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TERNARY CODES AND THETA-FUNCTIONS

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Abstract: There is a remarkable connection between the theory of modular forms, in particular theta functions, and coding theory, the connection being established by defining a suitable lattice corresponding to the given code. First we determine some transformation formulas for certain theta functions. Then it is proved that the theta function of the lattice corresponding to the code can be expressed in terms of the Hamming weight enumerator. In particular if the code is self-dual and the length is a multiple of 8 this theta function is a modular form for some subgroup of the modular group. And using the structure of this space of modular forms we can describe the weight distribution of the code.

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1. Introduction

In [3] Leech and Sloane constructed a lattice corresponding to a binary code. This lattice was unimodular, and it was an even unimodular lattice if the code was doubly even. Ebeling (see [1]) used this fact to describe a connection between the Hamming weight enumerator of the code and the theta function of the corresponding lattice. If C is a self-dual doubly even code this theta function is a modular form for the modular group $SL_2(\mathbb{Z})$. In [1] a generalization to p-ary codes is also given. For any odd prime p the ring of algebraic integers in the cyclotomic field of the p-th roots of unity is used instead of the ring $\mathbb{Z}$. The concept of a lattice over this ring of algebraic integers is introduced, and a

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construction of lattices corresponding to codes over $\mathbb{F}_p$ is given. Again, under certain conditions for the code C the corresponding theta function is a modular form for the full modular group. In particular the case of self-dual ternary codes was studied in [2]. In this paper we consider ternary codes, and follow the original construction by Leech and Sloane for the binary case. If we denote the standard reduction modulo 3 by ρ, i.e. $\rho : \mathbb{Z} \to \mathbb{F}_3$, and generalize this in an obvious way to $\rho : \mathbb{Z}^n \to \mathbb{F}_3^n$, then we define $\Lambda_C = \frac{1}{\sqrt{3}}\rho^{-1}(C)$. This is a lattice over $\mathbb{Z}$ of rank n, where n is the length of the code C. The theta function associated to this lattice can be expressed in terms of the weight enumerator, too. If C is a self-dual ternary code of length $n \equiv 0 \bmod 8$ this theta function is a modular form of weight $\frac{1}{2}n$, not for the full modular group, but for a subgroup if index 3, the so-called theta-group, instead. The structure of the linear space of modular forms of a given weight for the theta-group can be found in [5]. In particular a basis for this space is given explicitly. By using this basis we can describe the weight distribution for such codes.

2. Theta Functions

A lattice of rank n is a subset $\Lambda \subset \mathbb{R}^n$ such that $\Lambda = \mathbb{Z}\underline{e}_1 \oplus \ldots \oplus \mathbb{Z}\underline{e}_n$ for some basis $\{\underline{e}_1, \ldots \underline{e}_n\}$ of $\mathbb{R}^n$. The standard inner product $\underline{x} \cdot \underline{y} = \sum_{i=1}^n x_i y_i$ on $\mathbb{R}^n$ defines a non-degenerate, positive definite symmetric bilinear form on Λ. The dual Λ^* of a lattice Λ is defined by $\Lambda^* = \left\{\underline{x} \in \mathbb{R}^n | \underline{x} \cdot \underline{y} \in \mathbb{Z} \text{ for all } \underline{y} \in \Lambda\right\}$. A lattice is called *integral* if $\Lambda \subset \Lambda^*$, and it is called *unimodular* if $\Lambda = \Lambda^*$. To any lattice Λ we can associate its so called theta-function. It is defined for $z \in \mathbb{H}$ by

$$\vartheta(z, \Lambda) = \sum_{\underline{x} \in \Lambda} q^{\underline{x} \cdot \underline{x}},$$

where $q = e^{\pi i z}$. For an integral lattice Λ the numbers $\underline{x} \cdot \underline{x}$ are integers for all $\underline{x} \in \Lambda$, and in that case the theta function is a modular form of weight $\frac{1}{2}n$.

From the theory of theta functions the following general transformation formula is well-known.

Proposition 1.

$$\vartheta\left(-\frac{1}{z}, \Lambda\right) = \left(\frac{z}{i}\right)^{\frac{n}{2}} \frac{1}{\operatorname{vol}(\mathbb{R}^n/\Lambda)} \vartheta(z, \Lambda^*),$$

where $\operatorname{vol}(\mathbb{R}^n/\Lambda) = |\det(\underline{e}_1, \ldots, \underline{e}_n)|$,

the volume of the fundamental parallellotope $\{\alpha_1\underline{e}_1 + \ldots + \alpha_n\underline{e}_n | 1 \leq \alpha_i \leq 1\}$.

See for instance [1].

Now let p be an odd prime. Define the following theta-like functions:

$$\vartheta_{j,p}(z) = \sum_{m=-\infty}^{\infty} e^{\pi i z\left(m+\frac{j}{p}\right)^2}, \text{ for } j = 0, 1, \cdots, p-1.$$

Then we have for all p that $\vartheta_{0,p} = \vartheta_0(z) = \sum_{m=-\infty}^{\infty} e^{\pi i z m^2} = 1 + 2\sum_{m=1}^{\infty} e^{\pi i z n^2}$ is the classical Jacobi theta function. Since $\vartheta_0(z)$ is the theta-function corresponding to the standard lattice $\mathbb{Z} \subset \mathbb{R}$ we have as a special case of Proposition 1 that

$$\vartheta_0\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \vartheta_0(z).$$

Furthermore we have $\vartheta_{j,p}(z) = \vartheta_{p-j,p}(z)$ for $j = 1, \cdots, \frac{p-1}{2}$. Next we put

$$\overline{\vartheta}_p(z) = \sum_{j=1}^{\frac{p-1}{2}} \vartheta_{j,p}(z).$$

Now we are interested in the functions $\vartheta_0(pz)$ and $\overline{\vartheta}_p(pz)$; in particular we want to determine the effect of the transformation $z \to -\frac{1}{z}$ on these functions

Proposition 2.

(i) $\vartheta_0\left(-\frac{p}{z}\right) = \frac{1}{\sqrt{p}}\left(\frac{z}{i}\right)^{\frac{1}{2}} \left\{\vartheta_0(pz) + 2\overline{\vartheta}_p(pz)\right\}$.

(ii) $\overline{\vartheta}_p\left(-\frac{p}{z}\right) = \frac{1}{\sqrt{p}}\left(\frac{z}{i}\right)^{\frac{1}{2}} \left\{\frac{p-1}{2}\vartheta_0(pz) - \overline{\vartheta}_p(pz)\right\}$.

Proof. (i) The function $\vartheta_0(pz) = \sum_{m=-\infty}^{\infty} e^{\pi i p z m^2}$ can be seen as the theta function associated to the lattice $\sqrt{p}\,\mathbb{Z}$ in $\mathbb{Z}$. From Proposition 1 we get

$$\begin{aligned}\vartheta_0\left(-\frac{p}{z}\right) &= \frac{1}{\sqrt{p}}\left(\frac{z}{i}\right)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} e^{\pi i z \frac{1}{p} m^2} = \frac{1}{\sqrt{p}}\left(\frac{z}{i}\right)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} e^{\pi i p z\left(\frac{1}{p}m\right)^2} \\ &= \frac{1}{\sqrt{p}}\left(\frac{z}{i}\right)^{\frac{1}{2}} \sum_{j=0}^{p-1} \vartheta_{j,p}(pz) = \frac{1}{\sqrt{p}}\left(\frac{z}{i}\right)^{\frac{1}{2}} \left(\vartheta_0(pz) + 2\overline{\vartheta}_p(pz)\right).\end{aligned}$$

(ii) First we remark that it is easy to see that $\vartheta_0(z) = \sum_{j=0}^{p-1} \vartheta_{j,p}\left(p^2 z\right) = \vartheta_0\left(p^2 z\right) + 2\overline{\vartheta}_p\left(p^2 z\right)$. This can also be written as

$$\overline{\vartheta}_p(pz) = \frac{1}{2}\left(\vartheta_0\left(\frac{1}{p}z\right) - \vartheta_0(pz)\right).$$

So we have $\overline{\vartheta}_p\left(-\frac{p}{z}\right) = \frac{1}{2}\left(\vartheta_0\left(-\frac{1}{pz}\right) - \vartheta_0\left(-\frac{p}{z}\right)\right)$. From $\vartheta_0\left(-\frac{1}{pz}\right) = \left(\frac{pz}{i}\right)^{\frac{1}{2}}\vartheta_0(pz)$ and the result in i. the formula for $\overline{\vartheta}_p\left(-\frac{p}{z}\right)$ follows easily. □

From now on we take $p = 3$ and we denote $\vartheta_{j,3}(z)$ by $\vartheta_j(z)$. Then we have $\overline{\vartheta}_3(z) = \vartheta_1(z)$ and from Proposition 2 we get the following corollary.

Corollary 1. *(i)* $\vartheta_0\left(-\frac{3}{z}\right) = \frac{1}{\sqrt{3}}\left(\frac{z}{i}\right)^{\frac{1}{2}}\left(\vartheta_0(3z) + 2\vartheta_1(3z)\right)$.
(ii) $\vartheta_1\left(-\frac{3}{z}\right) = \frac{1}{\sqrt{3}}\left(\frac{z}{i}\right)^{\frac{1}{2}}\left(\vartheta_0(3z) - \vartheta_1(3z)\right)$.

The so-called theta group Γ_θ is the subgroup of the full modular group $\Gamma = SL_2(\mathbb{Z})$ generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. These matrices correspond to the transformations $z \to z+2$ and $z \to -\frac{1}{z}$ of the complex upper half plane. The group Γ_θ is a subgroup of index 3 of Γ. The function $\vartheta_0(3z)$ is invariant under the transformation $z \to z+2$, and for the function $\vartheta_1(3z)$ we have $\vartheta_1(3(z+2)) = e^{\frac{2\pi i}{3}}\vartheta_1(3z)$. Together with Corollary 1 the following proposition follows easily.

Proposition 3. *The function* $\vartheta_1(3z)^6\left(\vartheta_1(3z)^3 - \vartheta_0(3z)^3\right)^6$ *is a modular form of weight* 12 *for the group* Γ_θ.

Proposition 4. *There is no homogeneous non-zero polynomial* $F(X,Y)$ *such that* $F(\vartheta_0(3z), \vartheta_1(3z))$ *is identically* 0 *on* $\mathbb{H}$.

Proof. Suppose such a polynomial $F(X,Y)$ exists. Then $f(Z) = F(1,Z)$ is a non-zero polynomial in one variable, and $f\left(\frac{\vartheta_1(3z)}{\vartheta_0(3z)}\right)$ is identically 0 on $\mathbb{H}$. This means that the function $\frac{\vartheta_1(3z)}{\vartheta_0(3z)}$ can take as values only the finitely many roots of the polynomial $f(Z)$. The function is meromorphic, and must therefore be constant. The function vanishes at $i\infty$, and therefore this constant must be 0. But that means that $\vartheta(3z) = 0$ for all $z \in \mathbb{H}$, which is a contradiction. □

Denote by $M_k(\Gamma_\theta)$ the space of all modular forms for Γ_θ of weight k. In the next section we will be interested in particular in the spaces $M_k(\Gamma_\theta)$ where $k \equiv 0 \bmod 4$. In order to describe the structure of these spaces we need two more functions:

$$\Theta_0(z) = \sum_{m=-\infty}^{\infty} q^{\left(m+\frac{1}{2}\right)^2} \quad \text{and} \quad \Theta_1(z) = \vartheta_0(z+1).$$

From [5] we get the following proposition.

Proposition 5. *(i)* $\dim M_{4k}(\Gamma_\theta) = 1 + k$.
(ii) The following functions constitute a basis for

$$M_{4k}(\Gamma_\theta) : \vartheta_0(z)^{8k-8r}\left(\Theta_0(z)\Theta_1(z)\right)^{4r},$$

where $0 \le r \le k$.

3. Ternary Codes

Let C be a linear ternary code of length n, i.e. C is a subspace of $\mathbb{F}_3^n$. If we denote the standard reduction modulo 3 by ρ, i.e. $\rho : \mathbb{Z} \to \mathbb{F}_3$, and generalize this in an obvious way to $\rho : \mathbb{Z}^n \to \mathbb{F}_3^n$, then we define

$$\Lambda_C = \frac{1}{\sqrt{3}}\rho^{-1}(C).$$

It is easy to see that Λ_C is a lattice of rank n. To this lattice we associate a theta function in the way described in the previous section. There is a remarkable relation between this theta-function and the Hamming weight enumerator of this code. The Hamming weight enumerator of a linear code C of length n is defined by

$$W_C(X,Y) = \sum_{c\in C} X^{n-w(c)}Y^{w(c)} = \sum_{i=0}^{n} A_i X^{n-i}Y^i.$$

Here A_i is the number of code words in C of weight i, where the Hamming weight of a codeword is the number of nonzero coordinates.

Proposition 6. $\vartheta(z,\Lambda_C) = W_C\left(\vartheta_0(3z), \vartheta_1(3z)\right)$.

Proof. For $c \in C$ let $\underline{c}$ be the vector with the same entries 0, 1, or 2, but now considered to be an element of $\mathbb{Z}^n$. Then $\rho^{-1}(c) = \underline{c} + 3\mathbb{Z}^n$. We get

$$\sum_{\underline{x}\in\frac{1}{\sqrt{3}}(\underline{c}+3\mathbb{Z}^n)} e^{\pi i z \underline{x}\cdot\underline{x}} = \sum_{\underline{n}\in\mathbb{Z}^n} e^{\frac{1}{3}\pi i z(\underline{c}+3\underline{n})\cdot(\underline{c}+3\underline{n})} = \vartheta_0(3z)^{n-w(c)}\vartheta_1(3z)^{w(c)},$$

from which the result follows. □

This expression for the weight enumerator and the transformation formulas for the theta functions from the previous section yield a very simple proof of the MacWilliams identity, i.e the relation between the weight enumerator of a code and its dual.

$A_3 = 16$
$A_6 = 64$

Table 1: Weight distribution for $n = 8$

Corollary 2. *For any ternary code C the weight enumerator of C and that of the dual code $C^{\perp}$ are related by the identity*

$$W_{C^{\perp}}(X,Y) = \frac{1}{\#C} W_C(X+2Y, X-Y).$$

Proof. From a generator matrix for the code we get that $\operatorname{vol}(\mathbb{R}^n/\Lambda_C) = \frac{3^{\frac{n}{2}}}{\#C}$. Furthermore $W_{C^{\perp}}(\vartheta_0(3z), \vartheta_1(3z)) = \vartheta(z, \Lambda_C^*) = \left(\frac{i}{z}\right)^{\frac{n}{2}} \operatorname{vol}(\mathbb{R}^n/\Lambda_C)\, \vartheta\left(-\frac{1}{z}, \Lambda_C\right)$.

By using the fact that W_C is a homogeneous polynomial and Corollary 1 we get that

$$W_{C^{\perp}}(\vartheta_0(3z), \vartheta_1(3z)) = \frac{1}{\#C} W_C(\vartheta_0(3z) + 2\vartheta_1(3z), \vartheta_0(3z) - \vartheta_1(3z)).$$

The result then follows from Proposition 4. □

Next let C be a self-dual ternary code, i.e. $C = C^{\perp}$. Then we have that the length n of C has to be even, and the dimension of C is equal to $\frac{1}{2}n$. All weights in C are multiples of 3, so $A_j = 0$ for all $j \not\equiv 0 \bmod 3$. Besides $\Lambda_C^* = \Lambda_C$, and $\operatorname{vol}(\mathbb{R}^n/\Lambda_C) = 1$. For a self-dual ternary code C we have

$$W_C(X,Y) = \frac{1}{\sqrt{3}^n} W_C(X+2Y, X-Y).$$

From this identity one can derive relations between the coefficients of the weight enumerator. We will look into this matter in a different way, using Proposition 6 and the basis for $M_{4k}(\Gamma_\theta)$ given in Proposition 5.

Proposition 7. *Let C be a ternary self-dual code of length n, where $n \equiv 0 \bmod 8$.*

(i) $\vartheta(z, \Lambda_C) \in M_{\frac{1}{2}n}(\Gamma_\theta)$

The coefficient A_{3j} is a linear combination of $A_0 (=1), A_3, \ldots, A_{3\left(\frac{1}{8}n-1\right)}$ for all $j \geq \frac{1}{8}n$.

$A_6 = 224 + 5A_3$
$A_9 = 2720 - 21A_3$
$A_{12} = 3360 + 23A_3$
$A_{15} = 256 - 8A_3$

Table 2: Weight distribution for $n = 16$

$A_9 = 4048 + 249A_3 - 6A_6$
$A_{12} = 61824 - 308A_3 + 15A_6$
$A_{15} = 242880 - 564A_3 - 20A_6$
$A_{18} = 198352 + 1029A_3 + 15A_6$
$A_{21} = 24288 - 386A_3 - 6A_6$
$A_{24} = 48 - 21A_3 + A_6$

Table 3: Weight distribution for $n = 24$

$A_{12} = \frac{419858581056}{5200603} + \frac{33976422195}{5200603}A_3 + \frac{790311419}{5200603}A_6 - \frac{87872167}{5200603}A_9$
$A_{15} = \frac{5886013304832}{5200603} - \frac{65275207056}{5200603}A_3 - \frac{9561179092}{5200603}A_6 + \frac{870828020}{5200603}A_9$
$A_{18} = \frac{51504554041920}{5200603} + \frac{873379002699}{5200603}A_3 + \frac{73470083165}{5200603}A_6 - \frac{7006368820}{5200603}A_9$
$A_{21} = \frac{49611192626688}{5200603} - \frac{7743727782096}{5200603}A_3 - \frac{510450249238}{5200603}A_6 + \frac{50501041850}{5200603}A_9$
$A_{24} = \frac{372867202063680}{5200603} + \frac{49473565996743}{5200603}A_3 + \frac{3222490650215}{5200603}A_6 - \frac{321799212862}{5200603}A_9$
$A_{27} = -\frac{1500982498332928}{5200603} - \frac{240881215032540}{5200603}A_3 - \frac{15739274154184}{5200603}A_6 + \frac{1573736033228}{5200603}A_9$
$A_{30} = \frac{1244562578886912}{5200603} + \frac{198309291399452}{5200603}A_3 + \frac{12962529337112}{5200603}A_6 - \frac{1296219649852}{5200603}A_9$

Table 4: Weight distribution for $n = 32$

Proof. (i) It is obvious that $\vartheta(z+2, \Lambda_C) = \vartheta(z, \Lambda_C)$. From Proposition 1 and the remarks above we get that $\vartheta\left(-\frac{1}{z}, \Lambda_C\right) = z^{\frac{1}{2}n}\vartheta(z, \Lambda_C)$.

(ii) First we remark that in the expansion of

	$A_0 = 1$	A_3	A_6
A_{15}	$\frac{128975512832176}{76405887}$	$\frac{154831902376057}{993276531}$	$\frac{37861790291453}{4966382655}$
A_{18}	$\frac{1287137390073440}{76405887}$	$-\frac{752572455773974}{993276531}$	$-\frac{110243599209904}{993276531}$
A_{21}	$\frac{2907261721446320}{8489543}$	$\frac{1714464099789809}{110364059}$	$\frac{138751604925743}{110364059}$
A_{24}	$-\frac{4743375504966160}{25468629}$	$-\frac{53746647029833339}{331092177}$	$-\frac{4267703325984199}{331092177}$
A_{27}	$\frac{292731080978675920}{25468629}$	$\frac{488821679898169135}{331092177}$	$\frac{39437307852586759}{331092177}$
A_{30}	$-\frac{703335270337660624}{8489543}$	$-\frac{1347349860853132527}{110364059}$	$-\frac{544146186853179227}{551820295}$
A_{33}	$\frac{15254524489494530080}{25468629}$	$\frac{28946508974996279962}{331092177}$	$\frac{2337814890466849309}{331092177}$
A_{36}	$-\frac{82048153788240787600}{25468629}$	$-\frac{155729883020014028503}{331092177}$	$-\frac{12577138626445482958}{331092177}$
A_{39}	$\frac{22898419066591158800}{8489543}$	$\frac{43461801482976493787}{110364059}$	$\frac{3510086604055039919}{110364059}$

Table 5: Weight distribution for $n = 40$

	A_9	A_{12}
A_{15}	$-\frac{316987690021}{4966382655}$	$-\frac{138197914441}{4966382655}$
A_{18}	$\frac{640134052385}{993276531}$	$\frac{437505640061}{993276531}$
A_{21}	$-\frac{649538104003}{110364059}$	$-\frac{589279327789}{110364059}$
A_{24}	$\frac{18612775833590}{331092177}$	$\frac{18320752533476}{331092177}$
A_{27}	$-\frac{169606606660883}{331092177}$	$-\frac{169300677489335}{331092177}$
A_{30}	$\frac{2335820472398239}{551820295}$	$\frac{2335469514690619}{551820295}$
A_{33}	$-\frac{10033654381450262}{331092177}$	$-\frac{10033562006732879}{331092177}$
A_{36}	$\frac{53979162440604143}{331092177}$	$\frac{53979138933059576}{331092177}$
A_{39}	$-\frac{15064750158016831}{110364059}$	$-\frac{15064749275104359}{110364059}$

Table 6: Continuation: Weight distribution for $n = 40$

$$\sum_{j=0}^{\lfloor \frac{1}{3}n \rfloor} A_{3j} \vartheta_0(3z)^{n-3j} \vartheta_1(3z)^{3j}$$

$$= \sum_{j=0}^{\lfloor \frac{1}{3}n \rfloor} A_{3j} \left(1 + 2q^3 + 2q^{12} + \dots\right)^{n-3j} q^j \left(1 + q + q^5 \dots\right)^{3j}$$

the coefficient of q^j has the form

$$A_{3j} + \text{combination of } A_0, \dots, A_{3j-3} .$$

Therefore we consider only the powers of q up till $q^{\lfloor \frac{1}{3}n \rfloor}$. Using the basis from Proposition 5 we can write

$$\sum_{r=0}^{\frac{1}{8}n} \alpha_r \vartheta_0(z)^{n-8r} \left(\Theta_0(z)\Theta_1(z)\right)^{4r}$$
$$= \sum_{j=0}^{\lfloor \frac{1}{3}n \rfloor} A_{3j} \left(1 + 2q^3 + 2q^{12} + \dots\right)^{n-3j} q^j \left(1 + q + q^5 \dots\right)^{3j} ,$$

for some constants $\alpha_0, \dots, \alpha_{\frac{1}{8}n}$. By looking at the coefficients of $1, q, \dots, q^{\lfloor \frac{1}{3}n \rfloor}$ on the left and right hand side we get a system of $\lfloor \frac{1}{3}n \rfloor$ linear equations for the numbers $\alpha_0, \dots, \alpha_{\frac{1}{8}n}$. By using Gauss Jordan elimination the left hand side of the last $\lfloor \frac{1}{3}n \rfloor - \frac{1}{8}n$ equations is reduced to 0, while on the right hand side we get combinations of $A_0, A_3, \dots, A_{3\lfloor \frac{1}{3}n \rfloor}$. This can be considered as a system of homogeneous equations for $A_0, A_3, \dots, A_{3\lfloor \frac{1}{3}n \rfloor}$. We add one more equation, i.e. $A_0 + A_3 + \dots + A_{3\lfloor \frac{1}{3}n \rfloor} = 3^{\frac{1}{2}n}$. From this extended system of linear equations the $A_{\frac{3}{8}n}, \dots, A_{3\lfloor \frac{1}{3}n \rfloor}$ can be solved in terms of $A_0, \dots, A_{3\left(\frac{1}{8}n-1\right)}$. □

4. Tables

We follow the strategy of the proof of part (ii) of Proposition 7. We use the following expansions:

$$\vartheta_0(3z) = 1 + 2q^3 + 2q^{12} + 2q^{27} + 2q^{48} + \dots ,$$
$$\vartheta_1(3z) = q^{\frac{1}{3}} \left(1 + q + q^5 + q^8 + q^{16} + \dots\right) ,$$
$$\Theta_0(z) = 2q^{\frac{1}{4}} \left(1 + q^2 + q^6 + q^{12} + q^{20} + \dots\right) ,$$
$$\Theta_1(z) = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} + \dots .$$

For $n = 8, 16, 24, 32$ and 40 we get the following tables where the weight distribution $\{A_i\}$ is expressed in terms of $A_0, \ldots, A_{3(\frac{1}{8}n-1)}$. For these calculations the computer programme *Maple* was used.

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