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CURVES OVER $\mathbb{F}_q$ AND LINEAR SERIES

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Abstract: Here we introduce a few problems concerning the existence of smooth curves over $\mathbb{F}_q$ (q as low as possible) equipped with certain linear series defined over $\mathbb{F}_q$.

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1. Curves over a Finite Field and Linear Series

Fix a prime p and a p-power q. Let X be a smooth and geometrically connected projective curve of genus $g \geq 2$ defined over $\mathbb{F}_q$. For any q-power q' let $\alpha(X, q')$ denote the minimal integer t such that there is a degree t morphism $X \to \mathbf{P}^1$ defined over $\mathbb{F}_q$. Hence $\alpha(q'') \geq \alpha(X, q')$ for all $q'' \geq q'$ and $\alpha(X, q')$ is the gonality $\mathrm{gon}(X)$ of $X(\bar{\mathbb{F}})_q$ when $q' \gg 0$. Since the canonical sheaf of any projective variety Y is defined over the field of definition of Y, we have $2 \leq \alpha(X, q') \leq 2g - 2$. If $\mathrm{gon}(X)$ is computed by a unique line bundle, then $\alpha(X, q) = \alpha(X, q') = \mathrm{gon}(X)$ for all q powers q' (Proposition 1). Hence $\alpha(X, q) = 2$ if X is hyperelliptic. If $\mathrm{gon}(X)$ is computed by exactly t linear systems, then $\alpha(X, q') = \mathrm{gon}(X)$ for all q powers q' such that $q' \geq q^t$ (Proposition 2). Let $\beta(X, q)$ be the minimal q-power q' such that $\alpha(X, q') = \mathrm{gon}(X)$. Now

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assume g even. It is well-known that a sufficiently general smooth genus g curve over $\bar{\mathbb{F}}_q$ has gonality $(g+2)/2$ and a finite number, $\eta(g) := g!/((g/2)!(g/2+1)!)$, of degree $(g+2)/2$ line bundles computing its gonality. The integer $\eta(g)$ does not depend from p. Let $\eta(g,q)$ be the minimal q-power q' such that there is a smooth and geometrically connected curve defined over $\mathbb{F}_q$, with gonality $(g+2)/2$, having exactly $\eta(g)$ degree $(g+2)/2$ line bundle computing its gonality, each of them defined over $\mathbb{F}_{q'}$; set $\eta(g,q) = +\infty$ if such curve does not exist. Let $\eta[g;p]$ denote the minimal p-power q such that $\eta(g,q) = q$, i.e. such that there is a smooth and geometrically connected curve defined over $\mathbb{F}_q$, with gonality $(g+2)/2$, having exactly $\eta(g)$ degree $(g+2)/2$ line bundle computing its gonality, each of them defined over $\mathbb{F}_q$. Concerning these invariants in this note we prove the following result.

Theorem 1. *Fix a prime power q and an odd integer $g \geq 3$ such that $q+1-2g\sqrt{q} > 0$. Set $\eta(g) := g!/((g/2)!(g/2+1)!)$. Then either $\eta(g,q) = +\infty$ or $\eta(g,q) = q^{\eta(g)}$.*

Remark 1. Let Y be any projective variety and F any coherent algebraic sheaf on Y, both defined over a field K. Since any field extension is flat, $\dim_K(H^0(Y,F)) = \dim_L(H^0(Y_L,F_L))$ for any field extension L of K ([2], III.9.3).

Proposition 1. *Assume that* $\mathrm{gon}(X)$ *is computed by a unique line bundle L. Then L is defined over $\mathbb{F}_q$ and hence $\alpha(X,q) = \alpha(X,q') = \mathrm{gon}(X)$ for all q-powers q' and $\beta(X,q) = q$.*

Proof. By Remark 1 it is sufficient to prove that L is defined over $\mathbb{F}_q$. Hence it is sufficient to prove that the degree $\mathrm{gon}(X)$ morphism $f : X \to \mathbf{P}^1$ induced by $|L|$ is defined over $\mathbb{F}_q$. This is true because its uniqueness implies that it is invariant for the action of the absolute Galois group of $\mathbb{F}_q$. □

Proposition 2. *Fix an integer $g \geq 2$. Let X be a smooth and geometrically connected genus g curve such that the gonality of $X(\bar{\mathbb{F}}_q)$ is computed by exactly t line bundles. Then all these line bundles are defined over $\mathbb{F}_{q^t}$ if either $q+1-2g\sqrt{q} > 0$ or $X(\mathbb{F}_q) \neq \emptyset$.*

Proof. Let $L_1,\dots,L_t$ denote the isomorphic classes over $\bar{\mathbb{F}}_q$ of the line bundles computing the gonality of $X(\bar{\mathbb{F}}_q)$. Notice that the set $W^1_d(X)$ is defined over $\mathbb{F}_q$. Hence the absolute Galois group $\widehat{Z}$ of $\mathbb{F}_q$ acts on the set $\{1,\dots,t\}$. Thus we get a group homomorphism $u : \widehat{Z} \to S_t$. Fix any $\sigma \in \mathrm{Ker}(u)$. Hence $\sigma^*(L_i) \cong L_i$ over $\bar{\mathbb{F}}_q$ for all i. We have $\sharp(X(\mathbb{F}_q)) \geq q+1-2g\sqrt{q} > 0$ (Hasse-Weil inequality). Thus each L_i is defined over the stabilzer $\mathbb{F}_{q'}$ of σ ([3], Example

1.17). Since this is true for every $\sigma \in \text{Ker}(u)$ and each element of S_t has order at most t, we get that each L_i is defined over $\mathbb{F}_{q^t}$. □

Proof of Theorem 1. The callical formula for the integer $\eta(g)$ is given in [1], p. 257. Assume $\eta(g,q) \ne +\infty$. Hence there is a smooth and geometrically connected curve X defined over $\mathbb{F}_q$ such that the curve $X_{\bar{\mathbb{F}}_q}$ has gonality $(g+2)/2$ and its gonality is computed over $\bar{\mathbb{F}}_q$ by exactly $\eta(g)$ linear series. Apply Proposition 2. □

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