International Journal of Pure and Applied Mathematics
Volume 30 No. 1 2006, 33-41

PRIMITIVE LIFTING OF SOME ELEMENTS IN FREE NILPOTENT LIE ALGEBRAS

Zeynep Özkurt
Department of Mathematics
Çukurova University
Adana, 01330, TURKEY
e-mail: zyapti@mail.cu.edu.tr

Abstract: Let F_n be a free Lie algebra of rank n and $\gamma_c(F_n)$ the c-th term of the lower central series of F_n. We prove that for each $1 \leq i, k \leq n$ and $m \geq 1$, every element of $F_n/\gamma_c(F_n)$ of the form $x_i + [x_i, x_k^m]$ can be lifted to a primitive element of F_n.

AMS Subject Classification: 17B01, 17B40
Key Words: primitive lifting, automorphism

1. Introduction

Let F_n be a free Lie algebra with the free generating set $X = \{x_1, ..., x_n\}$ over a field K of characteristic 0. We identify any free nilpotent Lie algebra of rank $n \geq 2$ and class k with $F_n/\gamma_k(F_n)$, where $\gamma_k(F_n)$ is the k-th term of the lower central series of F_n. Denote by F_n' the commutator subalgebra $[F_n, F_n]$ of F_n. An element of a free Lie algebra is said to be primitive if it can be included in some set of free generators of this algebra, see [3]. Lifting primitivity of free Lie algebras is a natural problem which is linked to the study of the automorphism group $AutF_n$. Cohn [2] proved that the automorphism group $AutF_n$ of the algebra F_n is generated by the following automorphisms: (i) automorphisms which are induced by the action of the general linear group GL_n $(= GL_n(K))$ on the subspace of F_n spanned by $\{x_1, ..., x_n\}$; (ii) automorphisms of the form $x_1 \to x_1 + f(x_2, ..., x_n)$, $x_k \to x_k$, $k \neq 1$, where the polynomial $f(x_2, ..., x_n)$

Received: May 27, 2006

does not depend on x_1. An automorphism of $F_n/\gamma_k(F_n)$ is called tame if it can be induced by an automorphism of F_n. The question of lifting automorphisms are naturally related to the problem of finding appropriate necessary and (or) sufficient condition(s) for an endomorphism of the free Lie algebra F_n to be an automorphism (see [1]). An element $[[...[[a,b],b],...],b]$ of F_n will be denoted by $\left[a,b^k\right]$.

In this paper, we give a series of elements of $F_n/\gamma_c(F_n)$ $(c \geq 3)$ which can be lifted to primitive elements of F_n. Here our main result is the following theorem.

Theorem. *Every element of $F_n/\gamma_{m+2}(F_n)$ of the form $x_i + [x_i, x_k^m]$ can be lifted to a primitive element of F_n, where $1 \leq i,k \leq n$ and $m \geq 1$.*

2. Lifting Primitivity of a Single Element of $F_n/\gamma_c(F_n)$

Let F_n be the free Lie algebra generated by the set $X = \{x_1, ..., x_n\}$.

Lemma 2.1. *Every endomorphism of F_n of the form*

$$\begin{aligned} \varphi_{ijk} &: & x_i &\to x_i + [x_i, x_j] + v\,, \\ x_k &\to & x_k &+ u\,, \\ x_l &\to & x_l&, \quad l \neq i,k\,, \end{aligned}$$

is an automorphism where $v \in \gamma_3(F_n)$, $u \in F_n'$ and $1 \leq i,j,k \leq n$.

Proof. Consider endomorphisms θ_{kj}, θ_{ki} of F_n given as

$$\begin{aligned} &\theta_{kj} : x_i \to x_i + [x_k, x_j]\,, \quad x_l \to x_l, \quad l \neq i\,, \\ &\theta_{ki} : x_k \to x_k + x_i, \quad x_l \to x_l, \ l \neq k. \end{aligned}$$

Then clearly θ_{kj}, θ_{ki} are automorphisms of F_n. Set $\gamma = \theta_{ki}\theta_{kj}\theta_{ki}^{-1}\theta_{kj}^{-1}$.

Since

$$\begin{aligned} &\theta_{ki}\theta_{kj}\theta_{ki}^{-1}\theta_{kj}^{-1}(x_i) \\ &\quad = \theta_{ki}\theta_{kj}\theta_{ki}^{-1}(x_i - [x_k, x_j]) = \theta_{ki}\theta_{kj}(x_i - [x_k, x_j] + [x_i, x_j]) \\ &\quad\quad = \theta_{ki}(x_i - [x_k, x_j]) = x_i + [x_i, x_j] + [[x_k, x_j], x_j] + [[x_i, x_j], x_j] \end{aligned}$$

and

$$\begin{aligned} &\theta_{ki}\theta_{kj}\theta_{ki}^{-1}\theta_{kj}^{-1}(x_k) = \theta_{ki}\theta_{kj}\theta_{ki}^{-1}(x_k) = \theta_{ki}\theta_{kj}(x_k - x_i) \\ &\quad\quad = \theta_{ki}(x_k - x_i - [x_k, x_j]) = x_k - [x_k, x_j] - [x_i, x_j]\,. \end{aligned}$$

Then $\gamma = \theta_{ki}\theta_{kj}\theta_{ki}^{-1}\theta_{kj}^{-1}$ is of the required form:

$$\begin{array}{rcl} \gamma & : & x_i \to x_i + [x_i, x_j] + [[x_k, x_j], x_j] + [[x_i, x_j], x_j], \\ x_k & \to & x_k - [x_k, x_j] - [x_i, x_j], \\ x_l & \to & x_l, \quad l \neq i, k, \end{array}$$

where $1 \leq i, j, k \leq n$. □

Lemma 2.2. *Every endomorphism of F_n of the form*

$$\begin{array}{rcl} \gamma_{ij} & : & x_i \to x_i + [x_i, x_k^m] + v, \\ x_j & \to & x_j - [x_j, x_k^m] + u, \\ x_l & \to & x_l, \quad l \neq i, j \end{array}$$

is an automorphism, where $v,\ u \in \gamma_{m+2}(F_n)$ and $1 \leq i, j, k \leq n$, $m \geq 1$.

Proof. The case $m = 1$ is obtained by Lemma 1.

Let $m = 2t + 1$, $t = 1, 2, \ldots.$ Consider endomorphisms $\theta_{ijk}, \theta_{jik}$ of F_n given as

$$\theta_{ijk} : x_i \to x_i + \left[x_j, x_k^{m-n}\right], \quad x_l \to x_l, \quad l \neq i,$$

$$\theta_{jik} : x_j \to x_j + [x_i, x_k^n], \quad x_l \to x_l, \quad l \neq j.$$

Then clearly $\theta_{ijk}, \theta_{jik}$ are automorphisms of F_n. Set $\gamma_{ij} = \theta_{ijk}\theta_{jik}\ \theta_{ijk}^{-1}\theta_{jik}^{-1}$. Since

$$\begin{aligned} &\theta_{jik}\theta_{ijk}\theta_{jik}^{-1}\theta_{ijk}^{-1}(x_i) \\ &= \theta_{jik}\theta_{ijk}\theta_{jik}^{-1}(x_i - \left[x_j, x_k^{m-n}\right]) = \theta_{jik}\theta_{ijk}(x_i - \left[x_j, x_k^{m-n}\right] + [x_i, x_k^m]) \\ &= \theta_{jik}(x_i + [x_i, x_k^m] + \left[\left[x_j, x_k^{m-n}\right], x_k^m\right]) \\ &= x_i + [x_i, x_k^m] + \left[x_j, x_k^{2m-n}\right] + \left[x_i, x_k^{2m}\right] \end{aligned}$$

and

$$\begin{array}{rcl} \theta_{jik}\theta_{ijk}\theta_{jik}^{-1}\theta_{ijk}^{-1}(x_j) & = & \theta_{jik}\theta_{ijk}\theta_{jik}^{-1}(x_j) = \theta_{jik}\theta_{ijk}(x_j - [x_i, x_k^n]) \\ & = & \theta_{jik}(x_j - [x_i, x_k^n] - [x_j, x_k^m]) \\ & = & x_j - [x_j, x_k^m] - \left[x_i, x_k^{m+n}\right]. \end{array}$$

Then $\gamma_{ij} = \theta_{jik}\theta_{ijk}\theta_{jik}^{-1}\theta_{ijk}^{-1}$ is of the required form:

$$\begin{array}{rcl} \gamma_{ij} & : & x_i \to x_i + [x_i, x_k^m] + \left[x_j, x_k^{2m-n}\right] + \left[x_i, x_k^{2m}\right], \\ x_j & \to & x_j - [x_j, x_k^m] - \left[x_i, x_k^{m+n}\right], \end{array}$$

$$x_l \to x_l, \quad l \neq i, j.$$

Let $m = 2t$, $t = 1, 2, ...$ Consider endomorphisms $\theta_{ijk}, \theta_{jik}$ of F_n given as

$$\theta_{ijk} : x_i \to x_i + \left[x_j, x_k^{\frac{m}{2}}\right], \quad x_l \to x_l, \ l \neq i,$$
$$\theta_{jik} : x_j \to x_j + \left[x_i, x_k^{\frac{m}{2}}\right], \quad x_l \to x_l, \ l \neq j.$$

Then clearly $\theta_{ijk}, \theta_{jik}$ are automorphisms of F_n. Set $\gamma_{ij} = \theta_{ijk}\theta_{jik}\theta_{ijk}^{-1}\theta_{jik}^{-1}$. Since

$$\begin{aligned}
\theta_{ijk}\theta_{jik}\theta_{ijk}^{-1}\theta_{jik}^{-1}(x_i) &= \theta_{ijk}\theta_{jik}\theta_{ijk}^{-1}(x_i) = \theta_{ijk}\theta_{jik}(x_i - \left[x_j, x_k^{\frac{m}{2}}\right]) \\
&= \theta_{ijk}(x_i - \left[x_j, x_k^{\frac{m}{2}}\right] - [x_i, x_k^m]) \\
&= x_i - [x_i, x_k^m] - \left[x_j, x_k^{\frac{3m}{2}}\right]
\end{aligned}$$

and

$$\begin{aligned}
\theta_{ijk}\theta_{jik}\theta_{ijk}^{-1}\theta_{jik}^{-1}(x_j) &= \theta_{ijk}\theta_{jik}\theta_{ijk}^{-1}(x_j - \left[x_i, x_k^{\frac{m}{2}}\right]) \\
&= \theta_{ijk}\theta_{jik}(x_j - \left[x_i, x_k^{\frac{m}{2}}\right] + [x_j, x_k^m]) \\
&= \theta_{ijk}(x_j + [x_j, x_k^m] - \left[x_i, x_k^{\frac{3m}{2}}\right]) \\
&= x_j + [x_j, x_k^m] - \left[x_i, x_k^{\frac{3m}{2}}\right] - \left[x_j, x_k^{2m}\right],
\end{aligned}$$

$\gamma_{ij} = \theta_{ijk}\theta_{jik}\theta_{ijk}^{-1}\theta_{jik}^{-1}$ is of the required form:

$$\begin{aligned}
&\gamma_{ij} : \ x_i \to x_i - [x_i, x_k^m] - \left[x_j, x_k^{\frac{3m}{2}}\right], \\
&x_j \to x_j + [x_j, x_k^m] - \left[x_i, x_k^{\frac{3m}{2}}\right] - \left[x_j, x_k^{2m}\right], \\
&x_l \to x_l, \quad l \neq i, j.
\end{aligned}$$ □

As a corollary to Lemma 1 and Lemma 2 we obtain the following main result.

Theorem. *For each $1 \leq i, j \leq n$ and $m \geq 1$, every element of $F_n/\gamma_{m+2}(F_n)$ of the form $x_i + [x_i, x_k^m]$ can be lifted to a primitive element of F_n.*

Proof. By Lemma 1 and Lemma 2 every element of $F_n/\gamma_{m+2}(F_n)$ of the form $x_i + [x_i, x_k^m] + v$ is primitif element of F_n, where $v \in \gamma_{m+2}(F_n)$. □

3. Some Examples

In this section we costruct a series of elements of $F_3/\gamma_3(F_3)$ which can be lifted to primitive elements of F_3.

1) The endomorphism of F_3 of the form

$$\begin{array}{rcl} \varphi_{123} & : & x_1 \to x_1 + [x_1, x_2] + v\,, \\ x_2 & \to & x_2\,, \\ x_3 & \to & x_3 + u\,, \end{array}$$

is an automorphism, where $v \in \gamma_3(F_3)$, $u \in F_3'$.

Consider endomorphisms θ_{32}, θ_{31} of F_3 given as

$$\theta_{32} : x_1 \to x_1 + [x_3, x_2]\,, \quad x_i \to x_i, \quad i \neq 1,$$

$$\theta_{31} : x_3 \to x_3 + x_1, \quad x_i \to x_i, \quad i \neq 3.$$

Then clearly θ_{32}, θ_{31} are automorphisms of F_3. Set $\gamma = \theta_{31}\theta_{32}\theta_{31}^{-1}\theta_{32}^{-1}$. Since

$$\begin{aligned} &\theta_{31}\theta_{32}\theta_{31}^{-1}\theta_{32}^{-1}(x_1) = \theta_{31}\theta_{32}\theta_{31}^{-1}(x_1 - [x_3, x_2]) \\ &\quad = \theta_{31}\theta_{32}(x_1 - [x_3, x_2] + [x_1, x_2]) = \theta_{31}(x_1 + [x_1, x_2] + [[x_3, x_2]\,, x_2]) \\ &\qquad\qquad = x_1 + [x_1, x_2] + [[x_3, x_2]\,, x_2] + [[x_1, x_2]\,, x_2] \end{aligned}$$

and

$$\begin{array}{rcl} \theta_{31}\theta_{32}\theta_{31}^{-1}\theta_{32}^{-1}(x_3) & = & \theta_{31}\theta_{32}\theta_{31}^{-1}(x_3) = \theta_{31}\theta_{32}(x_3 - x_1) \\ & = & \theta_{31}(x_3 - x_1 - [x_3, x_2]) \\ & = & x_3 - [x_3, x_2] - [x_1, x_2]\,. \end{array}$$

Then $\gamma = \theta_{31}\theta_{32}\theta_{31}^{-1}\theta_{32}^{-1}$ is of the required form:

$$\begin{array}{rcl} \gamma & : & x_1 \to x_1 + [x_1, x_2] + [[x_3, x_2]\,, x_2] + [[x_1, x_2]\,, x_2]\,, \\ x_2 & \to & x_2\,, \\ x_3 & \to & x_3 - [x_3, x_2] - [x_1, x_2]\,. \end{array}$$

2) The endomorphism of F_3 of the form

$$\begin{array}{rcl} \varphi_{132} & : & x_1 \to x_1 + [x_1, x_3] + v\,, \\ x_2 & \to & x_2 + u\,, \\ x_3 & \to & x_3\,, \end{array}$$

is an automorphism, where $v \in \gamma_3(F_3)$, $u \in F_3$.

Let

$$\theta_{23} : x_1 \to x_1 + [x_2, x_3]\,, \quad x_i \to x_i, \quad i \neq 1\,,$$

and

$$\theta_{21} : x_2 \to x_2 + x_1, \quad x_i \to x_i, \quad i \neq 2.$$

Then clearly θ_{23}, θ_{21} are automorphisms of F_3. Set $\gamma = \theta_{21}\theta_{23}\theta_{21}^{-1}\theta_{23}^{-1}$. Since

$$\begin{aligned}
\theta_{21}\theta_{23}\theta_{21}^{-1}\theta_{23}^{-1}(x_1) &= \theta_{21}\theta_{23}\theta_{21}^{-1}(x_1 - [x_2, x_3]) \\
&= \theta_{21}\theta_{23}(x_1 - [x_2, x_3] + [x_1, x_3]) = \theta_{21}(x_1 + [x_1, x_3] + [[x_2, x_3]\,, x_3]) \\
&= x_1 + [x_1, x_3] + [[x_2, x_3]\,, x_3] + [[x_1, x_3]\,, x_3]\ ,
\end{aligned}$$

and

$$\begin{aligned}
\theta_{21}\theta_{23}\theta_{21}^{-1}\theta_{23}^{-1}(x_2) &= \theta_{21}\theta_{23}\theta_{21}^{-1}(x_2) = \theta_{21}\theta_{23}(x_2 - x_1) \\
&= \theta_{21}(x_2 - x_1 - [x_2, x_3]) = x_2 - [x_2, x_3] - [x_1, x_3]\,.
\end{aligned}$$

Then $\gamma = \theta_{21}\theta_{23}\theta_{21}^{-1}\theta_{23}^{-1}$ is of the required form:

$$\begin{aligned}
\gamma \quad : \quad & x_1 \to x_1 + [x_1, x_3] + [[x_2, x_3]\,, x_3] + [[x_1, x_3]\,, x_3]\,, \\
x_2 \quad \to \quad & x_2 - [x_2, x_3] - [x_1, x_3]\,, \\
x_3 \quad \to \quad & x_3.
\end{aligned}$$

3) The endomorphism of F_3 of the form

$$\begin{aligned}
\varphi_{213} \quad : \quad & x_1 \to x_1\,, \\
x_2 \quad \to \quad & x_2 + [x_2, x_1] + v\,, \\
x_3 \quad \to \quad & x_3 + u\,,
\end{aligned}$$

is an automorphism, where $v \in \gamma_3(F_3)$, $u \in F_3$.

Let θ_{31}, θ_{32} of F_3 be the automorphisms of F_3 defined as

$$\theta_{31} : x_2 \to x_2 + [x_3, x_1]\,, \quad x_i \to x_i, \quad i \neq 2,$$

$$\theta_{32} : x_3 \to x_3 + x_2, \quad x_i \to x_i, \quad i \neq 3.$$

Set $\gamma = \theta_{31}\theta_{32}\theta_{31}^{-1}\theta_{32}^{-1}$. Since

$$\begin{aligned}
\theta_{32}\theta_{31}\theta_{32}^{-1}\theta_{31}^{-1}(x_2) &= \theta_{32}\theta_{31}\theta_{32}^{-1}(x_2 - [x_3, x_1]) \\
&= \theta_{32}\theta_{31}(x_2 - [x_3, x_1] + [x_2, x_1]) = \theta_{32}(x_2 + [x_2, x_1] + [[x_3, x_1]\,, x_1])
\end{aligned}$$

$$= x_2 + [x_2, x_1] + [[x_3, x_1], x_1] + [[x_2, x_1], x_1] \ ,$$

and

$$\theta_{32}\theta_{31}\theta_{32}^{-1}\theta_{31}^{-1}(x_3) = \theta_{32}\theta_{31}\theta_{32}^{-1}(x_3) = \theta_{32}\theta_{31}(x_3 - x_2)$$
$$= \theta_{32}(x_3 - x_2 - [x_3, x_1]) = x_3 - [x_3, x_1] - [x_2, x_1] \, .$$

Then $\gamma = \theta_{21}\theta_{23}\theta_{21}^{-1}\theta_{23}^{-1}$ is of the required form:

$$\begin{array}{rcl} \gamma & : & x_1 \to x_1, \\ x_2 & \to & x_2 + [x_2, x_1] + [[x_3, x_1], x_1] + [[x_2, x_1], x_1] \, , \\ x_3 & \to & x_3 - [x_3, x_1] - [x_2, x_1] \, . \end{array}$$

4) The endomorphism of F_3 of the form

$$\begin{array}{rcl} \varphi_{231} & : & x_1 \to x_1 + u \, , \\ x_2 & \to & x_2 + [x_2, x_3] + v \, , \\ x_3 & \to & x_3 \, , \end{array}$$

is an automorphism, where $v \in \gamma_3(F_3)$, $u \in F_3$.

Consider the automorphisms θ_{13}, θ_{12} of F_3 given as

$$\theta_{13} : x_2 \to x_2 + [x_1, x_3] \, , \quad x_i \to x_i, \quad i \neq 2,$$

$$\theta_{12} : x_1 \to x_1 + x_2, \quad x_i \to x_i, \quad i \neq 1.$$

Set $\gamma = \theta_{12}\theta_{13}\theta_{12}^{-1}\theta_{13}^{-1}$. Hence

$$\theta_{12}\theta_{13}\theta_{12}^{-1}\theta_{13}^{-1}(x_2) = \theta_{12}\theta_{13}\theta_{12}^{-1}(x_2 - [x_1, x_3])$$
$$= \theta_{12}\theta_{13}(x_2 - [x_1, x_3] + [x_2, x_3]) = \theta_{12}(x_2 + [x_2, x_3] + [[x_1, x_3], x_3])$$
$$= x_2 + [x_2, x_3] + [[x_1, x_3], x_3] + [[x_2, x_3], x_3] \ ,$$

and

$$\theta_{12}\theta_{13}\theta_{12}^{-1}\theta_{13}^{-1}(x_1) = \theta_{12}\theta_{13}\theta_{12}^{-1}(x_1) = \theta_{12}\theta_{13}(x_1 - x_2)$$
$$= \theta_{12}(x_1 - x_2 - [x_1, x_3]) = x_1 + [x_1, x_3] - [x_2, x_3] \ ,$$

$\gamma = \theta_{12}\theta_{13}\theta_{12}^{-1}\theta_{13}^{-1}$ is of the required form:

$$\gamma \quad : \quad x_1 \to x_1 + [x_1, x_3] - [x_2, x_3] \, ,$$

$$\begin{aligned} x_2 &\to x_2 + [x_2, x_3] + [[x_1, x_3], x_3] + [[x_2, x_3], x_3], \\ x_3 &\to x_3. \end{aligned}$$

5) The endomorphism of F_3 of the form

$$\begin{aligned} \varphi_{312} &: x_1 \to x_1, \\ x_2 &\to x_2 + u, \\ x_3 &\to x_3 + [x_3, x_1] + v, \end{aligned}$$

is an automorphism, where $v \in \gamma_3(F_3)$, $u \in F_3$.

Let the automorphisms of F_3, θ_{21}, θ_{23} given as

$$\theta_{21} : x_3 \to x_3 + [x_2, x_1], \quad x_i \to x_i, \quad i \neq 3,$$

$$\theta_{23} : x_2 \to x_2 + x_3, \quad x_i \to x_i, \quad i \neq 2.$$

Set $\gamma = \theta_{23}\theta_{13}\theta_{23}^{-1}\theta_{13}^{-1}$. Since

$$\begin{aligned} \theta_{23}\theta_{21}\theta_{23}^{-1}\theta_{21}^{-1}(x_3) &= \theta_{23}\theta_{21}\theta_{23}^{-1}(x_3 - [x_2, x_1]) \\ &= \theta_{23}\theta_{21}(x_3 - [x_2, x_1] + [x_3, x_1]) = \theta_{23}(x_3 + [x_3, x_1] + [[x_2, x_1], x_1]) \\ &= x_3 + [x_3, x_1] + [[x_2, x_1], x_1] + [[x_3, x_1], x_1], \end{aligned}$$

and

$$\begin{aligned} \theta_{23}\theta_{21}\theta_{23}^{-1}\theta_{21}^{-1}(x_2) &= \theta_{23}\theta_{21}\theta_{23}^{-1}(x_2) = \theta_{23}\theta_{21}(x_2 - x_3) \\ &= \theta_{23}(x_2 - x_3 - [x_2, x_1]) = x_2 - [x_2, x_1] - [x_3, x_1]. \end{aligned}$$

Then $\gamma = \theta_{23}\theta_{13}\theta_{23}^{-1}\theta_{13}^{-1}$ is of the required form:

$$\begin{aligned} \gamma &: x_1 \to x_1, \\ x_2 &\to x_2 - [x_2, x_1] - [x_3, x_1], \\ x_3 &\to x_3 + [x_3, x_1] + [[x_2, x_1], x_1] + [[x_3, x_1], x_1]. \end{aligned}$$

6) The endomorphism of F_3 of the form

$$\begin{aligned} \varphi_{312} &: x_1 \to x_1 + u, \\ x_2 &\to x_2, \\ x_3 &\to x_3 + [x_3, x_2] + v, \end{aligned}$$

is an automorphism, where $v \in \gamma_3(F_3)$, $u \in F_3$.

Consider endomorphisms θ_{12}, θ_{13} of F_3 given as

$$\theta_{12} : x_3 \to x_3 + [x_1, x_2]\,, \quad x_i \to x_i, \quad i \neq 3,$$

$$\theta_{13} : x_1 \to x_1 + x_3, \quad x_i \to x_i, \quad i \neq 1.$$

Then clearly θ_{21}, θ_{23} are automorphisms of F_3. Set $\gamma = \theta_{13}\theta_{21}\theta_{13}^{-1}\theta_{21}^{-1}$. Since

$$\begin{aligned}\theta_{13}\theta_{12}\theta_{13}^{-1}\theta_{12}^{-1}(x_3) &= \theta_{13}\theta_{12}\theta_{13}^{-1}(x_3 - [x_1, x_2]) \\ &= \theta_{13}\theta_{12}(x_3 - [x_1, x_2] + [x_3, x_2]) = \theta_{13}(x_3 + [x_3, x_2] + [[x_1, x_2]\,, x_2]) \\ &= x_3 + [x_3, x_2] + [[x_1, x_2]\,, x_2] + [[x_3, x_2]\,, x_2]\ ,\end{aligned}$$

and

$$\begin{aligned}\theta_{13}\theta_{12}\theta_{13}^{-1}\theta_{12}^{-1}(x_1) &= \theta_{13}\theta_{12}\theta_{13}^{-1}(x_1) = \theta_{13}\theta_{12}(x_1 - x_3) \\ &= \theta_{13}(x_1 - x_3 - [x_1, x_2]) = x_1 - [x_1, x_2] - [x_3, x_2]\,.\end{aligned}$$

Then $\gamma = \theta_{13}\theta_{21}\theta_{13}^{-1}\theta_{21}^{-1}$ is of the required form:

$$\begin{aligned}\gamma \quad &: \quad x_1 \to x_1 - [x_1, x_2] - [x_3, x_2]\,, \\ x_2 &\to x_2, \\ x_3 &\to x_3 + [x_3, x_2] + [[x_1, x_2]\,, x_2] + [[x_3, x_2]\,, x_2]\,.\end{aligned}$$

References

[1] R.M. Bryant, V. Drensky, Obstructions to lifting automorphisms of free algebras, *Communication in Algebra*, **12** (1993), 4361-4389.

[2] P.M. Cohn, Subalgebras of free associative algebra, *Proc. London Math. Soc.*, **14**, No. 3 (1964), 618-632.

[3] G.P. Kukin, Primitive elements of free elements of free Lie algebras, *Algebrai-Logika*, **9**, No. 4 (1970), 458-472,