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THE ARC–SINE LAW FOR THE FIRST INSTANT AT WHICH A DIFFUSION PROCESS EQUALS THE ULTIMATE VALUE OF A FUNCTIONAL

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Abstract: We study the distribution of the first instant θ at which a diffusion process equals the ultimate value of a functional, showing that θ follows a compound arc-sine law.

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1. Introduction

In this note, we generalize to a certain class of one-dimensional diffusion processes, the arc–sine law discovered by P. Levy for Brownian motion (BM) B_t. This classical result (see [8]) states that the first instant θ at which BM attains the maximum in an interval $[0, T]$, has the following distribution:

$$P(\theta \leq t) = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{T}}, \quad 0 < t < T\,. \tag{1.1}$$

Thus, θ possesses the density:

$$f_\theta(t) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{t(T-t)}}\,, & \text{if } 0 < t < T\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{1.1'}$$

Another Levy's arc–sine law states that the time spent by BM on the positive

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half-line during the time interval $[0,T]$, obeys the law (1.1), while the so-called intermediate arc–sine law holds for the sojourn time on $[0,+\infty)$ of conditional BM during the interval $[0,T]$ (see [9]).

Our aim is to show a compound arc–sine law for the first time $\theta \in [0,T]$ at which a diffusion $X(t)$ equals the ultimate value S_T of a functional of the process in the interval $[0,T]$, i.e. $X(\theta) = S_T$. This distribution can be used e.g. in nonparametric statistics. We consider two cases: in the first one, under certain assumptions which will be precised later, $X(t)$ is the solution of a time-homogeneous stochastic differential equation (SDE) of the form:

$$dX(t) = b(X(t))dt + \sigma(X(t))dB_t,\ X(0) = X_0 \tag{1.2}$$

and S_t is the maximum process of $X(t)$ i.e. $S_t = \max_{s\in[0,t]} X(s)$; in the second case $X(t)$ is BM and S_t is the time average of $X(t)$ in the interval $[0,t]$, i.e. $S_t = \frac{1}{t}\int_0^t B_s ds$. In both cases, since S_T is a random quantity whose value depends on the entire path of the process $X(t)$ over the interval $[0,T]$, its ultimate value is at any time $t \in [0,T)$, unknown; so θ is not a stopping-time.

Section 2 will be devoted to study the first case, while the second one will be treated in Section 3.

2. The Case of the Maximum Functional

Let $X(t) \in I \doteq (r_1, r_2)$ be the solution of the SDE:

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dB_t\,, \\ X(0) = X_0\,, \end{cases} \tag{2.1}$$

where B_t is (standard) BM. Throughout this section we will suppose that the following conditions are satisfied:

A1. $b, \sigma : I \to \mathbf{R}$ are continuous functions and a constant $K > 0$ exists, such that, for every $x, y \in I$:

$$|b(x) - b(y)| \le K|x-y|\,, \quad b^2(x) + \sigma^2(x) \le K(1+x^2)\,.$$

A2. σ is a non-negative, bounded function and it is differentiable for every x belonging to the interior of I. Moreover, there exists a strictly increasing function $g : \mathbf{R}^+ \to \mathbf{R}$ such that $g(0) = 0$, $\int_{0^+} g^{-2}(u)du = +\infty$ and $|\sigma(x) - \sigma(y)| \le g(|x-y|)$, for every $x, y \in I$.

Conditions A1 and A2 ensure that there exists a unique non-explosive solution of (2.1) (see e.g. [4], [5], [6]); A2 holds, for instance, if $\sigma(\cdot)$ is Lipschitz-continuous, or Hölder-continuous of order $\geq 1/2$.

For $x, y \in I = (r_1, r_2)$, let $\tau_y(x) \doteq \inf\{t > 0 : X(t) = y | X(0) = x\}$ be the first-hitting time of X to y when starting from x. We recall that the diffusion $X(t)$ is said to be *recurrent* if for any $x,\ y \in I$, it results $P(\tau_y(x) < \infty) = 1$, or equivalently (see e.g. [5], and [2]) if the probability of the process coming back to x infinitely often is one. A necessary condition for recurrence is (see e.g. [5]):

$$\liminf_{t\to\infty} X(t) = r_1 \quad \text{and} \quad \limsup_{t\to\infty} X(t) = r_2.$$

Let us consider now the infinitesimal generator L associated to the diffusion (2.1):

$$Lh(x) = b(x)h'(x) + \frac{1}{2}h''(x)\sigma^2(x), \quad h \in C^2(I)\,, \tag{2.2}$$

and let $u(x) \in C^2(I)$ be the solution of the problem:

$$\begin{cases} Lu(x) = 0, \quad x \in I\,, \\ u(0) = 0; \quad u'(0) = 1\,. \end{cases} \tag{2.3}$$

Setting, for $x \in I$:

$$\xi(x) = \exp\left(-\int_0^x \frac{2b(z)}{\sigma^2(z)}dz\right),$$

it is easily seen that the function $u(x)$ is given by:

$$u(x) = \int_0^x \xi(t)dt\,. \tag{2.4}$$

It is called the *scale* function. If the boundaries r_1 and r_2 of I are unattainable (see e.g. [4], [5]) (for $I = (-\infty, +\infty)$ this means that diffusion $X(t)$ does not explode), the recurrence of $X(t)$ is equivalent to the conditions (see [5]):

$$\lim_{x\to r_1} u(x) = -\infty, \quad \lim_{x\to r_2} u(x) = +\infty.$$

For instance, BM is recurrent, being in this case $u(x) = x$.

The function u given by (2.4) is strictly increasing, and the process $Y(t) \doteq u(X(t))$ is a local martingale; in fact, by Itô's formula it follows:

$$dY(t) = u'(u^{-1}(Y(t)))\sigma(u^{-1}(Y(t)))dB_t\,. \tag{2.5}$$

We denote by

$$\langle Y \rangle_t = \int_0^t [u'(X(s))\sigma(X(s))]^2 ds \tag{2.6}$$

the quadratic variation of the process $Y(t)$.

Finally, we say that the diffusion process $X(t) \in I$, which is the solution of the SDE (2.1), is conjugated to BM if there exists an increasing differentiable function $v : I \longrightarrow \mathbf{R}$ with $v(0) = 0$, such that the process $Z(t) \doteq v(X(t))$ is BM.

Notice that, if $X(t)$ is conjugated to BM, then X is recurrent.

For given $T > 0$, let $S_T = \max_{t \in [0,T]} X(t)$ and let θ be the first instant in the interval $[0, T]$ at which $X(t)$ equals the ultimate value S_T of the functional in the interval $[0, T]$, i.e. $X(\theta) = S_T$. The following holds.

Theorem 2.1. *Assuming that the solution $X(t)$ of (2.1) is recurrent, let us suppose that the quadratic variation $\rho(t) \doteq \langle Y \rangle_t$ of the local martingale $Y(t)$ associated to $X(t)$, is deterministic and verifies*

$$\rho(\infty) = \infty\,. \tag{2.7}$$

Then:

$$P(\theta \le t) = \frac{2}{\pi} \arcsin \sqrt{\frac{\rho(t)}{\rho(T)}}, \quad t \in [0, T]\,, \tag{2.8}$$

and the density of θ is:

$$f_\theta(t) = \begin{cases} \frac{1}{\pi} \frac{\rho'(t)}{\sqrt{\rho(t)(\rho(T) - \rho(t))}}\,, & \text{if } 0 < t < T\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.8$'$}$$

Thus, θ follows a compound arc–sine law.

Proof. Thanking to (2.7), by using a random time–change (see e.g. [10], Theorem 1.6, p. 170), we obtain that there exists a Wiener process $\tilde{B}_t$ such that a.s.

$$Y(t) = Y_0 + \tilde{B}_{\langle Y \rangle_t} = Y_0 + \tilde{B}_{\rho(t)}\,, \tag{2.9}$$

where $Y_0 = u(X_0)$. Thus, by using our notations:

$$u(S_T) = \max_{t \in [0,T]} u(X(t)) = Y_0 + \max_{t \in [0,T]} \tilde{B}_{\rho(t)} = Y_0 + \max_{s \in [0, \rho(T)]} \tilde{B}_s\,.$$

Now, from $X(\theta) = S_T$ it follows that $Y(\theta) = u(X(\theta)) = u(S_T)$ and then:

$$Y(\theta) = Y_0 + \tilde{B}_{\rho(\theta)} = u(X(\theta)) = Y_0 + \max_{s \in [0, \rho(T)]} \tilde{B}_s\,, \tag{2.10}$$

i.e. $\tilde{B}_{\rho(\theta)} = \max_{t \in [0, \rho(T)]} \tilde{B}_t$. Therefore, by Levy's result (1.1) on BM, $\rho(\theta)$ follows the arc–sine law:

$$P(\rho(\theta) \le t) = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{\rho(T)}}, \quad t \in (0, \rho(T))\,. \tag{2.11}$$

Since $P(\theta \le t) = P(\rho(\theta) \le \rho(t))$, (2.8) follows from (2.11). $\square$

In particular, if $X(t)$ is conjugated to BM, then $\rho(t) = t$ and so θ follows the arc–sine law.

Let us consider now the case when $\rho(t)$ is not deterministic; we obtain:

Theorem 2.2. *Under the assumptions of Theorem 2.1, we still assume that $\rho(\infty) = \infty$, but we suppose that the quadratic variation $\rho(t)$ of $Y(t)$ is non deterministic, and there exist two deterministic, continuous increasing functions $\alpha(t)$ and $\beta(t)$, with $\alpha(0) = \beta(0) = 0$, such that for every $t < T$:*

$$\alpha(t) \le \langle Y \rangle_t \le \beta(t) \tag{2.12}$$

Then, for $0 < t < \beta^{-1}(\alpha(T))$ it holds:

$$\frac{2}{\pi} \arcsin \sqrt{\frac{\alpha(t)}{\beta(T)}} \le P(\theta \le t) \le \frac{2}{\pi} \arcsin \sqrt{\frac{\beta(t)}{\alpha(T)}} \, . \tag{2.13}$$

Proof. Since $\alpha(t) \le \rho(t) \le \beta(t)$, it results:

$$Y_0 + \max_{t \in [0, \alpha(T)]} \tilde{B}_t \le u(S_T) \le Y_0 + \max_{t \in [0, \beta(T)]} \tilde{B}_t \tag{2.14}$$

If we denote by $\tilde{\theta}_\alpha$ and $\tilde{\theta}_\beta$, the first instant at which $\tilde{B}_t$ attains its maximum in the interval $[0, \alpha(T)]$ and in the interval $[0, \beta(T)]$, respectively, we obtain:

$$\tilde{\theta}_\alpha \le \rho(\theta) \le \tilde{\theta}_\beta \, . \tag{2.15}$$

Thus:

$$\frac{2}{\pi} \arcsin \sqrt{\frac{t}{\beta(T)}} \le P(\rho(\theta) \le t) \le \frac{2}{\pi} \arcsin \sqrt{\frac{t}{\alpha(T)}}, \quad t \in (0, \alpha(T)) \, . \tag{2.16}$$

Since $P(\theta \le t) = P(\rho(\theta) \le \rho(t))$, (2.13) easily follows from (2.16).

In particular, $E(\tilde{\theta}_\alpha) \le E(\rho(\theta)) \le E(\tilde{\theta}_\beta)$ and so $\alpha(T)/2 \le E(\rho(\theta)) \le \beta(T)/2$. $\square$

We report below a few examples of diffusion processes for which our results hold. Other examples can be derived by those contained in [3].

Example 2.1. (Feller Process) Let us consider the process $X(t) \in [0, +\infty)$ which is the solution of the SDE:

$$dX(t) = \frac{1}{4} dt + \sqrt{X(t) \vee 0} \; dB_t, \; X(0) \ge 0 \tag{2.17}$$

(note that, although $\sqrt{x}$ is not Lipschitz-continuous, the solution is unique because $\sqrt{x}$ is Hölder-continuous (see e.g. condition A2)). As easily seen, the process $X(t)$ is conjugated to Brownian motion by means of the function

$v(x) = 2\sqrt{x}$ i.e. $v(X(t)) = 2\sqrt{X(t)} \equiv B_t$. Therefore, θ follows the arc–sine law.

Example 2.2. (Wright and Fisher-Like Process) Let $X(t)$ be the solution of the SDE:

$$dX(t) = \left(\frac{1}{4} - \frac{1}{2}X(t)\right) dt + \sqrt{X(t)(1 - X(t)) \vee 0}\; dB_t,$$
$$X(0) \in [0,1]\,. \quad (2.18)$$

It is a particular case of the Wright and Fisher diffusion equation for population genetics, and it is also used in certain diffusion models for neural activity; it can be shown (see e.g. [1]) that $X(t)$ remains in the interval $[0,1]$ for every time $t \geq 0$. As it is easy to see, $X(t)$ is conjugated to BM by means of the function $v(x) = 2\arcsin\sqrt{x}$, i.e. $v(X(t)) = 2\arcsin\sqrt{X(t)} \equiv B_t$. Thus, θ follows the arc–sine law.

Example 2.3. (Integral Process with Deterministic Integrand) Let $X(t)$ be the solution of the SDE:

$$dX(t) = \mu(t) dB_t, \quad X(0) = X_0\,,$$

where $\mu(t) \geq 0$ is a (deterministic) bounded continuous function; then the quadratic variation of X, $\langle X \rangle_t = \rho(t) = \int_0^t \mu^2(s)ds$, is deterministic and so, from (2.8)

$$P(\theta \leq t) = \frac{2}{\pi}\arcsin\sqrt{\frac{\int_0^t \mu^2(s)ds}{\int_0^T \mu^2(s)ds}}\,.$$

Example 2.4. (A Temporally Non-Homogeneous SDE) Let $Z(t)$ be the solution of the SDE:

$$\begin{cases} dZ(t) = -\frac{Z(t)}{1-t}dt + dB_t\,, & 0 \leq t \leq 1\,, \\ Z(0) = Z(1) = 0\,. \end{cases} \quad (2.19)$$

The diffusion $Z(t)$ is the Brownian bridge, i.e. BM conditioned to take the value 0 at time $t = 1$. The explicit solution of (2.19) is:

$$Z(t) = (1-t)\int_0^t \frac{1}{1-s} dB_s\,. \quad (2.20)$$

Set

$$X(t) = \frac{Z(t)}{1-t}\,,\ 0 \leq t \leq 1\,. \quad (2.21)$$

The diffusion $X(t)$ turns out to be a local martingale with deterministic quadratic variation $\langle X \rangle_t = \rho(t) = \frac{t}{1-t}$, $0 \leq t \leq 1$. So, by a random time–change it results

$X(t) = \tilde{B}(\frac{t}{1-t})$, for a suitable BM $\tilde{B}$.

The first instant θ at which $X(t)$ attains its maximum has distribution (from (1.1)):

$$P(\theta \leq t) = \frac{2}{\pi} \arcsin \sqrt{\frac{t(1-T)}{T(1-t)}} \ , \ 0 \leq t < T < 1 \, . \tag{2.22}$$

3. The Case of the Integral Functional

Throughout this section S_t is the integral functional of BM, i.e. $S_t = \frac{1}{t}\int_0^t B_s ds$ is the time average of the process B_s in the interval $[0,t]$. Let $T > 0$, we look for the the first instant $\theta \in [0,T]$ such that $B_\theta = S_T$. First, we state the following representation formula.

Lemma 3.1. *Let $X(t)$ be the solution of the stochastic differential equation:*

$$dX(t) = \mu(t) dB_t, \quad X(0) = X_0 \, , \tag{3.1}$$

where $\mu(t) \geq 0$ is a (deterministic) bounded continuous function. Then:

$$\int_0^T X(s)ds = TX_0 + \int_0^T \mu(s)\,(T-s)\,dB_s \, . \tag{3.2}$$

Moreover, if $\int_0^{+\infty} \mu^2(s)ds = +\infty$, there exists a BM W_t such that:

$$\int_0^T X(s)ds = W_{a_T} \, , \tag{3.3}$$

where $a_T = \int_0^T (\mu(s)(T-s))^2\, ds$. In particular, if $X(t) = B_t$ (i.e. $\mu(s) \equiv 1$) it holds $a_T = T^3/3$, and then:

$$\int_0^T B_s ds = \int_0^T (T-s)\,dB_s = W_{T^3/3} \, . \tag{3.2'}$$

Proof. By Itô's formula, we get:

$$\int_0^T X(s)ds = TX(T) - \int_0^T s\mu(s)dB_s \, . \tag{3.4}$$

By using that:

$$X(T) = X_0 + \int_0^T \mu(s)dB_s$$

and substituting in (3.4), we easily obtain (3.2).

Now, let us consider the martingale $M(t)$ having stochastic differential $dM(t) = \mu(t)(T-t)dB_t$ and such that $M(0) = TX_0$; the quadratic variation of $M(t)$ is:

$$\langle M \rangle_t = \int_0^t (\mu(s)(T-s))^2 \, ds = a_t \, .$$

It results $\langle M \rangle_\infty = \infty$. Indeed:

$$\int_0^{+\infty} (\mu(s)(T-s))^2 \, ds > \int_{T+1}^{+\infty} (\mu(s)(T-s))^2 \, ds$$
$$> \int_{T+1}^{+\infty} \mu^2(t)dt = \int_0^{+\infty} \mu^2(t)dt - \int_0^{T+1} \mu^2(t)dt = \infty,$$

by the assumption.

Let $\tau_t = \inf\{s : \langle M \rangle_s > t\}$; since $\langle M \rangle_\infty = \infty$, we have that $M(\tau_t) \doteq W_t$ is BM (see e.g. [10]) and $M(t)$ is obtained by a random time–change as $M(t) = W_{a_t}, \ t > 0$. In conclusion, we have obtained:

$$\int_0^T X(s)ds = M(T) = W_{a_T}$$

and (3.3) follows. □

The main result of this section is the following result.

Theorem 3.1. *The first instant $\theta \in [0,T]$ at which $B_\theta = S_T$ has distribution:*

$$P(\theta \le t) = \begin{cases} \dfrac{2}{\pi} \arccos \sqrt{(T-t)^3 - t^3}\,, & \textit{if } 0 \le t \le T/2\,, \\ 1\,, & t > T/2\,, \end{cases} \tag{3.5}$$

i.e. the density of θ is:

$$f_\theta(t) = \begin{cases} \dfrac{3}{\pi} \dfrac{1}{\sqrt{2t^3 - 3t^2 + 3t}} \dfrac{1 - 2t + 2t^2}{\sqrt{(1-t)^3 - t^3}}\,, & \textit{if } \ 0 \le t \le \frac{1}{2}\,, \\ 0\,, & \textit{otherwise}\,. \end{cases} \tag{3.6}$$

Proof. We have

$$\theta = \inf \left\{ t > 0 : \frac{1}{T} \left(\int_0^T B_s ds - TB_t \right) = 0 \right\},$$

that is θ is the first zero of the equation in the unknown t:

$$\int_0^T (B_s - B_t) ds = 0\,. \tag{3.7}$$

Equation (3.7) can be rewritten as:

$$\int_0^t (B_s - B_t)ds + \int_t^T (B_s - B_t)ds = 0\,,$$

i.e.

$$-\int_0^t B_{t-s}\ ds + \int_t^T B_{s-t}\ ds = 0\,.$$

Therefore, by changing variable, θ results to be the first t for which

$$\int_0^{T-t} B_s ds - \int_0^t B_s ds = 0\,,$$

or equivalently, by using (3.2′), the first zero of the equation

$$W_{r(t)} = 0\,, \tag{3.8}$$

where

$$r(t) = \frac{(T-t)^3}{3} - \frac{t^3}{3}\,. \tag{3.9}$$

Now, let $\alpha \in \left[0, \frac{T}{2}\right]$. Since $r(t) \geq 0$ for $0 \leq t \leq \frac{T}{2}$, we have:

$$P\{\theta > \alpha\} = P\{W_{r(t)} \neq 0,\ \forall t \in (0,\alpha)\} = P\left\{W_s \neq 0,\ \forall s \in \left(r(\alpha), \frac{1}{3}\right)\right\}.$$

This is the probability that BM does not vanish in the interval $(r(\alpha), \frac{1}{3})$; thus, by using e.g. Theorem 3.25 of [7], we conclude that, for $0 \leq \alpha \leq T/2$:

$$P\{\theta > \alpha\} = \frac{2}{\pi} \arcsin \sqrt{\frac{r(\alpha)}{1/3}} = \frac{2}{\pi} \arcsin \sqrt{(T-\alpha)^3 - \alpha^3}\,,$$

and so

$$P\{\theta \leq \alpha\} = \frac{2}{\pi} \arccos \sqrt{(T-\alpha)^3 - \alpha^3}\,. \tag{3.10}$$

In particular, putting $\alpha = T/2$, we get $P(\theta \leq T/2) = 1$ which means that the random variable θ is concentrated on the interval $(0, T/2)$. By taking the derivative in (3.10), formula (3.6) for the density of θ is obtained and the proof is concluded. □

Remark. We are not able to generalize Theorem 3.1 to a process $X(t)$ having stochastic differential $dX(t) = \mu(t)dB_t$. Although the process itself and $\int_0^t X(s)ds$ can be expressed in terms of two suitable Wiener processes, the arguments used in the proof of Theorem 3.1 cannot be reproduced, since the Wiener processes above result to be different (they are the same only in the case when $\mu(t) = \mu$ is constant, i.e. $X(t)$ is a Brownian motion with variance

$\mu^2 t)$.

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