lines before Theorem 1.4), contradicting [7]. □

Proof of Theorem 3. By Lefschetz principle we reduce to the case $\mathbb{K} = \mathbb{C}$. Every automorphism of X preserves ω_X and hence $(\omega_X^*)^{\otimes t}$ for all $t \gg 0$. Since $(\omega_X^*)^{\otimes t}$ is very ample, we see that the complex Lie group $\mathrm{Aut}(X)$ is a subgroup of a linear group and in particular it is algebraic and affine and its act algebraically on X. $H^0(X, TX)$ is the tangent space at Id of the connected linear algebraic group $G := \mathrm{Aut}^0(X)$ (see [1], p. 40). In a similar way we check that for any $P \in X_{reg}$ the linear space $H^0(X, TX(-P))$ is the tangent space at Id of the stabilizer G_P of P in G. Since TX is spanned, we have $h^0(X, TX(-P)) = h^0(X, TX) - \dim(X)$ for all $P \in X_{reg}$. Thus all orbits of all smooth points of X have dimension $\dim(X)$. Since X is integral, we get that G act transitively on X_{reg}. Since the action of G on X_{reg} is algebraic, we get $X_{reg} \cong G/\Gamma$ for some finite subgroup $\Gamma \cong G_P$. Thus X_{reg} is affine, contradicting the projectivity of X and the assumption $\dim(\mathrm{Sing}(X)) \leq \dim(X) - 2$. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] D.N. Akhiezer, *Lie Group Actions in Complex Analysis*, Friedr. Vieweg and Sohn (1995).

[2] M.F. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc.*, **7** (1957), 414-452.

[3] E. Ballico, On singular varieties with ample tangent bundle, *Indag. Math., N.S.*, **4**, No. 1 (1993), 1-10.

[4] W. Barth, C.A.M. Peters, A. Van de Ven, *Compact Complex Surfaces*, Springer (1984).

[5] G. Dloussky, Une construction élémentaire des surfaces d'Inoue-Hirzebruch, *Math. Ann.*, **280**, No. 4 (1988), 663-682.

[6] C. Gellhaus, P. Heinzner, Komplexe Flächen mit holomorphen Vectorfeldern, *Abh. Math. Sem. Hamburg*, **60** (1990), 37-46.

[7] M. Hochster, The Zariski-Lipman conjecture in the graded case, *J. Algebra*, **47**, No. 2 (1977), 411-424.

[8] H. Laufer, Minimally elliptic singularities, *Amer. J. Math.*, **99**, No. 6 (1977), 1257-1295.

[9] J. Lipman, Free derivation modules on algebraic varieties, *Amer. J. Math.*, **87** (1965), 874-898.

[10] V.B. Mehta, V. Srinivas, Varieties in positive characteristic with trivial tangent bundle, *Compositio Math.*, **64**, No. 2 (1987), 191-212.

[11] K. Oeljeklaus, M. Toma, D. Zaffran, Une caractérisation des surfaces d'Inoue-Hirzebruch, *Ann. Inst. Fourier, Grenoble*, **51**, No. 5 (2001), 1243-1257.

[12] E. Platte, Differentielle Eigenschaften der Invarianten regularer Algebrenen, *J. Algebra*, **62**, No. 1 (1980), 1-12.

[13] J. Potters, On almost homogeneous compact analytic surfaces, *Invent. Math.*, **8** (1969), 244-266.

[14] K. Saito, Einfach-elliptische Singularitäten, *Invent. Math.*, **23**, No-s: 3,4 (1974), 289-325.

[15] G. Scheja, U. Storch, Über differentielle Abhängigkeit bei idealen analytischer Algebren, *Math. Z.*, **114** (1970), 101-112.

[16] G. Scheja, U. Storch, Differentielle Eigenschaften des Lokaliserungen analytischer Algebren, *Math. Ann.*, **197** (1972), 137-170.

[17] A. Teleman, Projectively flat surfaces and Bogomolov's Theorem on class VV_0 surfaces, *Int. J. Math.*, **5**, No. 2 (1994), 253-264.