International Journal of Pure and Applied Mathematics
Volume 31 No. 1 2006, 1-8

SPLINE GENERALIZED SPHERICAL FUNCTIONS ON THE SPHERE

E. Aghdassi[1 §], S.F. Rzaev[2]

[1,2]Department of Applied Mathematics
Faculty of Mathematical Sciences
University of Tabriz
Tabriz, IRAN
[1]e-mail: esaghdassi@tabrizu.ac.ir
[2]e-mail: rzseymur@hotmail.com

Abstract: In this paper we consider periodical functions defined on

$$\sigma = (0 \leq \phi_1 \leq 2\pi; 0 \leq \theta \leq \pi, 0 \leq \phi_2 \leq 2\pi)$$

in $L_2(\sigma)$ Hilbert space quipped with spacial scalar product and corresponding norm. In space $W_2^r(B,\sigma)$ the convex programming problem which is called Favard problem is studied where B is a prescribed operator.

AMS Subject Classification: 42C10, 43A77, 43A90
Key Words: Hilbert space, self-adjoint operator, k-functional

1. Introduction

Schoenberg [4, 5] considered the following problem. Let the functions with $r-1$ continuous derivatives and r-th square integrable derivative, be minimized

$$\int_a^b [f^{(r)}(x)]^2 \, dx$$

subject to conditions $f(x_i) = f_i$, $i = 1, 2, \dots, n$.

The splines in the case were called natural splines by Shoenberg. This is a quadratic programming problem.

We note that if in place of Shoenberg problem consider the problem

Received: May 20, 2006

§Correspondence author

$$\int_\Delta (Px)^2\, dt \to \inf, \quad x(\tau_i)=\xi_i \quad i=1,\dots,n, \quad x(\cdot)\in W_2^r(\Delta)\,,$$

where $W_2^r(\Delta) = \{x|\ x_{(\cdot)}^{(r)} \in L_2(\Delta)\}$ – Sobolev spaces, and P is a differential operator with constant coefficients (or a pseudo differential operator) of order r, then another series of "spline-like functions" is obtained (see [7]).

A differential Laplace-Beltrami operator on the sphere P have been studied by Freeden [1], Shure, Parker, and Backus [6], and Wahba [9, 10].

Our point of view in this paper instead operator P to take another operator.

We consider periodical functions $u(\phi_1,\theta,\phi_2)$ with period 2π by ϕ_1 and by ϕ_2 in the $\sigma=(0\le\phi_1\le 2\pi; 0\le\theta\le\pi, 0\le\phi_2\le 2\pi)$. We denote by $L_2(\sigma)$ Hilbert space with scalar product

$$(u,v)=\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} u(\phi_1,\theta,\phi_2)\cdot\overline{v(\phi_1,\theta,\phi_2)}\sin\theta\ d\phi_1 d\theta d\phi_2.$$

We denote by B_1, B_2 and B_3 following operators:

$$B_1 = e^{i\phi_2}(\coth\theta\frac{\partial}{\partial\phi_2} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi_1} + i\cdot\frac{\partial}{\partial\theta}),$$

$$B_2 = e^{i\phi_2}(-\coth\theta\frac{\partial}{\partial\phi_2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi_1} + i\frac{\partial}{\partial\theta}), \qquad B_3 = i\frac{\partial}{\partial\phi_2}\,,$$

in the family functions $u(\phi_1,\theta,\phi_2)$ for which $B_1^{k_1}B_2^{k_2}B_3^{k_3}u$ $(k_1+k_2+k_3=1,2,\dots,r)$ is continuous. We introduce norm

$$\|u(\phi_1,\theta,\phi_2)\|_{W_2^r(B,\sigma)} = \sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k}\|B_1^{k_1}B_2^{k_2}B_3^{k_3}u\|_{L_2(\sigma)}\,.$$

Complete this family functions aspect introduced norm space, we denote it by $W_2^r(B,\sigma)$.

From relation communications for operators B_1, B_2, B_3 follows that, all expression in the form

$$B_{i_1}^{\gamma_1}B_{i_2}^{\gamma_2}\dots B_{i_m}^{\gamma_m}u(i_1,i_2,\dots,i_m=1,2,3),$$

where $\gamma_1+\gamma_2+\dots+\gamma_m=r$, are linear combination of $B_1^{k_1}B_2^{k_2}B_3^{k_3}u$, $k_1+k_2+k_3\le r$.

In $W_2^r(B,\sigma)$ we introduce scalar product

$$[u,v]=\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k}(B_1^{k_1}B_2^{k_2}B_3^{k_3}u)\overline{(B_1^{k_1}B_2^{k_2}B_3^{k_3}v)} \times\sin\theta\ d\phi_1\, d\theta\, d\phi_2.$$

Thus, $W_2^r(B,\sigma)$ will be a Hilbert space (see [3]).

2. Main Results

Generalized spherical functions $t^l_{mn}(\phi_1,\theta,\phi_2)$ on σ, which coincide with the eigenspace of the operator

$$\Delta_2 = -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi_1^2} - 2\cos\theta\frac{\partial^2}{\partial\phi_1\partial\phi_2} + \frac{\partial^2}{\partial\phi_2^2}\right)\right],$$

corresponding to the eigenvalue $l(l+1)$.

In this paper we consider following problem:

$$\frac{1}{8\pi^2}\int_\sigma |\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k}(B_1^{k_1}B_2^{k_2}B_3^{k_3}u)^2|d\sigma \to \inf, \quad u(\tau_i)=\xi_i,\ u\in W_2^r(B,\sigma),$$

where $d\sigma = \sin\theta\, d\phi_1\, d\theta\, d\phi_2$. This is a convex programming problem. It is called the Favard problem.

We note that B_1, B_2, and B_3 operators satisfy the following conditions (see [8])

$$B_1 t^l_{mn} = \alpha_{n+1} t^l_{m,n+1}, \quad B_2 t^l_{mn} = \alpha_n t^l_{m,n-1}, \quad B_3 t^l_{mn} = n t^l_{mn}, \qquad (1)$$

where $\alpha_n = \sqrt{(l+n)(l-n+1)}$ and $\Delta_2 = (B_1B_2 - B_3 + B_3^2)$.

Given arbitrary points $\eta_1,\eta_2,\ldots,\eta_n\in\sigma$, the problem we address now is the construction of a smooth function defined over σ which interpolates the given data points $(\eta_i,y_i)\in\sigma\times R$, $i=1,2,\ldots,N$. We consider integral the following form.

$$\mu(f) = \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k}|B_1^{k_1}B_2^{k_2}B_3^{k_3}f|^2\sin\theta\,d\phi_1\,d\theta\,d\phi_2$$

and also we consider problem minimization (see [2])

$$\mu(f)\to \min_{f\in W_2^r(B,\sigma),\ f(\eta_i)=y_i}. \qquad (2)$$

For generalized spherical functions (see [3]) by formulas (1)

$$B_3^{k_3}B_2^{k_2}B_1^{k_1}\overline{B_1^{k_1}B_2^{k_2}B_3^{k_3}t^l_{mn}(\phi_1,\theta,\phi_2)} = (-1)^{k_1k_2k_3}(\lambda^l_{mn})_{k_1+k_2+k_3}\overline{t^l_{mn}(\phi_1,\theta,\phi_2)},$$

where $(\lambda^l_{mn})_{k_1+k_2+k_3}\geq 0$ $(\lambda^l_{mn})_0 = 1$.

We denote

$$\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k}(\lambda_{mn}^l)_{k_1+k_2+k_3}=(\eta_{mn}^l)_r.$$

Then

$$\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k}(\lambda_{mn}^l)_{k_1+k_2+k_3}t_{mn}^l=(\eta_{mn}^l)_r\overline{t_{mn}^l}$$

It is known that (see [3, 8]) if $f\in L_2(\sigma)$

$$f=\sum_{l=0}^{\infty}\sum_{m,n=-l}^{l}a_{mn}^l\sqrt{2l+1}t_{mn}^l(\phi,\theta,\phi_2)\,. \tag{3}$$

Then

$$\sum_{k=0}^{r}(-1)^k\sum_{k_1+k_2+k_3=k}[B_3^{k_3}B_2^{k_2}B_1^{k_1}\overline{(B_1^{k_1}B_2^{k_2}B_3^{k_3}f)}]$$
$$=\sum_{l=1}^{\infty}\sum_{m,n=-l}^{l}a_{mn}^l\sqrt{2l+1}(\eta_{mn}^l)_r\overline{t_{mn}^l(\eta)}\,, \tag{4}$$

where $(\eta_{mn}^l)_r\geq 1$.

We introduce the class functions

$$W_r^2(\sigma)=\left\{f\in L_2(\sigma)\left|\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l}(\eta_{mn}^l)_r^2a_{mn}^2<\infty\right.\right\}.$$

In the work [3] is proved that $W_2^r(B,\sigma)=W_2^r(\sigma)$.

Lemma 1. *$\forall f\in W_2^r(\sigma)$ Fourier series (3) converge on the σ uniformly, and this series is continuity on σ.*

For the proof of the lemma see [3].

Lemma 2. *If $f\in W_2^r(\sigma)$, and $B_1^{k_1}B_2^{k_2}B_3^{k_3}f=0$ on σ, then function f is constant.*

Indeed, by virtue (3), (4) and Parseval equality we have

$$\left\|\sum_{k=0}^{r}(-1)^k\sum_{k_1+k_2+k_3=k}B_3^{k_3}B_2^{k_2}B_1^{k_1}f\right\|^2=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=-l}^{l}a_{mn}^2(2l+1)(\eta_{mn}^l)^2=0.$$

From this $a_{mn}^l=0$ as $l>0$ $f\equiv a_{00}^0t_{00}^0(\eta)=\frac{a_{00}^0}{8\pi}$

We note that, problem (2) has a generalized form considered in [2]. We used the notations of the work [2]. So, the role of Hilbert space H is played by

$L_2(\sigma)$, $X = W_2^r$, $T := \sum_{k=0}^r \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3}$, and a functional $L_i(f)$ is defined by formula $L_i(f) = f(\eta_i)$; $i = 1, 2, \dots, N$, $\eta_i \in \sigma$.

For solving problem (2) we introduce Green functions operator's for $\sum_{k=0}^r \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3}$

$$G(\eta_1, \eta_2) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \frac{(2l+1)}{(\eta_{mn}^l)_r^2} t_{mn}^l(\eta_1) t_{mn}^l(\eta_2)$$

by virtue of addition formula (see [8]) for generalized spherical functions we have

$$G(\eta_1, \eta_2) = \frac{1}{8\pi^2} \sum_{l=0}^{\infty} \frac{2l+1}{(\eta_{mn})_r^2} P_{mn}^l(\cos\theta),$$

where $\cos\theta$ is defined by formula

$$\cos\theta = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\phi_2\,,$$

$$\tan\phi = \frac{\sin\theta_2 \sin\phi_2}{\cos\theta_1 \sin\theta_2 \cos\phi_2 + \sin\theta_1 \cos\theta_2}\,,$$

$$\tan\phi_2 = \frac{\sin\theta_1 \sin\phi_2}{\sin\theta_1 \cos\theta_2 \cos\phi_2 + \cos\theta_1 \sin\theta_2}\,,$$

$$P_{mn}^l(x) = 2^{-m} \left[\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!} \right]^{\frac{1}{2}} (1-x)^{\frac{m-n}{2}} (1+x)^{\frac{m+n}{2}} P_{l-m}^{(m-n, m+n)}(x)$$

and $P_n^{(\alpha,\beta)}(z)$ is Jacobi polynomials.

There is no difficult to see that, this series is convergent and it contains the class $W_2^r(\sigma)$ (see [3]). In particular, we have following lemma.

Lemma 3. *Let $\eta \in \sigma$. For any function $f \in W_2^r(B, \sigma)$ the total representation*

$$f(\eta_1) = \frac{1}{8\pi^2} \int_\sigma f(q) dq + \int_G \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3} G(\eta_1 \eta_2) \times \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3} f d\eta_2\,. \quad (5)$$

To prove this lemma we use that Fourier series $f \in W_2^r(B, \sigma)$. They converge to $f(g)$ for each point $g \in \sigma$ (see Lemma 1).

The natural spline in the σ is defined to be

$$S(\eta) = c_0 + \sum_{k=0}^{N} d_k G(\eta, \eta_k) \qquad \eta, \eta_k \in \sigma, c \equiv \text{constant}$$

with the condition $\sum_{k=1}^{N} d_k = 0$

Finally we mention the following result.

Lemma 4. *For any natural spline $S(\eta)$ and any function $f \in W_2^r(B,\sigma)$ the equality*

$$\frac{1}{8\pi^2}\int_\sigma \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3} S(\eta) \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3} f(\eta) d\eta$$

$$= \sum_{k=1}^{N} d_k f(\eta_k) \quad (6)$$

is true

Proof. By using the formula (5), we have

$$\int_\sigma \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3} S(\eta) \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3} f(\eta) d\eta$$

$$= \sum_{k=1}^{N} d_k \int_\sigma \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3} G(\eta, \eta_k)$$

$$\times \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} B_1^{k_1} B_2^{k_2} B_3^{k_3} f(\eta) d\eta = \sum_{k=1}^{N} d_k \left[f(\eta_k) - \int_\sigma f(\eta) d\eta \right]$$

$$= \sum_{k=1}^{N} d_k f(\eta_k).$$

We prove this lemma by using condition $\sum_{k=1}^{N} d_k = 0$. The lemma is proved. □

Now we are able to solve the interpolation problem for natural splines.

Theorem 5. *Let points $\eta_1, \eta_2, \dots, \eta_N \in \sigma$ and $\eta_i \neq \eta_j$ for $i \neq j$. Then the interpolation problem*

$$S(\eta_i) = y_i, \qquad i = 1, 2, \dots, N\,,$$

is unique for any y_i.

Proof. Let c and d_k satisfy the homogenous system

$$c + \sum_{k=1}^{N} d_k G(\eta_i, \eta_k) = 0\,, \quad i = 1, 2, \dots, N\,, \quad \sum_{k=1}^{N} d_k = 0\,. \quad (7)$$

Then $S(\eta_i) = 0$ and by virtue of (6) (substituting in place of f the splines S)

we have

$$\frac{1}{8\pi^2}\int_\sigma \left[\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k} B_1^{k_1}B_2^{k_2}B_3^{k_3}S(\eta)\right]^2 d\eta = 0\,,$$

consequently

$$\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k} B_1^{k_1}B_2^{k_2}B_3^{k_3}S(\eta)$$
$$= \sum_{k=1}^{N} d_k \left[\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k} B_1^{k_1}B_2^{k_2}B_3^{k_3}G(\eta,\eta_k)\right] = 0\,, \quad \forall \eta\in\sigma\,. \quad (8)$$

It is easy to establish the following relation

$$\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k} B_1^{k_1}B_2^{k_2}B_3^{k_3}G(\eta,\eta_1) = \sum_{l=0}^{\infty}\frac{2l+1}{(\eta_{mn}^l)_r}P_{mn}^l(\cos\theta).$$

From this and (8) we get

$$\sum_{l=0}^{\infty}\frac{2l+1}{(\eta_{mn}^l)_r}\sum_{k=1}^{N} d_k P_{kk}^l(\cos\theta) = 0. \quad (9)$$

Since for different l the term is orthogonal on the σ

$$\sum_{k=1}^{N} d_k P_{kk}^l(\cos\theta) = 0\,. \quad (10)$$

Relation (10) holds and $l = 0$.

Finally, homogenous system (7) has nonzero solution. Therefore the proof is complete. □

We denote S_* natural spline, which $S_*(\eta_i) = y_i$, $i = 1, 2, \ldots, N$.

Theorem 6. *Uniqueness solution of problem (9) is the spline S_*.*

Proof. To solve the problem (2) it is sufficient to have the following relation (see [2])

$$\int_\sigma \sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k} B_1^{k_1}B_2^{k_2}B_3^{k_3}S_*(\eta)\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k} B_1^{k_1}B_2^{k_2}B_3^{k_3}h(\eta)d\eta = 0,$$
$$\forall h\in W_2^r(B,\sigma)\,,$$

which $h(\eta_i) = 0$, $i = 1, \ldots, N$.

The property (6) gives easily the uniqueness solution of the theorem. If $\sum_{k=0}^{r}\sum_{k_1+k_2+k_3=k} B_1^{k_1}B_2^{k_2}B_3^{k_3}h = 0$, $h\in W_2^r(B,\sigma)$, then by Lemma (2), function $h \equiv$constant; or $h = 0$. The theorem proved. □

Acknowledgment

This research has been supported by Tabriz University. We would like to thank the research office of Tabriz University, for its support.

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