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EQUIVALENT CONDITIONS FOR CONTRACTIVE TYPE MAPPINGS IN METRIC SPACES

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Abstract: In this paper the pairwise equivalences for 30 contractive conditions involving (φ, p, q)-contractive type mappings in metric spaces are established, which generalize and improve the equivalence results given by Jachymski.

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1. Introduction and Preliminaries

Throughput this paper, we assume that $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{N}$ denotes the set of all positive integers. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Put

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$$O_T(x) = \{T^i x : i \in \mathbb{N} \cup \{0\}\}, \quad O_T(x,y) = O_T(x) \cup O_T(y),$$
$$\delta(O_T(x,y)) = \sup\{d(a,b) : a,b \in O_T(x,y)\}, \quad \forall x,y \in X$$

and

$$\Phi = \{\varphi : \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ with } \varphi(t) < t, \quad \forall t > 0\}.$$

The mapping T is called to be ϕ-*contractive* if there exists $\phi \in \Phi$ satisfying

$$d(Tx,Ty) \le \phi(d(x,y)), \quad \forall x,y \in X.$$

T is said to be (φ,p,q)-*contractive* if there exist $p,q \in \mathbb{N}$ and $\varphi \in \Phi$ such that

$$d(T^p x, T^q y) \le \varphi(\delta(O_T(x,y))), \quad \forall x,y \in X.$$

In the past 30 years, many authors studied the existence of fixed points for a lot of kinds of contractive type mappings, which deal with increasing and right continuous functions, increasing and right upper semi-continuous functions, increasing and upper semi-continuous functions, and upper semi-continuous functions [1-20]. In 1995, Jachymski [9] earned three equivalent conditions for contractive type mappings in metric spaces. These equivalent conditions refer $M_{ij}(x,y)$, which is defined as follows:

$$M_{ij}(x,y) = \max\{d(Sx,Ty), d(Sx,A_i x), d(Ty,A_j y), \frac{1}{2}[d(Sx,A_j y) + d(Ty,A_i x)]\}, \quad \forall x,y \in X, i,j \in \mathbb{N},$$

where $S,T,A_i : X \to X$ are mappings for $i \in \mathbb{N}$. In 1997, he established an equivalence among eight ϕ-contractive conditions [10].

In this paper, we replace $M_{ij}(x,y)$ by the orbital diameter $\delta(O_T(x,y))$ and ϕ-contractive mapping by (φ,p,q)-contractive mapping, respectively, and establish 30 equivalent conditions for (φ,p,q)-contractive mappings on a metric space (X,d), which extend Theorem 1 in [9,10] to more general cases.

Lemma 1.1. (see [10]) *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy that $\varphi(t) < t$ and $\limsup_{s\to t} \varphi(s) < t$ for $t > 0$. Then there exists a strictly increasing and continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \le \psi(t) < t$ for $t > 0$.*

Lemma 1.2. (see [10]) *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be right upper semi-continuous and $\varphi(t) < t$ for $t > 0$. Then there exists a right continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \le \psi(t) < t$ for $t > 0$.*

Lemma 1.3. (see [10]) *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be strictly increasing and $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$. Then $\lim_{s\to t^+} \varphi(s) \ne t$ for $t > 0$.*

Lemma 1.4. (see [2]) *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an upper semi-continuous function with $\varphi(t) < t$ for $t > 0$ and $\varphi(0) = 0$. Then there exists a strictly increasing and continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \le \psi(t) < t$ for $t > 0$.*

Lemma 1.5. (see [2]) *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a lower semi-continuous function with $\varphi(t) > t$ for $t > 0$. Then there exists a strictly increasing and continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \ge \psi(t) > t$ for $t > 0$.*

Lemma 1.6. (see [8]) *Let Q be a subset of $\mathbb{R}_+^2$. Then the following statements are equivalent:*

(1) *For any $\varepsilon > 0$, there exist $\delta > 0$ and $\varepsilon_0 \in (0, \varepsilon)$ such that $0 \le s < \varepsilon + \delta$ and $(s,t) \in Q$ imply $t \le \varepsilon_0$;*

(2) *There exists an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{s \to t^+} \varphi(s) < t$ for $t > 0$ and $(s,t) \in Q$ imply $t \le \varphi(s)$;*

(3) *There exists an increasing, right continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) < t$ for $t > 0$ and $(s,t) \in Q$ imply $t \le \varphi(s)$;*

(4) *There exists an increasing, left continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) > t$ for $t > 0$ and $(s,t) \in Q$ imply $\varphi(t) \le s$.*

Lemma 1.7. (see [9]) *Let Q be a subset of $\mathbb{R}_+^2$. Then the following conitions are equivalent:*

(1) *There exists a function $\alpha : (0, +\infty) \to (0, +\infty)$ such that, for any $\varepsilon > 0, \alpha(\varepsilon) > \varepsilon$ and*

(a) $\sup\{\alpha(s) : s \in (0, \varepsilon)\} \ge \alpha(\varepsilon)$ *and*

(b) $(s,t) \in Q$ *and* $0 \le s < \alpha(\varepsilon)$ *imply* $t < \varepsilon$*;*

(2) *There exist functions $\beta, \eta : (0, +\infty) \to (0, +\infty)$ such that, for any $\varepsilon > 0, \beta(\varepsilon) > \varepsilon, \eta(\varepsilon) < \varepsilon$ and $(s,t) \in Q$ and $0 \le s < \beta(\varepsilon)$ imply $t < \eta(\varepsilon)$;*

(3) *There exists a lower semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that φ is non-decreasing, $\varphi(t) < t$ for $t > 0$, and $(s,t) \in Q$ implies $t \le \varphi(s)$;*

(4) *There exists a lower semi-continuous function $\alpha : (0, +\infty) \to (0, +\infty)$ such that α is non-decreasing, for any $\varepsilon > 0, \alpha(\varepsilon) > \varepsilon$, and $(s,t) \in Q$ and $0 \le s < \alpha(\varepsilon)$ imply $t < \varepsilon$;*

(5) *There exists a lower semi-continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ is non-decreasing, $\psi(t) > t$ for $t > 0$, and $(s,t) \in Q$ implies $\psi(t) \le s$.*

2. Main Results

Our main results are as follows.

Theorem 2.1. *Let T be a self mapping of a metric space (X, d) and $\delta(O_T(x)) < +\infty$ for every $x \in X$. Then the following statements are equivalent:*

(1) *There exists an increasing and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is (φ, p, q)-contractive.*

(2) *There exists an increasing and right continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is (φ, p, q)-contractive.*

(3) *There exists an increasing and right upper semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is (φ, p, q)-contractive.*

(4) *There exists an increasing and upper semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is (φ, p, q)-contractive.*

(5) *There exists a strictly increasing and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is (φ, p, q)-contractive.*

(6) *There exists a strictly increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$ and T is (φ, p, q)-contractive.*

(7) *There exists a strictly increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{s\to t^+} \varphi(s) < t$ for $t > 0$ and T is (φ, p, q)-contractive.*

(8) *There exists a right continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is (φ, p, q)-contractive.*

(9) *There exists a right upper semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is (φ, p, q)-contractive.*

(10) *There exists an upper semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that T is (φ, p, q)-contractive.*

(11) *There exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\limsup_{s\to t} \varphi(s) < t$ for $t > 0$ and T is (φ, p, q)-contractive.*

(12) *There exists a mapping $f : X \times X \to \mathbb{R}_+$ such that $\sup\{f(x, y) : a \le \delta(O_T(x, y)) \le b, x, y \in X\} < 1$ for $b > a > 0$ and*

$$d(T^p x, T^q y) \le f(x, y)\delta(O_T(x, y)), \quad \forall x, y \in X.$$

(13) *There exists a mapping $g : X \times X \to \mathbb{R}_+$ such that $\inf\{g(x, y) : a \le \delta(O_T(x, y)) \le b, x, y \in X\} > 0$ for $b > a > 0$ and*

$$d(T^p x, T^q y) \le \delta(O_T(x, y)) - g(x, y), \quad \forall x, y \in X.$$

(14) *There exists a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) > 0$ for $t > 0$ and*

$$d(T^p x, T^q y) \le \delta(O_T(x, y)) - \psi(\delta(O_T(x, y))), \quad \forall x, y \in X.$$

(15) *There exists an increasing and left continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) > t$ for $t > 0$ and*

$$\psi(d(T^p x, T^q y)) \le \delta(O_T(x, y)), \quad \forall x, y \in X.$$

(16) *There exists an increasing and lower semi-continuous function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi(t) > t$ *for* $t > 0$ *and*

$$\psi(d(T^p x, T^q y)) \leq \delta(O_T(x, y)), \quad \forall x, y \in X.$$

(17) *There exists a strictly increasing and continuous function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi(t) > t$ *for* $t > 0$ *and*

$$\psi(d(T^p x, T^q y)) \leq \delta(O_T(x, y)), \quad \forall x, y \in X.$$

(18) *There exists a lower semi-continuous function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi(t) > t$ *for* $t > 0$ *and*

$$\psi(d(T^p x, T^q y)) \leq \delta(O_T(x, y)), \quad \forall x, y \in X.$$

(19) *For any* $\varepsilon > 0$, *there exist* $\alpha(\varepsilon) > 0$ *and* $\varepsilon_0(\varepsilon) \in (0, \varepsilon)$ *such that for any* $x, y \in X$,

$$0 \leq \delta(O_T(x, y)) < \varepsilon + \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) \leq \varepsilon_0(\varepsilon).$$

(20) *For any* $\varepsilon > 0$, *there exist* $\alpha(\varepsilon) > 0$ *and* $\varepsilon_0(\varepsilon) \in (0, \varepsilon)$ *such that for any* $x, y \in X$,

$$0 < \delta(O_T(x, y)) < \varepsilon + \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) \leq \varepsilon_0(\varepsilon).$$

(21) *For any* $\varepsilon > 0$, *there exist* $\alpha(\varepsilon) > 0$ *and* $\varepsilon_0(\varepsilon) \in (0, \varepsilon)$ *such that for any* $x, y \in X$,

$$\varepsilon \leq \delta(O_T(x, y)) < \varepsilon + \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) \leq \varepsilon_0(\varepsilon).$$

(22) *There exists an increasing and lower semi-continuous function* $\alpha : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0, \alpha(\varepsilon) > \varepsilon$ *and for* $x, y \in X$,

$$0 \leq \delta(O_T(x, y)) < \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \varepsilon.$$

(23) *There exists an increasing and lower semi-continuous function* $\alpha : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0, \alpha(\varepsilon) > \varepsilon$ *and for* $x, y \in X$,

$$0 < \delta(O_T(x, y)) < \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \varepsilon.$$

(24) *There exists an increasing and lower semi-continuous function* $\alpha : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0, \alpha(\varepsilon) > \varepsilon$ *and for* $x, y \in X$,

$$\varepsilon \leq \delta(O_T(x, y)) < \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \varepsilon.$$

(25) *There exists a function* $\alpha : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0$, $\sup\{\alpha(\delta(O_T(x,y))) : \delta(O_T(x,y)) \in (0,\varepsilon), x, y \in X\} \geq \alpha(\varepsilon) > \varepsilon$ *and for any* $x, y \in X$,

$$0 \leq \delta(O_T(x,y)) < \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \varepsilon.$$

(26) *There exists a function* $\alpha : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0$, $\sup\{\alpha(\delta(O_T(x,y))) : \delta(O_T(x,y)) \in (0,\varepsilon), x, y \in X\} \geq \alpha(\varepsilon) > \varepsilon$ *and for any* $x, y \in X$,

$$0 < \delta(O_T(x,y)) < \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \varepsilon.$$

(27) *There exists a function* $\alpha : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0$, $\sup\{\alpha(\delta(O_T(x,y))) : \delta(O_T(x,y)) \in (0,\varepsilon), x, y \in X\} \geq \alpha(\varepsilon) > \varepsilon$ *and for any* $x, y \in X$,

$$\varepsilon \leq \delta(O_T(x,y)) < \alpha(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \varepsilon.$$

(28) *There exist functions* $\beta, \eta : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0, \beta(\varepsilon) > \varepsilon, \eta(\varepsilon) < \varepsilon$ *and for any* $x, y \in X$,

$$0 \leq \delta(O_T(x,y)) < \beta(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \eta(\varepsilon).$$

(29) *There exist functions* $\beta, \eta : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0, \beta(\varepsilon) > \varepsilon, \eta(\varepsilon) < \varepsilon$ *and for any* $x, y \in X$,

$$0 < \delta(O_T(x,y)) < \beta(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \eta(\varepsilon).$$

(30) *There exist functions* $\beta, \eta : (0, +\infty) \to (0, +\infty)$ *such that for any* $\varepsilon > 0, \beta(\varepsilon) > \varepsilon, \eta(\varepsilon) < \varepsilon$ *and for any* $x, y \in X$,

$$\varepsilon \leq \delta(O_T(x,y)) < \beta(\varepsilon) \quad \textit{implies} \quad d(T^p x, T^q y) < \eta(\varepsilon).$$

Proof. Clearly, $(1) \Rightarrow (2) \Rightarrow (3), (2) \Rightarrow (8), (4) \Rightarrow (10), (5) \Rightarrow (1)$ and $(7), (8) \Rightarrow (9), (16) \Rightarrow (18), (17) \Rightarrow (15), (19) \Leftrightarrow (20) \Rightarrow (21), (22) \Leftrightarrow (23) \Rightarrow (24), (25) \Leftrightarrow (26) \Rightarrow (27)$, and $(28) \Leftrightarrow (29) \Rightarrow (30)$.

Put $Q = \{(\delta(O_T(x,y)), d(T^p x, T^q y)) : (x,y) \in X \times X\} \subseteq \mathbb{R}_+ \times \mathbb{R}_+$. By (3), (1) and (4) in Lemma 1.6, we derive that $(2) \Leftrightarrow (15) \Leftrightarrow (19)$. On the other hand, Lemma 1.7 ensures that $(4) \Leftrightarrow (16) \Leftrightarrow (22) \Leftrightarrow (25) \Leftrightarrow (28)$. By Lemma 1.1, we deduce that $(8) \Rightarrow (2)$ and $(11) \Rightarrow (5)$. It follows from Lemmas 1.2 and 1.5 that $(9) \Rightarrow (8)$ and $(18) \Rightarrow (17)$.

(3) $\Rightarrow$ (4). Note that φ is increasing and right upper semi-continuous. It follows that

$$\limsup_{s\to t}\varphi(s)=\lim_{\varepsilon\to 0^+}\sup_{s\in(t-\varepsilon,t+\varepsilon)}\varphi(s)=\lim_{\varepsilon\to 0^+}\sup_{s\in[t,t+\varepsilon)}\varphi(s)$$
$$=\limsup_{s\to t^+}\varphi(s)\le\varphi(t). \quad (2.1)$$

That is, φ is upper semi-continuous.

(10) $\Rightarrow$ (11). That φ is upper semi-continuous ensures that $\limsup_{s\to t}\varphi(s) \le \varphi(t) < t$ for $t>0$. Therefore, (10) implies (11).

(5) $\Rightarrow$ (6). Since φ is strictly increasing and $\varphi(t)<t$ for $t>0$, it follows that

$$\varphi^{n+1}(t)<\varphi^n(t)<\cdots<\varphi(t)<t,\ n\ge 1,\ t>0\,,$$

which implies that $\lim_{n\to\infty}\varphi^n(t)=w$ for some $w\ge 0$. We assert that $w=0$. Otherwise$w>0$. In view of the continuity of φ, we know that

$$w>\varphi(w)=\varphi(\lim_{n\to\infty}\varphi^n(t))=\lim_{n\to\infty}\varphi^{n+1}(t)=w,$$

which is a contradiction. Hence $\lim_{n\to\infty}\varphi^n(t)=0$ for $t>0$. That is, (6) holds.

(6) $\Rightarrow$ (11). By Lemma 1.3, we earn that

$$\lim_{s\to t^+}\varphi(s)\ne t \text{ for } t>0. \quad (2.2)$$

Note that φ is strictly increasing and $\varphi(s)<s$ for $s>0$. It follows that $\varphi(t)\le\lim_{s\to t^+}\varphi(s)\le t$ for $t>0$. Thus (2.2) yields that $\lim_{s\to t^+}\varphi(s)<t$ for $t>0$. Since φ is strictly increasing, by (2.1) we infer that

$$\limsup_{s\to t}\varphi(s)=\lim_{\varepsilon\to 0^+}\sup_{s\in[t,t+\varepsilon)}\varphi(s)\le\lim_{\varepsilon\to 0^+}\varphi(t+\varepsilon)$$
$$=\lim_{s\to t^+}\varphi(s)<t,\quad \forall t>0.$$

That is, (11) holds.

(7) $\Rightarrow$ (11). As in the proof of (6) $\Rightarrow$ (11), by Lemma 1.1, we deduce that

$$\limsup_{s\to t}\varphi(s)\le\lim_{s\to t^+}\varphi(s)<t,\ \forall t>0.$$

(5) $\Rightarrow$ (14). Set $\psi(t)=t-\varphi(t)$. It is easy to see $\psi(t)>0$ for $t>0$ and

$$d(T^px,T^qy)\le\delta(O_T(x,y))-\psi(\delta(O_T(x,y))),\ \forall x,y\in X.$$

(14) $\Rightarrow$ (13). Put $g(x,y) = \psi(\delta(O_T(x,y)))$. We claim that $\inf\{g(x,y) : a \le \delta(O_T(x,y)) \le b\} > 0$ for $b > a > 0$. Otherwise $\inf\{g(x,y) : a \le \delta(O_T(x,y)) \le b\} = 0$ for some $b > a > 0$. There exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset X \times X$ such that $a \le \delta(O_T(x_n, y_n)) \le b$ and $g(x_n, y_n) < \frac{1}{n}$ for $n \in \mathbb{N}$. Set $t_n = \delta(O_T(x_n, y_n))$ for $n \in \mathbb{N}$. It is clear that there exists a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$ such that $\{t_{n_k}\}_{k\in\mathbb{N}}$ converges to some $t_0 \in [a, b]$. Thus

$$g(x_{n_k}, y_{n_k}) = \psi(\delta(O_T(x_{n_k}, y_{n_k}))) = \psi(t_{n_k}) < \frac{1}{n_k}, \quad \forall k \in \mathbb{N}.$$

From the continuity of ψ, it follows that

$$0 < \psi(t_0) = \lim_{k\to\infty} \psi(t_{n_k}) = 0,$$

which is a contradiction.

(13) $\Rightarrow$ (12). Define $f : X \times X \to \mathbb{R}_+$ by

$$f(x,y) = \begin{cases} 0 & \text{for } \delta(O_T(x,y)) = 0\,, \\ 1 - \frac{g(x,y)}{\delta(O_T(x,y))} & \text{for } \delta(O_T(x,y)) \ne 0, \end{cases}$$

where $x, y \in X$. Let $b > a > 0$. From $\inf\{g(x,y) : a \le \delta(O_T(x,y)) \le b, x, y \in X\} > 0$, we derive that

$$\begin{aligned} &\sup\{f(x,y) : a \le \delta(O_T(x,y)) \le b, x, y \in X\} \\ &= \sup\left\{1 - \frac{g(x,y)}{\delta(O_T(x,y))} : a \le \delta(O_T(x,y)) \le b, x, y \in X\right\} \\ &= 1 - \inf\left\{\frac{g(x,y)}{\delta(O_T(x,y))} : a \le \delta(O_T(x,y)) \le b, x, y \in X\right\} \\ &\le 1 - \frac{1}{b}\inf\{g(x,y) : a \le \delta(O_T(x,y)) \le b, x, y \in X\} \\ &< 1 \end{aligned}$$

and

$$d(T^p x, T^q y) \le \delta(O_T(x,y)) - g(x,y) = f(x,y)\delta(O_T(x,y)), \quad \forall x, y \in X.$$

(12) $\Rightarrow$ (11). Define $A_n = \{(x,y) \in X \times X : \frac{1}{n} \le \delta(O_T(x,y)) \le n\}$ for $n \in \mathbb{N}$. Since each $\delta(O_T(x,y))$ is bounded, there exists $k \in \mathbb{N}$ such that $A_n \ne \emptyset$ for $n \ge k$. Set

$$a_n = \sup\{f(x,y) : (x,y) \in A_n\}\ n \ge k.$$

Now we define a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by:

$$\varphi(t) = \begin{cases} 0 & \text{for } t = 0, \\ a_k t & \text{for } t \in [\frac{1}{k}, k], \\ a_n t & \text{for } t \in [\frac{1}{n}, \frac{1}{n-1}) \cup (n-1, n], n > k. \end{cases}$$

Clearly, $d(T^p x, T^q y) \leq \varphi(\delta(O_T(x, y))),\ \forall x, y \in X$. In order to verify that

$$\limsup_{s \to t} \varphi(s) < t, \quad t > 0 \tag{2.3}$$

we have to consider following cases:

Case 1. Let $t \in (0, \frac{1}{k})$. It is clear that there exists an unique $m > k$ such that $t \in [\frac{1}{m}, \frac{1}{m-1})$. If $t \in (\frac{1}{m}, \frac{1}{m-1})$, we have

$$\limsup_{s \to t} \varphi(s) = \limsup_{s \to t} a_m s = a_m t < t;$$

if $t = \frac{1}{m}$, we get that

$$\limsup_{s \to (\frac{1}{m})^+} \varphi(s) = \limsup_{s \to (\frac{1}{m})^+} a_m s = \frac{a_m}{m} < \frac{1}{m}$$

and

$$\limsup_{s \to (\frac{1}{m})^-} \varphi(s) = \limsup_{s \to (\frac{1}{m})^-} a_{m+1} s = \frac{a_{m+1}}{m} < \frac{1}{m}.$$

Case 2. Let $t = \frac{1}{k}$. It is clear that

$$\limsup_{s \to (\frac{1}{k})^+} \varphi(s) = \limsup_{s \to (\frac{1}{k})^+} a_k s = \frac{a_k}{k} < \frac{1}{k}$$

and

$$\limsup_{s \to (\frac{1}{k})^-} \varphi(s) = \limsup_{s \to (\frac{1}{k})^-} a_{k+1} s = \frac{a_{k+1}}{k} < \frac{1}{k}.$$

Case 3. Let $t \in (\frac{1}{k}, k)$. It follows that

$$\limsup_{s \to t} \varphi(s) = \limsup_{s \to t} a_k s = a_k t < t.$$

Case 4. Let $t = k$. We attain immediately that

$$\limsup_{s \to k^-} \varphi(s) = \limsup_{s \to k^-} a_k s = a_k k < k$$

and

$$\limsup_{s\to k^+}\varphi(s)=\limsup_{s\to k^+}a_{k+1}s=a_{k+1}k<k.$$

Case 5. Let $t\in(k,+\infty)$. Obviously, there exists an unique $m\geq k$ such that $t\in(m,m+1]$. If $t\in(m,m+1)$, we gain that

$$\limsup_{s\to t}\varphi(s)=\limsup_{s\to t}a_{m+1}s=a_{m+1}t<t;$$

if $t=m+1$, it is not difficult to see that

$$\limsup_{s\to(m+1)^-}\varphi(s)=\limsup_{s\to(m+1)^-}a_{m+1}s=a_{m+1}(m+1)<m+1$$

and

$$\limsup_{s\to(m+1)^+}\varphi(s)=\limsup_{s\to(m+1)^+}a_{m+2}s=a_{m+2}(m+1)<m+1.$$

In conclusion, (2.3) holds.

(15) $\Rightarrow$ (17). Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be increasing, left continuous and $\psi(t)>t$ for $t>0$. It follows that

$$\liminf_{s\to t^-}\psi(s)=\lim_{\varepsilon\to 0^+}\inf_{s\in(t-\varepsilon,t]}\psi(s)\geq\lim_{\varepsilon\to 0^+}\psi(t-\varepsilon)=\psi(t).$$

That is, ψ is lower semi-continuous. By Lemma 1.5, there exists $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that φ is strictly increasing and continuous and $\psi(t)\geq\varphi(t)>t$ for $t>0$. Hence,

$$\varphi(d(T^px,T^qy))\leq\psi(d(T^px,T^qy))\leq\delta(O_T(x,y)),\quad\forall x,y\in X.$$

(21) $\Rightarrow$ (20). For any $\varepsilon>0$, there exist $\alpha(\varepsilon)>0$ and $\varepsilon_0(\varepsilon)\in(0,\varepsilon)$ such that for any $x,y\in X$,

$$0<\varepsilon\leq\delta(O_T(x,y))<\varepsilon+\alpha(\varepsilon)\quad\text{implies}\quad d(T^px,T^qy)\leq\varepsilon_0(\varepsilon).$$

Suppose that there exist $x,y\in X$ such that $\delta(O_T(x,y))\in(0,\varepsilon)$. Put $\epsilon'=\min\{\delta(O_T(x,y)),\varepsilon_0(\varepsilon)\}$. In view of (21) we know that there exist $\alpha(\epsilon')>0$ and $\epsilon_0(\epsilon')\in(0,\epsilon')$ such that

$$0<\epsilon'\leq\delta(O_T(x,y))<\epsilon'+\alpha(\epsilon')<\varepsilon+\alpha(\varepsilon)$$

implies

$$d(T^px,T^qy)\leq\epsilon_0(\epsilon')<\epsilon'\leq\varepsilon_0(\varepsilon).$$

Thus, (20) is fulfilled.

Similarly, we can prove that (24) $\Rightarrow$ (23), (27) $\Rightarrow$ (26) and (30) $\Rightarrow$ (29). This completes the proof. □

Remark 2.1. Theorem 2.1 generalizes Theorem 1 in [9] and Theorem 1 in [10].

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