International Journal of Pure and Applied Mathematics
Volume 29 No. 3 2006, 337-347

# ON A CLASS OF ANALYTIC FUNCTIONS RELATED TO HADAMARD PRODUCTS

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**Abstract:** In this paper, we introduce a new class of analytic functions which are analytic related to Hadamard products. Characterization properties which include coefficient bounds, growth and distortion, and closure theorem are given. Further, results on integral transforms are also discussed.

**AMS Subject Classification:** 30C45
**Key Words:** analytic functions, convex functions, starlike functions, Hadamard product, integral transform

## 1. Introduction and Preliminaries

Denote by $A$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$. Denote by $S^*(\alpha)$ the class of starlike functions $f \in A$ of order $\alpha$ $(0 \leq \alpha < 1)$ satifying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U$$

and let $C(\alpha)$ be the class of convex functions $f \in A$ of order $\alpha$ $(0 \leq \alpha < 1)$ such that $zf' \in S^*(\alpha)$.

Received: May 15, 2006 

If $f$ of the form (1) and $g(z)=z+\sum\limits_{n=2}^{\infty} b_n z^n$ are two functions in $A$, then the Hadamard product (or convolution) of $f$ and $g$ is denoted by $f*g$ and is given by

$$(f*g)(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n. \tag{2}$$

Ruscheweyh [4] using the convolution techniques, introduced and studied an important subclass of $A$, the class of prestarlike functions of order $\alpha$, which denoted by $\mathcal{R}(\alpha)$. Thus $f\in A$ is said to be prestarlike function of order $\alpha(0\leq\alpha<1)$ if $f*S_\alpha\in S^*(\alpha)$, where $S_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}}=z+\sum_{n=2}^{\infty}c_n(\alpha)z^n$ and $c_n(\alpha)=\frac{\Pi_{j=2}^{n}(j-2\alpha)}{(n-1)!}\quad(n\in\mathbf{N}\,\{1\}\quad \mathbf{N}:=\{1,2,3,\ldots\})$. We note that $\mathcal{R}(0)=C(0)$ and $\mathcal{R}(\frac{1}{2})=S^*(\frac{1}{2})$. Juneja et al [1] define the family $\mathcal{D}(\Phi,\Psi;\alpha)$ consisting of functions $f\in A$ so that

$$\operatorname{Re}\left(\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}\right)>\alpha,\quad z\in U\,, \tag{3}$$

where $\Phi(z)=z+\sum_{n=2}^{\infty}\Upsilon_n z^n$ and $\Psi(z)=z+\sum_{n=2}^{\infty}\gamma_n z^n$ analytic in $U$ such that $f(z)*\Psi(z)\neq 0$, $\Upsilon_n\geq 0$, $\gamma_n\geq 0$ and $\Upsilon_n>\gamma_n$ $(n\geq 2)$.

For suitable choices of $\Phi$ and $\Psi$, we can easily gather the various subclasses of $A$. For example $\mathcal{D}(\frac{z}{(1-z)^2},\frac{z}{1-z};\alpha)=S^*(\alpha)$, $\mathcal{D}(\frac{z+z^2}{(1-z)^3},\frac{z}{(1-z)^2};\alpha)=C(\alpha)$ and $\mathcal{D}(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}},\frac{z}{(1-z)^{2-2\alpha}};\alpha)=\mathcal{R}(\alpha)$.

Next we give a brief concept of subordination which will be used in the next section.

Let $f(z)$ and $F(z)$ be analytic in $U$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $U$, if there exists an analytic function $w(z)$ in $U$ such that $w(z)\leq|z|$ and $f(z)=F(w(z))$, denoted by $f\prec F$ or $f(z)\prec F(z)$. If $F(z)$ is univalent in $U$, then the subordination is equivalent to $f(0)=F(0)$ and $f(U)\subset F(U)$ (see [3]).

Now we define the following new class of analytic functions, and obtain some interesting results.

**Definition 1.1.** Given $0\leq\mu\leq 1$ and $0<\beta\leq 1$ and functions

$$\Phi(z)=z+\sum_{n=2}^{\infty}\Upsilon_n z^n,\quad \Psi(z)=z+\sum_{n=2}^{\infty}\gamma_n z^n$$

analytic in $U$ such that $\Upsilon_n\geq 0$, $\gamma_n\geq 0$ and $\Upsilon_n>\gamma_n$ $(n\geq 2)$, we say that

$f \in A$ is in $\mathcal{D}(\Phi, \Psi; \beta, \mu)$ if $f(z) * \Psi(z) \neq 0$ and

$$\left|\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right| < \beta\left|\mu\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} + 1\right|, \tag{4}$$

for all $z \in U$.

We note that when $\mu = 0$ and $\beta = 1 - \alpha$, we have

$$\left|\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right| < 1 - \alpha \tag{5}$$

which implies (3).

Also denote by $T$ (see [5]) the subclass of $A$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z_n. \tag{6}$$

Let $T^*(\alpha)$ and $C_T(\alpha)$ denote the subfamilies of $T$ that are starlike of order $\alpha$ and convex of order $\alpha$. Silverman [5] studied $T^*(\alpha)$ and $C_T(\alpha)$ and Siverman and Silvia [6] studied $R_T(\alpha) = T \cap R_\alpha$ and obtained many interesting results.

Now let us write

$$D_T(\Phi, \Psi; \alpha) = \mathcal{D}(\Phi, \Psi; \alpha) \cap T$$

and

$$\mathcal{D}_T(\Phi, \Psi; \beta, \mu) = \mathcal{D}(\Phi, \Psi; \beta, \mu) \cap T.$$

Note that $D_T(\Phi, \Psi; \alpha)$ has been extensively studied by Juneja et al [1].

## 2. Characterization Property

First of all, we consider the geometric property for the class $\mathcal{D}(\Phi, \Psi; \beta, \mu)$.

**Theorem 2.1.** *The function $f \in \mathcal{D}(\Phi, \Psi; \beta, \mu)$ if and only if*

$$\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} \prec \frac{1 + \beta z}{1 - \beta\mu z}$$

*where $\prec$ stands for the subordination.*

*Proof.* Let $f \in \mathcal{D}(\Phi, \Psi; \beta, \mu)$, then from (4) we have

$$\left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1\right|^2 < \beta^2 \left|\mu\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} + 1\right|^2.$$

By a simple calculation we obtain

$$\left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - \frac{1+\mu\beta^2}{1-\mu^2\beta^2}\right| < \frac{\beta(1+\mu)}{1-\mu^2\beta^2}.$$

Obviously, this is saying that $F(z) = (f(z)*\Phi(z))/(f(z)*\Psi(z))$ is contained in the disk whose center is $(1+\mu\beta^2)/(1-\mu^2\beta^2)$ and radius is $(\beta(1+\mu))/(1-\mu^2\beta^2)$. This also tells us that the funtion $w = p(z) = (1+\beta z)/(1-\mu\beta z)$ maps the unit disk to the disk

$$\left|w - \frac{(1+\mu\beta^2)}{(1-\mu^2\beta^2)}\right| < \frac{\beta(1+\mu)}{1-\mu^2\beta^2}.$$

Notice also that $F(0) = p(0)$, $G(U) \subset p(U)$, and $p(z)$ is univalent in $U$, we obtain the following conclusion

$$\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} \prec p(z) = \frac{1+\beta z}{1-\beta\mu z}.$$

Conversely, let

$$\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} \prec \frac{1+\beta z}{1-\beta\mu z},$$

then

$$\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} \prec \frac{1+\beta w(z)}{1-\beta\mu w(z)}, \tag{7}$$

where $w(z)$ is analytic in $U$, and $w(0) = 0$, $|w(z)| < 1$. By calculation we can easily obtain from (7) that

$$\left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1\right| < \beta \left|\mu\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} + 1\right|,$$

that is $f \in \mathcal{D}(\Phi, \Psi; \beta, \mu)$.

If $\mu = \beta = 1$, we have

$$\left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1\right| < \left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} + 1\right|.$$

It is obvious that
$$\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} \prec \frac{1+z}{1-z}.$$

Hence the proof of the theorem is complete. □

We shall now make a systematic study of the class $\mathcal{D}_T(\Phi,\Psi;\beta,\mu)$. It would be assumed throughout that $\Phi(z)$ and $\Psi(z)$ satisfy the conditions stated in Definition 1.1 and that $f(z)*\Psi(z)\neq 0$ for $z\in U$.

In the following theorem, we give a necessary and sufficient condition for a function $f$ to be in $\mathcal{D}_T(\Phi,\Psi;\beta,\mu)$.

**Theorem 2.2.** (Coefficient Bounds) *Let a function $f\in A$ be given by (1). If $0\leq\mu\leq 1$ and $0<\beta\leq 1$,*

$$\sum_{n=2}^{\infty}\frac{[(1+\beta\mu)\Upsilon_n-(1-\beta)\gamma_n]}{\beta(\mu+1)}|a_n|\leq 1, \tag{8}$$

*then*

$$f\in\mathcal{D}(\Phi,\Psi;\beta,\mu).$$

*Proof.* Assume that (8) holds true. It is sufficient to show that

$$\left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}-1\right|<\beta\left|\frac{\mu f(z)*\Phi(z)}{f(z)*\Psi(z)}+1\right|.$$

Consider

$$M(f)=|(f(z)*\Phi(z))-(f(z)*\Psi(z))|-\beta\left|\mu(f(z)*\Phi(z))+(f(z)*\Psi(z))\right|.$$

Then for $0<|z|=r<1$, we have

$$M(f)=\left|-\sum_{n=0}^{\infty}(\Upsilon_n-\gamma_n)a_nz^n\right|-\beta\left|(1+\mu)z-\sum_{n=0}^{\infty}(\mu\Upsilon_n+\gamma_n)a_nz^n\right|.$$

That is

$$rM(f)=\sum_{n=0}^{\infty}(\Upsilon_n-\gamma_n)|a_n|r^{n+1}-\beta(1+\mu)+\sum_{n=0}^{\infty}\beta(\mu\Upsilon_n+\gamma_n)|a_n|r^{n+1}$$

and so

$$rM(f)=\sum_{n=0}^{\infty}((1+\beta\mu)\Upsilon_n-(1-\beta)\gamma_n)|a_n|r^{n+1}-\beta(1+\mu). \tag{9}$$

The inequality in (9) holds true for all $r (0 < r < 1)$. Therefore, letting $r \to 1^-$ in (9) we obtain

$$M(f) = \sum_{n=0}^{\infty} ((1+\beta\mu)\Upsilon_n - (1-\beta)\gamma_n)|a_n| - \beta(1+\mu) \le 0$$

by (8). Hence $f \in \mathcal{D}(\Phi, \Psi; \beta, \mu)$. □

**Theorem 2.3.** (Coefficient Bounds) *Let a function $f$ be given by (6). Then $f \in \mathcal{D}_T(\Phi, \Psi; \beta, \mu)$ if and only if (8) is satisfied.*

*Proof.* Let $f \in \mathcal{D}_T(\Phi, \Psi; \beta, \mu)$ satisfies the coefficient inequality. Then

$$\frac{1}{\beta}\left|\frac{\dfrac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1}{\dfrac{\mu f(z)*\Phi(z)}{f(z)*\Psi(z)} + 1}\right| = \frac{1}{\beta}\left|\frac{\sum_{n=0}^{\infty}(\Upsilon_n - \gamma_n)a_n z^n}{(\mu+1)z - \sum_{n=0}^{\infty}(\mu\Upsilon_n + \gamma_n)a_n z^n}\right| < 1 \qquad (10)$$

for all $z \in U$. Since $\mathrm{Re}(z) \le |z|$ for all $z$, it follows from (10) that

$$\mathrm{Re}\left[\frac{1}{\beta}\left(\frac{\sum_{n=0}^{\infty}(\Upsilon_n - \gamma_n)a_n z^n}{(\mu+1)z - \sum_{n=0}^{\infty}(\mu\Upsilon_n + \gamma_n)a_n z^n}\right)\right] < 1 \quad (z \in U). \qquad (11)$$

We choose the values $z$ on the real axis so that $\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)}$ is real. Upon clearing the denominator in (11) and letting $r \to 1^-$ along real values leads to the desired inequality

$$\sum_{n=2}^{\infty}[(1+\beta\mu)\Upsilon_n - (1-\beta)\gamma_n]|a_n| \le \beta(\mu+1),$$

whch is (8). That (8) implies $f \in \mathcal{D}_T(\Phi, \Psi; \beta, \mu)$ is an immediate consequence of Theorem 2.2. Hence the theorem is proved. □

The result is sharp for functions $f$ given by

$$f(z) = z - \frac{\beta(\mu+1)z^n}{(1+\beta\mu)\Upsilon_n - (1-\beta)\gamma_n} \quad (n \ge 2).$$

**Corollary 1.** *Let a function $f$ defined by (6) belongs to the class $\mathcal{D}_T(\Phi, \Psi; \beta, \mu)$. Then*

$$a_n \le \frac{\beta(\mu+1)}{[(1+\beta\mu)\Upsilon_n - (1-\beta)\gamma_n]}, \quad n \ge 2.$$

For $\mu = 0$ and $\beta = 1 - \alpha$, we have result obtained by Juneja [1].

**Corollary 2.** (see [1]) *Let a function $f$ defined by (6) belongs to the class $\mathcal{D}_T(\Phi, \Psi; 1-\alpha, 0)$. Then*

$$\sum_{n=2}^{\infty} \frac{[\Upsilon_n - \alpha\gamma_n]}{1-\alpha}|a_n| \leq 1. \tag{12}$$

Next we consider the growth and distortion theorem for the class $\mathcal{D}_T(\Phi, \Psi; \beta, \mu)$. We shall omit the proof as the techniques are similar to various other papers.

**Theorem 2.4.** *Let the function $f$ defined by (6) be in the class $\mathcal{D}_T(\Phi, \Psi; \beta, \mu)$. Then*

$$|z| - |z|^2 \frac{\beta(1+\mu)}{[(1+\beta\mu)\Upsilon_2 - (1-\beta)\gamma_2]} \leq |f(z)| \leq |z| + |z|^2 \frac{\beta(1+\mu)}{[(1+\beta\mu)\Upsilon_2 - (1-\beta)\gamma_2]}\,, \tag{13}$$

*and*

$$1 - |z| \frac{2\beta(1+\mu)}{[(1+\beta\mu)\Upsilon_2 - (1-\beta)\gamma_2]} \leq |f'(z)| \leq 1 + |z| \frac{2\beta(1+\mu)}{[(1+\beta\mu)\Upsilon_2 - (1-\beta)\gamma_2]}\,. \tag{14}$$

*The bounds (13) and (15) are attained for functions given by*

$$f(z) = z - z^2 \frac{\beta(1+\mu)}{[(1+\beta\mu)\Upsilon_2 - (1-\beta)\gamma_2]}. \tag{15}$$

**Theorem 2.5.** *Let a function $f$ be defined by (6) and*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{16}$$

*be in the class $\mathcal{D}_T(\Phi, \Psi; \beta, \mu)$. Then the function $h$ defined by*

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{n=2}^{\infty} q_n z^n\,, \tag{17}$$

*where $q_n = (1-\lambda)a_n + \lambda b_n,\ \ 0 \leq \lambda \leq 1$ is also in the class $\mathcal{D}_T(\Phi, \Psi; \beta, \mu)$.*

*Proof.* The result follows easily from (8) and (17). □

We prove the following theorem by defining functions $f_j(z)$ $(j = 1, 2, \dots m)$ of the form

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \ \text{ for } \ a_{n,j} \geq 0, \ z \in U \,. \tag{18}$$

**Theorem 2.6.** (Closure Theorem) *Let the functions $f_j(z)(j = 1, 2 \dots m)$ defined by (18) be in the classes $\mathcal{D}_T(\Phi, \Psi; \beta_j, \mu)$ $(j = 1, 2, \dots m)$, respectively. Then the function $h(z)$ defined by $h(z) = z - \frac{1}{m} \sum\limits_{n=2}^{\infty} (\sum\limits_{j=1}^{m} a_{n,j}) z^n$ is in the class $\mathcal{D}_T(\Phi, \Psi; \beta, \mu)$, where*

$$\beta = \max_{1 \leq j \leq m} \{\beta_j\} \ \textit{with} \ \ 0 < \beta_j \leq 1. \tag{19}$$

*Proof.* Since $f_j \in \mathcal{D}_T(\Phi, \Psi; \beta_j, \mu)(j = 2, \dots m)$ by applying Theorem 2.2, we observe that

$$\begin{aligned} &\sum_{n=2}^{\infty} [(1 + \beta\mu)\Upsilon_n - (1 - \beta)\gamma_n](\frac{1}{m} \sum_{j=1}^{m} a_{n,j}) \\ &= \frac{1}{m} \sum_{j=1}^{m} \left( \sum_{n=2}^{\infty} [(1 + \beta\mu)\Upsilon_n - (1 - \beta)\gamma_n] a_{n,j} \right) \\ &\leq \frac{1}{m} \sum_{j=1}^{m} (\beta_j (1 + \mu)) \leq \beta(1 + \mu), \end{aligned}$$

which in view of Theorem 2.2, again implies that $h \in \mathcal{D}_T(\Phi, \Psi; \beta, \mu)$ and the proof is complete. □

## 3. Integral Transform of the Class $\mathcal{D}_T(\Phi, \Psi; \beta, \mu)$

For $f \in A$ we define the integral transform

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

where $\lambda$ is real valued, non-negative weight function normalized so that $\int_0^1 \lambda(t) dt = 1$. Since special cases of $\lambda(t)$ are particularly interesting such as $\lambda(t) = (1 + c)t^c$, $c > -1$, for which $V_\lambda$ is known as the Bernardi operator, and

$$\lambda(t) = \frac{(c + 1)^\delta}{\lambda(\delta)} t^c \left( \log \frac{1}{t} \right)^{\delta - 1}, \quad c > -1, \ \delta \geq 0 \,,$$

which gives the Komatu operator. For more details see [2].

First we show that the class $\mathcal{D}_T(\Phi,\Psi;\beta,\mu)$ is closed under $V_\lambda(f)$.

**Theorem 3.1.** *Let $f\in\mathcal{D}_T(\Phi,\Psi;\beta,\mu)$. Then $V_\lambda(f)\in\mathcal{D}_T(\Phi,\Psi;\beta,\mu)$.*

*Proof.* By definition, we have

$$\begin{aligned} V_\lambda(f) &= \frac{(c+1)^\delta}{\lambda(\delta)}\int_0^1(-1)^{\delta-1}t^c(\log t)^{\delta-1}\left(z-\sum_{n=2}^{\infty}a_nz^nt^{n-1}\right)dt \\ &= \frac{(-1)^{\delta-1}(c+1)^\delta}{\lambda(\delta)}\lim_{r\to0^+}\left[\int_r^1 t^c(\log t)^{\delta-1}\left(z-\sum_{n=2}^{\infty}a_nz^nt^{n-1}\right)dt\right], \end{aligned}$$

and a simple calculation gives

$$V_\lambda(f)(z)=z-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta a_nz^n.$$

We need to prove that

$$\sum_{n=2}^{\infty}\frac{[(1+\beta\mu)\Upsilon_n-(1-\beta)\gamma_n]}{\beta(1+\mu)}\left(\frac{c+1}{c+n}\right)^\delta a_n<1. \tag{20}$$

On the other hand by Theorem 2.3, $f\in\mathcal{D}_T(\Phi,\Psi;\beta,\mu)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[(1+\beta\mu)\Upsilon_n-(1-\beta)\gamma_n]}{\beta(1+\mu)}<1.$$

Hence $\frac{c+1}{c+n}<1$. Therefore (20) holds and the proof is complete. □

Next we provide a starlikeness condition for functions in $\mathcal{D}_T(\Phi,\Psi;\beta,\mu)$ under $V_\lambda(f)$.

**Theorem 3.2.** *Let $f\in\mathcal{D}_T(\Phi,\Psi;\beta,\mu)$. Then $V_\lambda(f)$ is starlike of order $0\le\tau<1$ in $|z|<R_1$, where*

$$R_1=\inf_n\left[\left(\frac{c+n}{c+1}\right)^\delta\frac{(1-\tau)[(1+\beta\mu)\Upsilon_n-(1-\beta)\gamma_n]}{(n-\tau)\beta(1+\mu)}\right]^{\frac{1}{n-1}}.$$

*Proof.* It is sufficient to prove

$$\left|\frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)}-1\right|<1-\tau. \tag{21}$$

For the left hand side of (21) we have

$$\left|\frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)}-1\right| = \left|\frac{\sum\limits_{n=2}^{\infty}(1-n)(\frac{c+1}{c+n})^{\delta}a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}(\frac{c+1}{c+n})^{\delta}a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)(\frac{c+1}{c+n})^{\delta}a_n |z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}(\frac{c+1}{c+n})^{\delta}a_n |z|^{n-1}}.$$

This last expression is less than $(1-\tau)$ since

$$|z|^{n-1} < \left(\frac{c+1}{c+n}\right)^{-\delta}\frac{(1-\tau)[(1+\beta\mu)\Upsilon_n-(1-\beta)\gamma_n]}{(n-\tau)\beta(1+\mu)}.$$

Therefore the proof is complete. □

Using the fact that $f$ is convex if and only if $zf'$ is starlike, we obtain the following theorem.

**Theorem 3.3.** *Let $f \in \mathcal{D}_T(\Phi,\Psi;\beta,\mu)$. Then $V_\lambda(f)$ is convex of order $0 \leq \tau < 1$ in $|z| < R_2$, where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta}\frac{(1-\tau)[(1+\beta\mu)\Upsilon_n-(1-\beta)\gamma_n]}{n(n-\tau)\beta(1+\mu)}\right]^{\frac{1}{n-1}}.$$

We omit the proof because it is easily derived.

Finally we have the following theorem.

**Theorem 3.4.** *Let $f \in \mathcal{D}_T(\Phi,\Psi;\beta,\mu)$. Then $V_\lambda(f)$ is close-to-convex of order $0 \leq \tau < 1$ in $|z| < R_3$, where*

$$R_3 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta}\frac{(1-\tau)[(1+\beta\mu)\Upsilon_n-(1-\beta)\gamma_n]}{n\beta(1+\mu)}\right]^{\frac{1}{n-1}}.$$

Again we omit the proof.

## Acknowledgments

The paper presented here is supported by IRPA 09-02-02-10029EAR Grant, Malaysia.

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