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ON THE VERTEX-DISTINGUISHING EDGE COLORING OF $P_m \vee P_n$

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Abstract: Let $G(V, E)$ be a connected graph. A k-proper edge coloring f of $G(V, E)$ is said to be a k-vertex-distinguishing edge coloring iff $C(u) \neq C(v)$ for $\forall u, v \in V(G)$, $u \neq v$, where $C(u) = \{f(uv)|uv \in E(G)\}$; and $\chi'_{vd}(G) = \min\{k|\text{there exists a } k\text{-VDEC of } G\}$ is called the vertex-distinguishing edge chromatic number. In this paper, we obtain the vertex-distinguishing edge chromatic number of the join graphs $P_m \vee P_n$.

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1. Introduction

A proper edge-coloring of G is called vertex-distinguishing, if for any two distinct vertices u and v of G the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. The minimal number of colors required for a vertex-distinguishing proper edge-coloring of G is called the vertex-distinguishing proper edge-coloring chromatic number of G (or observability), and is denoted by $\chi'_{vd}(G)$.

In this paper, we study the vertex distinguishing edge coloring of the graphs $P_m \vee P_n$.

Definition 1.1. (see [2]) Let $G(V, E)$ be a connected graph. A k-proper edge coloring f of $G(V, E)$ is said to be a k-vertex-distinguishing edge coloring

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(abbreviated as k-VDEC) iff $C(u) \neq C(v)$ for $\forall u, v \in V(G)$, $u \neq v$, where $C(u) = \{f(uv)|uv \in E(G)\}$; and $\chi'_{vd}(G) = \min k|$there exists a k-VDEC of $G\}$ is called the vertex-distinguishing edge chromatic number.

Definition 1.2. Let $G(V, E)$ be a simple graph and n_i the number of vertices with degree i. The combinatorial degree of graph G is defined as

$$\mu(G) = \max\left\{\min\left\{\lambda \middle| \binom{\lambda}{i} \geq n_i, \delta \leq i \leq \Delta\right\}\right\},$$

where δ and Δ denote the minimum and maximum degree of graph $G(V, E)$, respectively.

Conjecture 1.1. (see [2]) *For a connected graph $G(V, E)$ with $|V| \geq 3$, we have*

$$\mu(G) \leq \chi'_{vd}(G) \leq \mu(G) + 1.$$

It is clear that the left hand inequality of the Conjecture 1.1 is trivial.

Let G and H be two simple graphs. The join graph [1] of the graphs G and H, which is denote by $G \vee H$, is such a graph that $V(G \vee H) = V(G) \bigcup V(H)$ and $E(G \vee H) = E(G) \bigcup E(H) \bigcup \{uv|u \in V(G)$ and $v \in V(H)\}$.

In this paper, the general result with respect to the vertex-distinguishing edge chromatic number of $P_m \vee P_n$ is studied. The undefined terminologies and notations in this paper are refereed to references (see [1]).

2. Main Results

Lemma 2.1. *For a connected graph $G(V, E)$ with $|V| \geq 3$, we have*

$$\mu(G) \leq \chi'_{vd}(G).$$

Lemma 2.2. (see [3]-[4]) *For the complete graph K_n, we have*

$$\chi'_{vd}(K_n) = \begin{cases} n+1, & \text{if } n \equiv 0(\mod 2); \\ n, & \text{otherwise.} \end{cases}$$

Theorem 2.1. *If $n = 1$, then*

$$\chi'_{vd}(P_m \vee P_1) = \begin{cases} m+1, & \text{if } m = 2, 3; \\ m, & \text{if } m \geq 4. \end{cases}$$

This theorem is easily to be proved.

Theorem 2.2. *If n=2, then*

$$\chi'_{vd}(P_m \vee P_2) = \begin{cases} 5, & \text{if } m = 2; \\ m+2, & \text{if } m \geq 3. \end{cases}$$

Proof. We know

$$\mu(P_m \vee P_2) = \begin{cases} 5, & m = 2; \\ m+2, & m \geq 3. \end{cases}$$

Suppose that $P_m = v_1v_2\cdots v_m$, $P_2 = u_1u_2$ and $C = \{1, 2, \cdots, m+2\}$ $\overline{C}(w) = C \setminus C(w), w \in V(P_m \vee P_2)$.

Case 1. If $m = 2$, then

$$P_2 \vee P_2 = K_4$$

It is ture by lemma 2.2.

Case 2. If $m \geq 3$, we just prove that there exists a $(m+2)$-VDEC of the graph $P_m \vee P_2$.

Let f be defined as follows:

$$f(u_1v_i) = i, \quad i = 1, 2\cdots, m; \qquad f(u_2v_i) = 1+i, \quad i = 1, 2, \cdots, m;$$

$$f(u_1u_2) = m+2; \qquad f(v_iv_{i+1}) = i+3, \quad i = 1, 2\cdots, m-1.$$

Then, we obtain:

$$\overline{C}(u_1) = \{m+1\};\ \overline{C}(u_2) = \{1\};$$

$$\overline{C}(v_1) = \{3, 5, 6\cdots, m+2\};\ \overline{C}(v_m) = \{1, 2\cdots, m-1\};$$

$$\overline{C}(v_i) = \{1, 2\cdots, i-1, i+4, i+5\cdots, m+2\} i = 2, 3\cdots, m-1.$$

So, f is $(m+2) - VDEC$ of $P_m \vee P_2$.

Therefore Theorem 2.2 holds true. □

Theorem 2.3. *If $n = 3$, then*

$$\chi'_{vd}(P_m \vee P_3) = \begin{cases} 6, & \text{if } m = 3; \\ m+2, & \text{if } m \geq 4. \end{cases}$$

Proof. It is easy to see that the followig relation holds true

$$\mu(P_m \vee P_3) = \begin{cases} 6, & m = 3; \\ m+2, & m \geq 4. \end{cases}$$

Suppose that $P_m = v_1v_2\cdots v_m$ and $P_3 = u_1u_2u_3$.

Case 1. If $m = 3$, then using Lemma 2.1, it is sufficient to prove that there exists a 6-VDEC of the graph $P_3 \vee P_3$.

Let f be defined as follow:

$$f(u_1v_i) = i, \quad i = 1,2,3; \qquad f(u_2v_i) = 1+i, \quad i = 1,2,3;$$

$$f(u_3v_i) = i+2, \quad i = 1,2,3; \qquad f(v_1v_2) = 6; \qquad f(v_2v_3) = 1;$$

$$f(u_1u_2) = 5; \qquad f(u_2u_3) = 6.$$

Suppose $C = \{1,2,3,4,5,6\}$, $\overline{C}(w) = C \setminus C(w)$, $w \in V(P_3 \vee P_3)$. For f, we have:

$$\overline{C}(u_1) = \{4,6\};\ \overline{C}(u_2) = \{1\};$$

$$\overline{C}(u_3) = \{1,2\};\ \overline{C}(v_1) = \{4,5\};\ \overline{C}(v_2) = \{5\};\ \overline{C}(v_3) = \{2,6\}.$$

So, f is $6-VDEC$ of $P_3 \vee P_3$.

Case 2. If $m \geq 4$, then using Lemma 2.1 again, we just shall prove that there exists a $(m+2)$-VDEC of the graph $P_m \vee P_3$.

We define:

$$f(u_1v_i) = i, \quad i = 1,2\cdots,m; \qquad f(u_2v_i) = 1+i, \quad i = 1,2,\cdots,m;$$

$$f(u_3v_i) = i+2, \quad i = 1,2,\cdots,m; \quad f(v_iv_{i+1}) = i+4, \quad i = 1,2,\cdots,m-2;$$

$$f(v_{m-1}v_m) = 1; \quad f(u_1u_2) = m+2; \quad f(u_2u_3) = 1.$$

Suppose $C = \{1,2,\cdots m+1, m+2\}$, $\overline{C}(w) = C \setminus C(w)$, $w \in V(P_m \vee P_3)$. For f:

$$\overline{C}(u_1) = \{m+1\};\ \overline{C}(u_2) = \emptyset;\ \overline{C}(u_3) = \{2\}$$

$$\overline{C}(v_1) = \{4,6,7\cdots,m+2\};\ \overline{C}(v_m) = \{2,3\cdots,m-1\};$$

$$\overline{C}(v_i) = \{1,2\cdots,i-1,i+5,i+6\cdots,m+2\}, \quad i = 2,3\cdots,m-2.$$

$$\overline{C}(v_{m-1}) = \{2,3\cdots,i-1\}.$$

So, f is $(m+2)-VDEC$ of $P_m \vee P_3$. □

Theorem 2.4. *If $m > n \geq 4$, then*

$$\chi'_{vd}(P_m \vee P_n) = m+3.$$

Proof. It is easy to see the following equality

$$\mu(P_m \vee P_n) = m + 3\,.$$

Follows from the Lemma 2.1, we have $\chi'_{vd}(P_m \vee P_n) \geq \mu(P_m \vee P_n)$. Therefore, we just prove that there exists an $(m+3)$-VDEC f of graph $P_m \vee P_n$. In order to describe conveniently, we denote $P_m = v_1 v_2 \cdots\ v_m$, $P_n = u_1 u_2 \cdots u_n$ and $C = \{1, 2, \cdots m+1, m+2, 0\}$,$\overline{C}(w) = C \setminus C(w), w \in V(P_m \vee P_n)$.

Let f be defined as follow:

$$f(u_i v_j) = i + j - 1(\mathrm{mod}\,(m+3)), \qquad i = 1, 2 \cdots, n;\ j = 1, 2, \cdots, m;$$

$$f(u_i u_{i+1}) = m + 1 + i(\mathrm{mod}\,(m+3)), \qquad i = 1, 2 \cdots, n-1;$$

$$f(v_i v_{i+1}) = m + i - 1, \qquad i = 1, 2, 3, \cdots, m-1\,.$$

Therefore, we have

$$\overline{C}(u_1) = \{m+1, 0\}, \quad \overline{C}(u_n) = \{n-1, n-2\};$$

$$\overline{C}(v_1) = \{n+1, n+3, n+4 \cdots, m+2, 0\}, \quad \overline{C}(v_m) = \{n-2, n-3 \cdots, m-1\};$$

$$\overline{C}(v_i) = \{1, 2, \cdots, i-1, i+n+2, i+n+3 \cdots, m+2, 0\}, i = 2, 3 \cdots, m+2-n;$$

$$\overline{C}(v_{m+2-n+i}) = \{i+1, i+2 \cdots, m+1-n+i\};$$

$$\overline{C}(u_i) = \{i-1\}, i = 2, 3 \cdots, n-1.$$

It is obviously that f is an $(m+3)$-VDEC of $P_m \vee P_n$. □

Theorem 2.5. (see [5]) *If* $m = n \geq 4$, *then*

$$\chi'_{vd}(P_n \vee P_n) = \begin{cases} n+3, & \text{if } n = 4, 5, 6; \\ n+4, & \text{if } n \geq 7. \end{cases}$$

We find it special, when $n = 7$, $\mu(P_7 \vee P_7) = 10$, but $\chi'_{vd}(P_7 \vee P_7) = 11$

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