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HOLOMORPHIC TRIPLES ON SMOOTH ALGEBRAIC CURVES AND α-STABILITY

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Abstract: Let X be a smooth curve of genus $g \geq 2$. Here we study the α-stability of a holomorphic triple (E_2, E_1, ϕ) on X in the following cases:

(i) E_2 is a general stable bundle and E_1 is a general extension of a general stable bundle E_3 by E_2;

(ii) E_1 is a general stable bundle and E_2 is a maximal degree subbundle of E_1 for a prescribed rank.

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1. Introduction

Let X be a smooth and connected projective curve of genus $g \geq 2$. A holomorphic triple on X is a triple $T = (E_2, E_1, \phi)$ such that E_2 and E_1 are vector bundles on X and $\phi : E_2 \to E_1$. A subtriple of T is a triple (E_2', E_1', ϕ') on X such that there are inclusions (as sheaves) $i : E_2' \to E_2$, $j : E_1' \to E_1$ and $\phi \circ i = j \circ \phi'$. Set $\mathrm{rank}(T) := \mathrm{rank}(E_1) + \mathrm{rank}(E_2)$. Fix any $\alpha \in \mathbb{R}$. Define the α-degree of T by the formula $\deg_\alpha(T) := \deg(E_1) + \deg(E_2) + \alpha \cdot \mathrm{rank}(E_2)$ and the α-slope of T by the formula $\mu_\alpha(T) := \deg_\alpha(T)/\mathrm{rank}(T)$. The α-slope allows us to define the notions of α-semistable and α-stable triples on X. If

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$r_2 := \mathrm{rank}(E_2) < r_1 := \mathrm{rank}(E_1)$ and (E_2, E_1, ϕ) is α-stable, then $\alpha > 0$, $\alpha > \mu(E_1) - \mu(E_2)$ and $\alpha < (1 + (r_1 + r_2)/(r_1 - r_2))(\mu(E_1) - \mu(E_2))$ ([2], 3.17 and 3.18). For the theory of holomorphic triples, see [1], [2] and [4]. For all integers r, d such that $r > 0$ let $M(X; r, d)$ denote the moduli space of all stable vector bundles on X with rank r and degree d. $M(X; r, d)$ is an integral variety and $\dim(M(X; r, d)) = r^2(g - 1) + 1$. Fix any vector bundle E on X and any integer ρ such that $1 \le \rho < \mathrm{rank}(E)$. Let $S(E; \rho)$ denote the set of all rank ρ subsheaves of E with maximal degree. Set $\delta(E; \rho) := \deg(F)$ for any $F \in S(E; \rho)$. When E is a general element of $M(X; r, d)$ all integers $\delta(E, \rho)$, $1 \le \rho \le r-1$, are known by a theorem of Hirschowitz ([3], Section 4, or [6], Remark 3.14); indeed, let $\epsilon_{r,d,\rho}$ be the only integer such that $0 \le \epsilon_{r,d,\rho} \le r - 1$ and $\epsilon_{r,d,\rho} + \rho(r-\rho)(g-1) \equiv \rho \cdot d \pmod r$; then $\rho \cdot d - r \cdot \delta(E, \rho) = \rho(r-\rho)(g-1) + \epsilon_{r,d,\rho}$. Set $\delta_{r,d,\rho} := \delta(E, \rho)$ for a general $E \in M(X; r, d)$; roughly speaking, $\delta_{r,d,\rho}$ is an integer and $g - 1 \le (d - \delta_{r,d,\rho})/(r - \rho) - \delta_{r,d,\rho}/\rho \le g$. Here we will prove the following results.

Theorem 1. *Let X be a smooth curve of genus $g \ge 2$. Fix integers r_1, r_2, d_1 such that $1 \le r_2 < r_1$, a general $E \in M(X; r_1, d_1)$ and any $F \in S(E; r_2)$. Use the inclusion $j : F \to E$ to obtain a holomorphic triple $T = (F, E, j)$ on X. Set $r_3 := r_1 - r_2$, $d_2 := \delta_{r_1,d_1,r_2}$, $d_3 := d_1 - d_2$ and $\nabla(r_1, r_2) := \{(b_1, c, b_2) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : 0 \le b_2 \le c \le r_2, c \le b_1 \le c + r_1 - r_2, b_1 > 0$ and $((b_1, b_2) \ne (r_1, r_2)\}$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1 + b_2) = b_2(r_1 + r_2)$ set $\beta(b_1, c, b_2) := 0$ and $\gamma(b_1, c, b_2) := +\infty$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1+b_2) < b_2(r_1 + r_2)$ set $\gamma(b_1, c, b_2) := +\infty$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1 + b_2) > b_2(r_1 + r_2)$ set $\beta(b_1, c, b_2) := 0$. Set $\beta(b_1, c, 0) := 0$ for all $0 < b_1 < r_1$. Set $\beta(r_1, c, 0) := d_1/r_1 - d_2/r_2$. Set $\gamma(r_2, r_2, r_2) := 2(r_2 d_1 - r_1 d_2)/(r_2(r_1 - r_2))$. For all $(c + r_3, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$ set $\beta(c + r_3, c, b_2) := ((\delta_{r_2,d_2,b_2} + (b_1 d_1/r_1) + d_3)(r_1 + r_2) - (c + b_2 + r_3)(d_1 + d_2))/(r_2 r_3 + c r_2 - b_2 r_1)$. For all $(b_1, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$, $b_2 < b_1 < c + r_3$ and $r_2/(r_1 + r_2) > b_2/(b_1 + b_2)$ set $\beta(b_1, c, b_2) := ((\delta_{r_2,d_2,b_2} + (b_1 d_1/r_1) + \delta_{r_3,d_3,b_1-r_3})(r_1 + r_2) - (d_1 + d_2)(b_1 + b_2))/(r_2 b_1 - b_2 r_1)$. For all $(b_1, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$, $b_2 < b_1 < c + r_3$ and $r_2/(r_1 + r_2) < b_2/(b_1 + b_2)$ set $\gamma(b_1, c, b_2) := ((d_1 + d_2)(b_1 - b_2) - (\delta_{r_2,d_2,b_2} + (b_1 d_1/r_1) + \delta_{r_3,d_3,b_1-r_3})(r_1 + r_2))/(b_2 r_1 - r_2 b_1)$. For all $1 \le b_2 < r_2$ set $\gamma(b_2, b_2, b_2) := (2b_2(d_1 + d_2) - (r_1 + r_2)(\delta_{r_2,d_2,b_2} + (d_1 b_2/r_1)))/(b_2(r_1 - r_2))$. Fix $\alpha \in \mathbb{R}$ such that $\beta(b_1, c, b_2) < \alpha < \gamma(b_1, c, b_2)$ for all $(b_1, c, b_2) \in \nabla(r_1, r_2)$. Then T is α-stable.*

Theorem 2. *Let X be a smooth curve of genus $g \ge 2$. Fix integers r_2, r_3, d_2, d_3 such that $r_2 > 0$, $r_3 > 0$ and $d_2/r_2 < d_3/r_3$. Set $r_1 := r_2 + r_3$, $d_2 := d_2 + d_3$ and $\nabla(r_1, r_2) := \{(b_1, c, b_2) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : 0 \le b_2 \le c \le r_2, c \le$*

$b_1 \le c + r_1 - r_2, b_1 > 0$ and $((b_1, b_2) \neq (r_1, r_2)\}$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1 + b_2) = b_2(r_1 + r_2)$ set $\beta(b_1, c, b_2) := 0$ and $\gamma(b_1, c, b_2) := +\infty$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1 + b_2) < b_2(r_1 + r_2)$ set $\gamma(b_1, c, b_2) := +\infty$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1 + b_2) > b_2(r_1 + r_2)$ set $\beta(b_1, c, b_2) := 0$. Set $\beta(b_1, c, 0) := 0$ for all $0 < b_1 < r_1$. Set $\beta(r_1, c, 0) := d_1/r_1 - d_2/r_2$. Set $\gamma(r_2, r_2, r_2) := 2(r_2 d_1 - r_1 d_2)/(r_2(r_1 - r_2))$. For all $(c + r_3, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$ set $\beta(c + r_3, c, b_2) := ((\delta_{r_2,d_2,b_2} + (b_1 d_1/r_1) + d_3)(r_1 + r_2) - (c + b_2 + r_3)(d_1 + d_2))/(r_2 r_3 + c r_2 - b_2 r_1)$. For all $(b_1, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$, $b_2 < b_1 < c + r_3$ and $r_2/(r_1 + r_2) > b_2/(b_1 + b_2)$ set $\beta(b_1, c, b_2) := ((\delta_{r_2,d_2,b_2} + (b_1 d_1/r_1) + \delta_{r_3,d_3,b_1-r_3})(r_1 + r_2) - (d_1 + d_2)(b_1 + b_2))/(r_2 b_1 - b_2 r_1)$. For all $(b_1, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$, $b_2 < b_1 < c + r_3$ and $r_2/(r_1 + r_2) < b_2/(b_1 + b_2)$ set $\gamma(b_1, c, b_2) := ((d_1 + d_2)(b_1 - b_2) - (\delta_{r_2,d_2,b_2} + (b_1 d_1/r_1) + \delta_{r_3,d_3,b_1-r_3})(r_1 + r_2))/(b_2 r_1 - r_2 b_1)$. For all $1 \le b_2 < r_2$ set $\gamma(b_2, b_2, b_2) := (2b_2(d_1 + d_2) - (r_1 + r_2)(\delta_{r_2,d_2,b_2} + (d_1 b_2/r_1)))/(b_2(r_1 - r_2))$. Fix a general $(E_2, E_3) \in M(X; r_2, d_2) \times M(X; r_3, d_3)$ and let E_1 be a general exttension of E_2 by E_1, i.e. the general vector bundle E_1 fitting in an exact sequence

$$0 \to E_2 \xrightarrow{j} E_1 \xrightarrow{\pi} E_3 \to 0\,. \tag{1}$$

Then E is stable and the holomorphic triple $T := (E_2, E_1, j)$ induced by (1) is α-stable.

Notice that in the statement of Theorem 1 (resp. Theorem 2) our assumptions on α do not depend from the pairs (b_1, b_2) such that $r_2(b_1+b_2) = b_2(r_1+r_2)$ (resp. $\rho(b_1 + b_2) = b_2(r + \rho)$). In a very similar way we will also prove the following result.

Theorem 3. *Let X be a smooth curve of genus $g \ge 2$. Fix integers r_2, r_3, d_2, d_3 such that $r_2 > 0$, $r_3 > 0$ and $d_2/r_2 + g \le d_3/r_3$. Set $r_1 := r_2 + r_3$, $d_2 := d_2 + d_3$ and $\nabla(r_1, r_2) := \{(b_1, c, b_2) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : 0 \le b_2 \le c \le r_2, c \le b_1 \le c + r_1 - r_2, b_1 > 0$ and $((b_1, b_2) \neq (r_1, r_2)\}$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1 + b_2) = b_2(r_1 + r_2)$ set $\beta(b_1, c, b_2) := 0$ and $\gamma(b_1, c, b_2) := +\infty$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1 + b_2) < b_2(r_1 + r_2)$ set $\gamma(b_1, c, b_2) := +\infty$. If $(b_1, c, b_2) \in \nabla(r_1, r_2)$ and $r_2(b_1 + b_2) > b_2(r_1 + r_2)$ set $\beta(b_1, c, b_2) := 0$. Set $\beta(b_1, c, 0) := 0$ for all $0 < b_1 < r_1$. Set $\beta(r_1, c, 0) := d_1/r_1 - d_2/r_2$. Set $\gamma(r_2, r_2, r_2) := 2(r_2 d_1 - r_1 d_2)/(r_2(r_1 - r_2))$. For all $(c + r_3, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$ set $\beta(c + r_3, c, b_2) := ((\delta_{r_2,d_2,b_2} + \delta_{r_1,d_1,c} + d_3)(r_1 + r_2) - (c + b_2 + r_3)(d_1 + d_2))/(r_2 r_3 + c r_2 - b_2 r_1)$. For all $(b_1, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$, $b_2 < b_1 < c + r_3$ and $r_2/(r_1 + r_2) > b_2/(b_1 + b_2)$ set $\beta(b_1, c, b_2) := ((\delta_{r_2,d_2,b_2} + \delta_{r_1,d_1,c} + \delta_{r_3,d_3,b_1-r_3})(r_1 + r_2) - (d_1 + d_2)(b_1 + b_2))/(r_2 b_1 - b_2 r_1)$. For all $(b_1, c, b_2) \in \nabla(r_1, r_2)$ such that $1 \le b_2 < r_2$, $b_2 <$*

$b_1 < c + r_3$ and $r_2/(r_1+r_2) < b_2/(b_1+b_2)$ set $\gamma(b_1, c, b_2) := ((d_1+d_2)(b_1 - b_2) - (\delta_{r_2,d_2,b_2} + \delta_{r_1,d_1,c} + \delta_{r_3,d_3,b_1-r_3})(r_1+r_2))/(b_2 r_1 - r_2 b_1)$. For all $1 \le b_2 < r_2$ set $\gamma(b_2, b_2, b_2) := (2b_2(d_1+d_2) - (r_1+r_2)(\delta_{r_2,d_2,b_2} + \delta_{r_1,d_1,c}))/(b_2(r_1 - r_2))$. Fix a general $(E_2, E_3) \in M(X; r_2, d_2) \times M(X; r_3, d_3)$ and let E_1 be a general extension of E_2 by E_1, i.e. the general vector bundle E_1 fitting in the exact sequence (1). Then E is stable and the holomorphic triple $T := (E_2, E_1, j)$ induced by (1) is α-stable.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

Proof of Theorem 2. By [7], Theorem 0.1, E_1 is stable.

(i) Fix $\alpha \in \mathbb{R}$. We have $\mu_\alpha(T) = \alpha \cdot r_2/(r_1+r_2) + (d_1+d_2)/(r_1+r_2)$. Set $T_1 := (E_2, j(E_2), j)$ and $T_2 := (0, E_1, 0)$. Hence T_1 and T_2 are proper holomorphic subtriples of T. We have $\mu_\alpha(T_1) = \alpha/2 + d_2/r_2$. Thus $\mu_\alpha(T_1) < \mu_\alpha(T)$ if and only if $\alpha < 2(r_2 d_1 - r_1 d_2)/(r_2(r_1 - r_2)) = \gamma(r_2, r_2, r_2)$. We have $\mu_\alpha(T_2) = d_1/r_1$. Thus $\mu_\alpha(T_2) < \mu_\alpha(T)$ if and only if $\alpha > d_1/r_1 - d_2/r_2 = \beta(r_1, c, 0)$. Assume that all the inequalities listed in the statement of Theorem 2 are satisfied. In order to obtain a contradiction we assume the existence of a proper holomorphic subtriple $T' = (F_2, F_1, j|F_2)$ of T such that $\mu_\alpha(T') \ge \mu_\alpha(T)$. We also assume that T' has maximal α-slope among all proper holomorphic subtriples of T. Set $b_i := \mathrm{rank}(F_i)$ and $a_i := \deg(F_i)$. Since j is injective, we have $b_2 \le b_1$. Set $c := \mathrm{rank}(F_1 \cap E_2)$. Notice that $b_2 \le c \le r_2$, $b_1 \le c + r_3$ and $b_1 - c = \mathrm{rank}(\pi(F_1))$, where $\pi : E_1 \to E_3$ is the surjective map of the exact sequence (1). Set $G := \pi(F_1)$. If $c = b_1$, then $G = 0$. If $c < b_1$, then G is a rank $b_1 - c$ subsheaf of E_2. Set $A := E_2 \cap F_1$ and let $\widetilde{A}$ be the saturation of A in F_1. By the maximality of the α-slope of T the sheaf F_1 is saturated in E_1. Hence $\widetilde{A}$ is the saturation of A in E_1. Hence $\deg(F_1) = \deg(\widetilde{A}) + \deg(\pi(F_1))$. Since E_1 is stable, we have $\deg(\widetilde{A}) \le c d_1/r_1$ and this inequality is strict if $c < r_1$. We distinguish 3 cases.

(ii) Here we assume $b_2 = 0$. Hence $\mu_\alpha(T') = a_1/b_1$. Since E_1 is stable, we have $a_1/b_1 \le d_1/r_1$. Hence $\mu_\alpha(T') \le \mu_\alpha(T_2)$, contradicting the computation made in part (i) and the assumption $\alpha > \beta(r_1, c, 0)$.

(iii) Here we assume $b_2 = r_2$. Hence $c = b_2$ and E_2 is the saturation of F_2 in E_2. If $b_1 = r_2$, then $\mu_\alpha(T') \le \mu_\alpha(T_1)$ and we get a contradiction by part (i) and the assumption $\alpha < \gamma(r_2, r_2, r_2)$. If $b_1 = r_1$, then $\mu_\alpha(T') < \mu_\alpha(T)$, because each E_i, $i = 1, 2$, is the saturation of its subsheaf F_i and $T' \ne T$. Hence we may assume $r_2 < b_1 < r_1 = r_2 + r_3$. Since E_3 is general in $M(X; r_3, d_3)$, [7], Theorem 0.1, implies $\deg(G) \le \delta_{r_3,d_3,r_1-b_1}$. Hence $\mu_\alpha(T') \le \alpha \cdot r_2/(r_2+b_1) + (d_2 + (r_2 d_2/r_1) + \delta_{r_3,d_3,r_1-b_1})/(b_1+r_2)$, contradicting the inequality $\alpha < ((d_1 + d_2)(r_1+r_2) - (d_2 + (r_2 d_2/r_1) + \delta_{r_3,d_3,r_1-b_1})(r_2+b_1))/(r_2(r_1-b_1)) = \gamma(b_1, b_2, b_2)$.

(iv) Now assume $1 \le b_2 < r_2$. By [7], Theorem 0.1, we have $\deg(F_2) \le \delta_{r_2,d_2,b_2}$. If $b_1 = c + r_3$, then $\deg(F_1) \le \deg(\widetilde{A}) + d_3 \le (b_1 d_1/r_1) + d_3$ and hence $\mu_\alpha(T') \leq \alpha \cdot b_2/(r_3+c+b_2) + (\delta_{r_2,d_2,b_2} + (b_1d_1/r_1) + d_3)/(r_3+c+b_2)$; here the contradiction comes from the inequality $\alpha > \beta(c+r_3,c,b_2) := ((\delta_{r_2,d_2,b_2} + (b_1d_1/r_1) + d_3)(r_1+r_2) - (c+b_2+r_3)(d_1+d_2))/(r_2r_3+cr_2-b_2r_1)$. If $b_2 < b_1 < c+r_3$, then as in part (iii) we get $\deg(F_1) \leq \deg(\widetilde{A}) + \delta_{r_3,d_3,b_1-r_3} \leq (b_1d_1/r_1) + \delta_{r_3,d_3,b_1-r_3}$ and hence $\mu_\alpha(T') \leq \alpha \cdot b_2/(b_1+b_2) + (\delta_{r_2,d_2,b_2} + (b_1d_1/r_1) + \delta_{r_3,d_3,b_1-r_3})/(b_1+b_2)$. If $r_2/(r_1+r_2) = b_2/(b_1+b_2)$ we got a contradiction. If $r_2/(r_1+r_2) > b_2/(b_1+b_2)$ (resp. $r_2/(r_1+r_2) > b_2/(b_1+b_2)$) the contradiction come from the assumption $\alpha > ((\delta_{r_2,d_2,b_2} + (b_1d_1/r_1) + \delta_{r_3,d_3,b_1-r_3})(r_1+r_2) - (d_1+d_2)(b_1+b_2))/(r_2b_1 - b_2r_1) = \beta(b_1,c,b_2)$ (resp. $\alpha < ((d_1+d_2)(b_1 - b_2) - (\delta_{r_2,d_2,b_2} + (b_1d_1/r_1) + \delta_{r_3,d_3,b_1-r_3})(r_1+r_2))/(b_2r_1 - r_2b_1) = \gamma(b_1,c,b_2)$). Now assume $b_1 = b_2$. Hence $c = b_2$ and F_1 is contained in the saturation of F_2 in E_1. The maximality of $\mu_\alpha(T')$ implies $F_1 = \widetilde{A}$. Thus $\mu_\alpha(T') \leq \alpha/2 + (\delta_{r_2,d_2,b_2} + (d_1b_2/r_1))/2b_2$, contradicting the inequality $\alpha < (2b_2(d_1+d_2) - (r_1+r_2)(\delta_{r_2,d_2,b_2} + (d_1b_2/r_1)))/(b_2(r_1-r_2)) = \gamma(b_2,b_2,b_2)$. □

Proof of Theorem 3. Look at the proof of Theorem 2. By [7] and the proof of the quoted theorem of A. Hirschowitz ([3], [6], §3, [7]), E_1 is general in $M(X;r_1,d_1)$. Hence if $c < r_1$ we have $\deg(\widetilde{A}) \leq \delta_{r_1,d_1,c}$. Use this inequality instead of the inequality $\deg(\widetilde{A}) < cd_1/r_1$ used in the proof of Theorem 2. □

Proof of Theorem 1. By [7] and the proof of the quoted theorem of A. Hirschowitz ([3], [6], §3, [7]) F is stable and a general element of $M(\rho,\delta_{r,d,\rho})$. Furthermore, the same references show that F is saturated in E and that the vector bundle E/F is a general element of $M(X;r-\rho,d-\delta_{r,d,\rho})$. Hence we may apply the proof of Theorem 3. □

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