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SOME *FK* SPACES DEFINED BY A MODULUS FUNCTION

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Abstract: In this paper we define and study the spaces of entire and analytic sequences with respect to a modulus function. We also characterize some matrix transformations of $L(f)$ into some FK spaces.

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1. Introduction

Let w denote the space of all sequences $x = (x_k)$ real or complex. By l_∞, c and c_0 we denote the spaces of all bounded, convergent and null sequences. Let l_1 and l_p denote the spaces of all absolutely convergent and p-absolutely convergent series respectively. A sequence $x = (x_k)$ is said to be *analytic* if $\sup_k |x_k|^{1/k} < \infty$ and it is said to be *entire* if $\lim_k |x_k|^{1/k} = 0$. We write Λ and Γ for the spaces of all analytic and entire sequences respectively.

A function $f : [0,\infty) \to [0,\infty)$ is called a *modulus function* if: (i) $f(x) = 0$ iff $x = 0$, (ii) $f(x+y) \le f(x)+f(y)$, (iii) f is increasing, and (iv) f is continuous from the right at 0. It is immediate from (ii) and (iv) that f is continuous every where on $[0,\infty)$. A modulus function may be bounded or unbounded ([2], [5], [6]).

Ruckle [5] used the idea of a modulus function f to construct the following sequence space

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$$L(f) := \{x \in w : \sum_k f(|x_k|) \ < \infty\}.$$

This space is an FK-space (Section 3) and Ruckle proved that the intersection of all such $L(f)$ spaces is the space ϕ of sequences which have a finite number of non-zero entries. For FK-space theory we refer to Wilansky [7] and Mursaleen [3]. In [2], Maddox also defined and study some sequence spaces defined by a modulus.

If we take $f(x) = x$, then $L(f)$ is reduced to the space l_1. Ruckle showed that $L(f) \subset l_1$ and $L(f)^\alpha = l_\infty$ for any modulus f, where

$$L(f)^\alpha := \{a = (a_k) \in w : \sum_k |a_k x_k| < \infty \quad \text{for all} \quad x \in L(f)\}$$

denotes the α-dual of $L(f)$.

In this paper we define and study the spaces Λ and Γ with respect to a modulus f and determine some matrix transformations of $L(f)$ into $c, c_0, l_1, l_\infty, l_p, \Lambda$ and Γ.

2. Some New Sequence Spaces

Let f be a modulus function. We define the following sequence spaces.

$$\Gamma(f) := \{x \in w : (f(|x_k|))^{1/k} \to 0, k \to \infty\},$$

and

$$\Lambda(f) := \{x \in w : \sup_k (f|x_k|)^{1/k} < \infty\}.$$

If we take $f(x) = |x|$ then these spaces are respectively reduced to the spaces of entire and analytic sequences (see Rao and Subramanian [4]).

We further generalize these spaces by considering an arbitrary sequence $u = (u_k),\ u_k \neq 0 \quad (k = 1, 2, \cdots)$ as follows

$$\Gamma(f, u) := \{z \in w : ux \in \Gamma(f)\}$$

and

$$\Lambda(f, u) := \{x \in w : ux \in \Lambda(f)\}.$$

For $u = e = (1, 1, 1, \cdots)$, the spaces $\Gamma(f, u)$ and $\Lambda(f, u)$ are reduced to $\Gamma(f)$ and $\Lambda(f)$ respectively.

We also define

$$L(f, u) := \{x \in w : ux \in L(f)\}$$

which is a slight generalization of the space $L(f)$ defined by Ruckle [5].

Remark 1. Ruckle [5] proved that, for any modulus function, $L(f) \subset l_1$ and $L(f)^\alpha = l_\infty$, where

$$L(f)^\alpha := \{a_k \in w : ax = (a_k x_k)_{k=1}^\infty \in l_1 \quad \text{for all} \quad x \in L(f)\}$$

is the α-dual of $L(f)$. The following are easy consequences of this result:

(i) $\Gamma(f) \subset \Gamma, \quad \Lambda(f) \subset \Lambda$;

(ii) $L(f,u))^\alpha = (l_\infty)_{1/u} := \{x \in w : \sup_k |\frac{x_k}{u_k}| < \infty\}$;

(iii) $\Gamma(f)^\alpha = \Gamma^\alpha = \Lambda$, $\Lambda(f)^\alpha = \Lambda^\alpha = \Gamma$.

First we establish some basic results about our spaces.

Theorem 2. *$\Gamma(f)$ and $\Lambda(f)$ are linear spaces over $\mathbb{C}$.*

Proof. Let us consider the space $\Lambda(f)$. We make use of the following inequality (see Maddox [1]): If $a_k, b_k \in \mathbb{C}$ and $0 < p_k \le \sup p_k = H$ for each k, then

$$|a_k + b_k|^{p_k} \le K(|a_k|^{p_k} + |b_k|^{p_k}), \tag{1}$$

where $K = \max\{1, 2^{H-1}\}$.

Let $x, y \in \Lambda(f)$ and $\lambda, \mu \in \mathbb{C}$. For λ, μ there exist integers M_λ and N_μ such that $|\lambda| \le M_\lambda$ and $|\mu| \le N_\mu$. From (1) and definition of f, we have

$$(f(|\lambda x_k + \mu y_k|))^{1/k} \le K M_\lambda \{f(|x_k|)\}^{1/k} + K N_\mu \{f(|y_k|)\}^{1/k},$$

which implies that $lx + \mu y \in \Lambda(f)$. □

Theorem 3. *$\Gamma(f)$ and $\Lambda(f)$ are complete metric spaces with the metric d defined by*

$$d(x,y) = \sup_k \{f(|x_k - y_k|)\}^{1/k}, \tag{2}$$

where $x = (x_k), y = (y_k) \in \Lambda(f)$ or $\Gamma(f)$ whatever be the case.

Proof. It is easy to prove that $d(x,y) = 0 \Leftrightarrow x = y$ and $d(y,x) = d(x,y)$. By Minkowski's inequality

$$\{f(x_k - z_k)\}^{1/k} \le \{f(x_k - y_k)\}^{1/k} + \{f(y_k - z_k)\}^{1/k}.$$

Hence

$$d(x,z) \le d(x,y) + d(y,z)$$

for every $x, y, z \in \Lambda(f)$ or $\Gamma(f)$. Suppose that (x^n) is a Cauchy sequence in $\Gamma(f)$. Let $\epsilon > 0$ be given. Then there is a positive integer N such that $d(x^n, x^m) < \epsilon_1$ for every $n, m \ge N$. Thus

$$\sup_k \{f(x_k^n - x_k^m)\} < \epsilon = \epsilon_1^k, \quad \text{for} \quad n, m \ge N.$$

Therefore $|x_k^n - x_k^m| < \epsilon$ for $n, m \geq N$, i.e., $(x^n) = ((x_k^n))$ is a Cauchy sequence in $\mathbb{C}$. Suppose that $x^n \to x = (x_k)$ in $\mathbb{C}$, i.e., $|x_k^n - x_k| \to 0$ (as $n \to \infty$) for every k. Therefore, we have

$$d(x^n, x) < \epsilon \quad \text{for} \quad n \geq N,$$

that is, (x^n) converges to x in $\Gamma(f)$. Using the definition of f and by Minkowski's inequality, it is easy to see that $x \in \Gamma(f)$, since

$$\{f(|x_k|)\}^{1/k} \leq \{f(|x_k^n|)\}^{1/k} + \{f(|x_k^n - x_k|)\}^{1/k}.$$

Hence $\Gamma(f)$ is complete. □

Theorem 4. *Let f, g be modulus functions. Then:*
(i) $X(g) \subseteq X(f \circ g)$,
(ii) $X(f) \cap X(g) \subseteq X(f + g)$, where $X = \Lambda$ or Γ.

Proof. (i) Let us consider $X = \Lambda$. Let $\epsilon > 0$. Choose $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $t_k = g(|x_k|)$ and consider

$$\sup_k \{f(t_k)\}^{1/k} = \sup_{k, t_k \leq \delta} \{f(t_k)\}^{1/k} + \sup_{k, t_k > \delta} \{f(t_k)\}^{1/k}.$$

For $\delta < t_k$, we use the following inequality

$$t_k < \frac{t_k}{\delta} \leq 1 + [|\frac{t_k}{\delta}|],$$

where $[\cdot]$ denotes the integer part. Therefore, by the definition of f, we have

$$f(t_k) \leq f(1)(1 + [|\frac{t_k}{\delta}|]) \leq 2f(1)\frac{t_k}{\delta}.$$

Now, $x \in \Lambda(g)$ gives

$$\sup_k \{f(t_k)\}^{1/k} \leq \max(1, [\frac{2f(1)}{\delta}]) \sup_k (t_k)^{1/k} < \infty.$$

Hence $t_k \in \Lambda(f)$, i.e. $x \in \Lambda(f \circ g)$.

(ii) By (1) we have

$$\{(f + g)|x_k|\}^{1/k} \leq K(\{f(|x_k|)\}^{1/k} + \{g(|x_k|)\}^{1/k}).$$

Let $x = (x_k) \in \Lambda(f) \cap \Lambda(g)$ and taking the supremum over k on both sides, we get $x \in \Lambda(f + g)$. □

3. FK Spaces and Some Matrix Transformations

A sequence space X with linear topology is called a *k-space* if each of the maps $P_i : X \to \mathbb{C}$ defined by $P_i(x) = x_i$ is continuous for $i = 1, 2, \cdots$.

A Fréchet space is a complete linear metric space. In other words, a locally convex space is called a Fréchet space if it is metrizable and the underlying metric space is complete.

An FK-space is a k-space if X is a complete linear metric space. In other words, we say that X is FK-space if X is a Fréchet space with continuous coordinate projection, we mean that if $x^{(n)} \to x$ (as $n \to \infty$) in the metric of X, then $x_k^{(n)} \to x_k$ (as $n \to \infty$) for each $k \in \mathbb{N}$, i.e., for each $k \in$, the linear functional $P_k(x) = x_k$ is such that P_k is continuous on X.

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix and X and Y be two sequence spaces. Then we say that A transforms X into Y, written as $A \in (X, Y)$, if $Ax \in Y$ for $x \in X$, where $Ax = (A_n(x))$,

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$$

provided that the series $\sum_k a_{nk} x_k$ converges for all $n = 1, 2, \cdots$.

In this section, we give necessary and sufficient conditions for the matrix classes $(L(f), c)$, $(L(f), c_0)$, $(L(f), l_1)$, $(L(f), l_\infty)$ and $(L(f), l_p)$, $1 < p < \infty$.

These classes are directly obtained (Mursaleen [3]) from the known classes (l_1, c), (l_1, c_0), (l_1, l_1), (l_1, l_∞) and $(l_1, l_p), 1 < p < \infty$ since $L(f)^\alpha = L(f)^\beta = l_1$; where $L(f)^\beta$ denotes the β-dual of $L(f)$, i.e.,

$$L(f)^\beta = \{a = (a_k) : \sum_k a_k x_k \text{ converges for all } x \in L(f)\}.$$

Theorem 5. *a) $A \in (L(f), c)$ if, and only if:*
(i) $\sup_{n,k} |a_{nk}| < \infty$*;*
(ii) $\lim_n a_{nk}$ *exists for all k.*
b) $A \in (L(f), c_0)$ if, and only if, (i) holds and
(iii) $\lim_n a_{nk} = 0$ *for all k.*

Theorem 6. *$A \in (L(f), l_1)$ if, and only if,* $\sup_k \sum_n |a_{nk}| < \infty$.

Theorem 7. *$A \in (L(f), l_\infty)$ if, and only if, condition (i) of Theorem 5 holds.*

Theorem 8. *$A \in (L(f), l_p)), 1 < p < \infty$, if, and only if,* $\sup_k \sum_n |a_{nk}|^p < \infty$.

Theorem 9. *$A \in (L(f), \Gamma)$ if, and only if, condition (i) of Theorem 5 holds and*

$$\lim_n |a_{nk}|^{1/n} = 0, \quad \text{for all } k.$$

Theorem 10. *$A \in (L(f), \Lambda)$ if, and only if,* $\sup_{n,k} |a_{nk}|^{1/n} < \infty$.

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