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A LINEARIZED MOSER-TRUDINGER INEQUALITY ON THE UNIT DISC B_2

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Abstract: Classical Sobolev inequalities was used by Beckner to generalize the linearized Moser-Trudinger inequality for the sphere S^2. By using the same inequality we observe sharp form of the linearized Moser-Trudinger inequality on B_2 by using symmetrization and rearrangement.

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1. Introduction

As a limiting case of the Sobolev-Embedding Theorem, see [14], Moser introduced the Moser-Trudinger inequality in [10]. The linearized form of the inequality on the two dimensional sphere with sharp constant played an important role in some geometric analysis and non linear PDE problems (see [4], [5], [12], [13]). As an example, Onofri [12] used the inequality to solve the partial differential equation $\Delta u + e^u - 1 = 0$ (on S^2) which also can be understood as an log-determinant problem for the Laplacian and Osgood-Philips-Sharnack [13] showed analogue applications. In [3], Beckner genealized the linearized Moser-Trudinger inequality for the sphere S^n and his result [4] with sharp constant have provided the significent information of the geometric analysis.

To obtain similar results, the interest on the linearized Moser-Trudinger in-

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equality for the domain in $\Re^n$ is grown and from the Moser-Onofri inequality[12] on the sphere

$$\log \int_{S^2} e^F d\xi \leq \log_{S^2} F d\xi + \frac{1}{4}\int_{S^2} |\nabla F|^2 d\xi\,.$$

Beckner first introduced the following linearized Moser-Trudinger inequality

$$\log \frac{1}{\pi}\int_{|x|\leq 1} e^{2f} dx + (\frac{1}{\pi}\int_{|x|\leq 1} e^{2f} dx)^{-1} \leq 1 + \frac{1}{4\pi}\int_{|x|\leq 1} |\nabla f|^2 dx$$

for a non-negative function f with zero boundary-value on the unit disk in $\Re^2$. In [9], we also obtained the following inequality

$$\log \frac{1}{\pi}\int_{|x|\leq 1} e^{2f} dx \leq 1 + \frac{1}{4\pi}\int_{|x|\leq 1} |\nabla f|^2 dx\,,$$

by using Carleson-Chang's method [7], which was used to show the existence of the extremal function for an inequality of J. Moser.

In this paper, we obtain a linearized Moser-Trudinger inequality for the unit ball by using the following classical Sobolev inequality S^n

$$(\int_{S^n} |F(\xi)|^q d\xi)^{\frac{2}{q}} \leq \frac{q-2}{n}\int_{S^n} |\nabla F|^2 d\xi + \int_{S^n} |F(\xi)|^2 d\xi$$

as an equivalent form of the classical L^2 Sobolev inequality on $\Re^n$ (see [2]) and is is deeply related to geometric analysis. In the inequality $d\xi$ denotes normalized surface measure and ∇ is the conformal gradient on S^n and where $2 \leq q \leq \frac{n}{n-2}$ $(n \geq 3)$, $2 \leq q < \infty$ $(n = 1, 2)$. Beckner [2] recognized above inequality by transforming the classical L^2 Sobolev inequality to the sphere S^n.

In the proof of our results, as a result of linearized Moser-Trudinger inequality for the Euclidean domain, we used symmetrization and rearrangement technique and our method emphasizes the connection between the exponential class Sobolev inequalities and the classical Sobolev inequalities.

2. Preliminary Result

Here we present symmetrization technique (see [10]) and some basic results derived from it which also will be used in the proof of our theorem to change one dimensional result (7) to a general one. For each function $u \in C^1(\Omega)$, associate a function u^* depending on $|x|$ with

$$m(\{x : u^* > \rho\}) = m(\{x \in \Omega : |u(x)| > \rho\}) \tag{1}$$

for every $\rho \geq 0$. Thus u^* is an radial decreasing function defined in the ball

$$\Omega^* = \{x : |x| \leq R, \int_{|x|\leq R} dx = m(\Omega)\}.$$

Thus by the property of the distribution function

$$m\{x \in \Omega : |u(x)| > \rho\},$$

we have

$$\int_\Omega |u|^q dx = \int_{\Omega^*} |u^*|^q dx.$$

In the second step of the proof, we shall use symmetrization [10] and the result used by E.H. Lieb.

Lemma 1. (see E.M. Lieb [11]) *Let* $u \in H^1(\Re^n)$, *then*

$$u^* \in H^1(\Re^n)$$

and

$$\|\nabla u^*\|_2 \leq \|\nabla u\|_2. \tag{2}$$

3. Main Result

Theorem 1. *Let* Ω *be the domain containing the unit ball in* $\Re^2$ *and* $u \in C^1(\Omega)$ *then*

$$\log \frac{1}{\pi} \int_{|x|\leq 1} e^{2u} dx \leq \frac{2}{\pi} \int_{|x|\leq 1} u dx + \frac{1}{4\pi} \int_{|x|\leq 1} |\nabla u|^2 dx.$$

To obtain above exponential Sobolev inequality from the classical Sobolev inequality the first step is mapping the one variable problem on the sphere S^2 to the one variable problem on the ball B_2 and then take the limit $q \to \infty$.

On S^2, the polar angle θ $(0 \leq \theta \leq \pi)$ is defined by $\xi_1 = \cos\theta$, where $\xi = (\xi_1, \xi_2, \xi_3) \in S^2$. When we change the problem as an one dimensional one on $\Re^2$ from the classical Sobolev inequality

$$\int_{S^2} |F(\xi)|^q d\xi \leq (\frac{q-2}{2} \int_{S^2} |\nabla F|^2 d\xi + \int_{S^2} |F(\xi)|^2 d\xi)^{\frac{q}{2}} \tag{3}$$

with $2 \leq q < \infty$, we restrict F to be a function of polar angle θ and set

$$2(1 - \cos\theta) = |x|^2, \quad x \in \Re^2\,, \tag{4}$$

$$F(\theta) = f(\frac{|x|}{2})\,. \tag{5}$$

So, we also have

$$|\frac{d|x|}{d\theta}| \leq 1. \tag{6}$$

Since our function F depends only on θ, notice the fact that we can evaluate the integrals on the sphere by first integrating over the parallel $Z_\theta = \{\xi \in S^2 : e \cdot \xi = \cos\theta\}$ orthogonal to $e = (1, 0, 0)$ and where the measure of Z_θ is $2\pi \sin\theta$ (i.e. $\int_{S^2} |F(\xi)| d\xi = \frac{1}{2} \int_0^\pi F(\theta) \sin\theta d\theta$). Then, by (3), (4) and (5), f is radial and

$$\int_{S^2} |\nabla F|^2 d\xi \leq \frac{1}{2} \int_0^2 |\frac{df}{dr}(\frac{r}{2})|^2 r dr = \frac{1}{4\pi} \int_{B_2} |\nabla f|^2 dx.$$

Similarly, from (2), we obtain the following inequality for the radial function f defined on B_2

$$\frac{1}{\pi} \int_{B_2} |f|^q dx \leq (\frac{q-2}{8\pi} \int_{B_2} |\nabla f|^2 dx + \frac{1}{\pi} \int_{B_2} |f|^2 dx)^{\frac{q}{2}}\,, \tag{7}$$

where $2 \leq q < \infty$.

Now, in the following lemma we obtain an exponential-class Sobolev inequality from (6) by choosing an argument which was used by Beckner [3] in the study of the same type inequalities for the sphere.

Lemma 2.

$$\frac{1}{\pi} \int_{|x| \leq 1} e^g dx \leq \exp(\frac{1}{\pi} \int_{|x| \leq 1} g dx + \frac{1}{16\pi} \int_{|x| \leq 1} |\nabla g|^2 dx) \tag{8}$$

for a radial function $g \in C^1(B_2)$.

Proof. In (6), without loss of generality set $f = 1 + \frac{1}{q} g$. Then the resulting inequality has the form

$$\begin{aligned}\frac{1}{\pi} \int_{B_2} |1 + \frac{1}{q} g|^q dx \quad &\leq \quad 1 + \frac{q}{2}[\frac{1}{\pi} \int_{B_2} g dx \\ &+ \frac{1}{2q\pi} \int_{B_2} g^2 dx + \frac{q-2}{16\pi q} \int_{B_2} |\nabla g|^2 dx]^{\frac{q}{2}}\end{aligned}$$

and since $2 \le q < \infty$, by taking the limit $q \to \infty$ we proved (7). □

Proof of Theorem 1. From $u|_{B_2}$ symmetrization produces a function $u^*(x)$ which is radial on B_2. By Lemma 2 above it suffices to prove the following inequality

$$\log \frac{1}{\pi} \int_{B_2} e^{2u^*} dx \le \frac{2}{\pi} \int_{B_2} u^* dx + \frac{1}{4\pi} \int_{B_2} |\nabla u^*|^2 dx \tag{9}$$

and we already proved it in Lemma 2. □

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