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THE COMPLEXITY OF LOOP ERASED WALK IN Z^3

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Abstract: Let S_n be a simple random walk (SRW) defined on Z^3. We construct a stochastic process from S_n by erasing loops of length at most N^α, where $\alpha \in (0, \infty]$ and N is the scaling parameter that will be taken to infinity in determining the limiting distribution. We call this process the N^α loop erased walk (N^α LEW). Under some assumptions we will prove that for $0 < \alpha < \frac{1}{1+2\zeta}$, the limiting distribution is Gaussian. Here ζ is the intersection exponent of random walks in Z^3. For $\alpha > 2$ the limiting distribution is equal to the limiting distribution of the loop erased walk ($\alpha = \infty$). It is known that $.25 < \zeta < .5$. We conjecture that for $\alpha < 2$, the limiting distribution of N^α LEW is Gaussian and hence the critical value is $\alpha_c = 2$. Our result implies that the complexity of simulating an N-step loop erased walk on Z^3 has a deterministic uniform upper bound $O(N^3)$ and lower bound $O(N^{3/2})$.

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1. Introduction

One of the most studied subjects in statistical mechanics and probability theory

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is phase transition and its associated critical phenomena. Models such as self-avoiding random walks, percolation, oriented percolation, loop-erased random walks have been extensively studied. All of these models have been proved to have a critical dimension d_c and exhibit an interesting behavior near d_c, so-called universality. For self-avoiding random walks, Hara and Slade [6] show that for $d > 4$ the scaling limit is Gaussian. It is believed that the scaling limit of self-avoiding random walk is non-Gaussian if $d < 4$. For percolation, Hara and Slade [7] show that the critical exponents take their mean-field values if d is sufficiently large and the same holds for spread-out model if $d > 6$. The critical dimension for this model is conjectured $d_c = 6$. For oriented percolation in $d+1$ dimensions, Nguyen and Yang [14, 15] show that the scaling limit is Gaussian if d is sufficiently large and the same holds for spread-out model for $d > d_c = 4$. For a long-range oriented percolation in $d+1$ dimensions with $1/n^2$-interaction, Chen and Shieh [3] show that the scaling limit is a Cauchy distribution if d is sufficiently large and the same holds for its spread-out model for $d > d_c = 2$. For loop-erased random walk, it is proved by Lawler [9] that the scaling limit is Gaussian if $d \geq 4$; it is conjectured that the scaling limit is non-Gaussian if $d < 4$. For above the critical dimensions, expansion methods are used to obtain scaling limits and the mean-field exponents, see e.g., Yang and Zeleke [17] for an expository of the expansion method. The scaling limits and critical exponents of these models are generally unknown below their respective critical dimensions, except for some trivial cases, e.g., self-avoiding random walk in 1 dimension.

A common feature of such models to have critical dimensions and to exhibit critical phenomena is that they must have a strong interaction between random variables which are far apart. A deterministic complexity of a correct simulation for the models is usually exponential.

For self-avoiding random walks below critical dimensions, some ideas of efficient algorithms have been considered; see e.g., [4, 12, 16]. An algorithm is used in Guttmann and Bursill [5] to generate loop-erased walks in 2 and 3 dimensions for numerical studies of critical exponents. However there is no discussion about the deterministic complexity of the algorithm.

A technique to overcome the strong interactions between random variables which are far apart is to introduce a cut-off for the interactions and gradually remove the cut-off when scaling limit is taken. One then studies the effect of such cut-offs by showing either the cut-off model converges to the original model in the scaling limit or otherwise converges to something else. If the cut-off model converges to the same scaling limit as that of the original model, then the complexity of computer simulations is reduced.

In this paper, we will carry out the above idea for loop-erased walk on Z^3 to obtain an upper and lower bound of the deterministic complexity of loop-erased walks. Here complexity is defined by the number of steps of simple random walk needed in order to generate an N-step walk such that it has the same limiting distribution as loop-erased walks. The simulated walks need not be loop free. We shall show that for loop-erased random walks in 3 dimensions, the complexity has a deterministic upper bound $O(N^3)$ and lower bound $O(N^{3/2})$.

Even though the limiting distribution of the loop erased walk in Z^3 is still unknown, a lot of work has been done in understanding the intersection of random walks in low dimensions, see [1, 2, 8, 9, 10, 11].

We will use the following notations for asymptotics. Suppose $f(x)$ and $g(x)$ are functions. The notation $f \approx g$ means that $\ln f \sim \ln g$, where we write $f \sim g$ if f and g are asymptotic, i.e. $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 1$.

Let S_n be a simple random walk taking values in Z^d. We call a time n loop free if $S[0,n] \cap S[n+1,\infty) = \phi$. Here $S[0,n] = \{S_j; 0 \le j \le n\}$. It is known that $P\{S[0,n] \cap S[n+1,2n] = \phi\} \approx n^{-\zeta}$, where $\zeta = \zeta_d$ is the intersection exponent if $d = 2$ or 3 (see [1]). Burdzy and Lawler [1] prove the existence of ζ. In [2] they prove the rigorous estimates $\frac{1}{4} \le \zeta_3 < \frac{1}{2}$ and $\frac{1}{2} + \frac{1}{8\pi} \le \zeta_2 < \frac{3}{4}$. Lawler [11] improve the estimate for ζ_3 to $\frac{1}{4} < \zeta < \frac{1}{2}$. Numerical studies suggest that $\zeta_3 \in (.28, .29)$, see [9].

To study the complexity of the loop-erased walk, we consider a stochastic process constructed from the SRW in Z^3 by erasing loops using only finite memory. At each step the first N^α loops will be erased (see Section 2 for the definition of N^α loops). Here $\alpha \in [0,\infty]$, and N is a scaling parameter which will be taken to ∞ in determining the limiting distribution of LEW. We call this process the N^α LEW. Note that $\alpha = 0$ is the case of SRW and $\alpha = \infty$ is that of LEW. Under some assumptions we will prove that the N^α LEW has a Gaussian distribution for $0 < \alpha < \frac{1}{1+2\zeta}$, where ζ is the intersection exponent of random walks in Z^3. For $\alpha > 2$ we will show that the N^α LEW has the same limiting distribution as the original LEW. It can be implied from our work that if there is a critical point α_c then it must be between $\frac{1}{1+2\zeta}$ and 2. The existence of α_c and the behavior of the N^α loop erased walk for $\frac{1}{1+2\zeta} < \alpha \le 2$ remain open. We conjecture that there exists a critical $\alpha_c = 2$ and that the behavior of the N^α loop erased walk for $\alpha < 2$ is Gaussian and for $\alpha > 2$ non Gaussian. For an N^α LEW, if we take $2N^{1+\alpha}$ steps of simple random walk, then there must be at least N steps of N^α LEW. Therefore, the complexity for LEW has upper bound $O(N^3)$ and lower bound $O(N^{3/2})$.

2. The N^α Loop Erased Walk

Let $\lambda = [S_i, S_{i+1}, ...S_j]$ be a segment of a path of an SRW. We say that λ forms an N^α loop if $S_i = S_j$ and $0 \leq |i-j| \leq N^\alpha$ for some fixed N and α. Let

$$\sigma_\alpha(0) = \sup\{j : S_j = 0, 0 \leq j \leq \ N^\alpha\},$$

and for $i > 0$

$$\sigma_\alpha(i) = \sup\{j > \sigma_\alpha(i-1) : S(j) = S(\sigma_\alpha(i-1)+1), |j - \sigma_\alpha(i-1) - 1| \leq \ N^\alpha\}.$$

Note that $\sigma_\alpha(i)$ is the last time the SRW visits site i in a "short" memory time, that is in at most N^α time frame. The N^α LEW is then defined by $\hat{S}_i^{(N)} = S_{\sigma_\alpha(i)}$. In this definition we have a loop erasing procedure in the sense that sites that are visited in "short" time frame are deleted from the path of the SRW. This loop erasing procedure allows us to erase short time memories while at the same time keeps sites that have already been visited after long time has elapsed. Long and short time memories are described in terms of N^α. However it is important to see that with this loop erasing procedure we may still have some small loops remaining after the first N^α loops have been erased. The following example demonstrates this fact. Let $S_0 = (0,0,0), S_1 = (1,0,0), S_2 = (2,0,0), S_3 = (2,1,0), S_4 = (1,1,0), S_5 = (1,0,0), S_6 = (1,-1,0), S_7 = (1,0,0), S_8 = (0,0,0)$. Suppose we erase sites whose return visit time is less than or equal to 4. With this condition, the points that remain are $S_0, S_1, S_2, S_3, S_4, S_7$ and S_8. The loops between S_0 and S_8 as well as S_1 and S_7 are both short loops. But long time has elapsed to form these loops. The loop erasing algorithm described above to generate $\hat{S}_N^N$ requires only finite memory depending on N^α.

Our goal is to find $\lim_{N\to\infty} \frac{\hat{S}_N^{(N)}}{N^\gamma}$, for some γ. We say that n belongs to an N^α loop if there exists i and j such that $i \leq n \leq j$ with $S_i = S_j$ and $|i-j| \leq N^\alpha$. For each n we say n is N^α loop free if n does not belong to an N^α loop. Suppose n is N^α loop free. Then loop erasing before n and after n are independent. If n is N^α loop free, then n is not erased. However the converse is not in general true. In order to analyze the behavior of $\hat{S}_N^{(N)}$ for large N we need to investigate how many steps of the SRW remain after the first N^α loops have been erased.

Notation. For the convenience of writing the notations, when α is fixed, from time to time we shall drop α in notations; for examples, $\sigma(i)$ means $\sigma_\alpha(i)$, $\rho(j)$ means $\rho_\alpha(j)$ and Y_n means $Y_{n,\alpha}$ as seen below.

Let $\rho_\alpha(j) = i$ if $\sigma(i) \le j < \sigma(i+1)$. Then, $\rho_\alpha(\sigma(i)) = i,$ if $\sigma(\rho_\alpha(j)) \le j$. Let $Y_{n,\alpha} = 1$ if $\sigma(i) = n$ for some $i \ge 0$, and $Y_{n,\alpha} = 0$ otherwise. Then $\rho(n) = \sum_{j=0}^{n} Y_j$ is the number of points remaining of the first n points after the first N^α loops are erased. Let $a_{n,\alpha} = E(Y_{n,\alpha})$ be the probability that the n^{th} point is not erased. For the asymptotic behavior of $\rho_\alpha(N)$, we have,

Theorem 2.1. *For $0 < \alpha < \frac{1}{1+2\zeta}$, $\frac{\rho_\alpha(N)}{Na_{N,\alpha}} \to 1$ in probability as $N \to \infty$.*

Lawler [9] has proved analogous results in higher dimensions for $\alpha = \infty$.

Our next result is about the limiting distribution of the N^α LEW. Let F_N be defined by $F_N = [\sigma_\alpha(N)a_{\sigma_\alpha(N)}]$. Here by $\lceil\cdot\rceil$ we mean the greatest integer function. Then we have the following result.

Theorem 2.2. *Let $0 < \alpha < \frac{1}{1+2\zeta}$.*

(a) $\lim_{N\to\infty} \frac{S_{F_N}}{\sqrt{N}} = \Phi$, in distribution , where Φ is a normal random variable.

(b) Suppose $a_{N,\alpha} \sim \mathrm{const}N^{-q}$, for some $q > 0$. Let $\tau_N = N^{-q/(1-q)}$.

Then $\lim_{N\to\infty} \frac{S_{\sigma_\alpha(N)}\sqrt{\tau_N}}{\sqrt{N}} = \Phi$, in distribution.

Clearly, q satisfies $0 < q \le \alpha\zeta$. However, we were unable to prove the existence of q. For a sufficiently large α we have the following theorem.

Theorem 2.3. *Let $c_N = (E(|S_{\sigma_\alpha(N)}|^2))^{1/2}$, and $d_N = (E(|S_{\sigma_\infty(N)}|^2))^{1/2}$. Suppose that $\frac{S_{\sigma_\alpha(N)}}{c_N}$ or $\frac{S_{\sigma_\infty(N)}}{d_N}$ converges in distribution, as N goes to ∞. If $\alpha > 2$, then $\lim_{N\to\infty} \frac{S_{\sigma_\alpha(N)}}{c_N} = \lim_{N\to\infty} \frac{S_{\sigma_\infty(N)}}{d_N}$, in distribution.*

3. Proofs

For $0 \le j < k < \infty$, we denote by $Z(j,k)$ the indicator function of the event "there is no N^α loop free point between j and k including j and k".

Lemma 3.1. *There exist constants c_1, c_2 such that if $\beta > \alpha$, then $E(Z(k-N^\beta, k)) \le c_1 e^{-c_2 N^{\beta-\alpha}}$.*

Proof. From Theorem 1.1 of [10] it follows that there is a c_3 such that in the interval $[k - 4N^\alpha, k]$ the probability of an N^α loop free point is at least c_3. Consider now an interval I of length N^β divided into $\frac{1}{4}N^{\beta-\alpha}$ small intervals of length $4N^\alpha$. Then the probability of no N^α loop free point in I is bounded by

$(1-c_3)^{\frac{1}{4}N^{\beta-\alpha}}$ which can be written in the form $c_1 e^{-c_2 N^{\beta-\alpha}}$. □

Proof of Theorem 2.1. For each N, choose $0 \leq j_0 < j_1 < j_2 < ... < j_m = N$, such that $j_i - j_{i-1} \sim N^{1-\alpha\zeta-\delta}$, uniformly in I. Then $m \sim N^{\alpha\,\zeta+\delta}$. Erase loops on each interval $[j_i, j_{i+1}]$ separately. Let $\tilde{Y}_k$ be the indicator function of the event "S_k is not erased in this finite loop-erasing". Let $K_0 = [0,0]$, and $\epsilon_1 > \delta$. Then, for $i = 1, ..., m$, define the intervals K_i and $K_i^{'}$ by $K_i = [j_i - N^{1-2\alpha\zeta-\epsilon_1}, j_i]$, $K_i^{'} = [j_i, j_i + N^{1-2\alpha\zeta-\epsilon_1}]$. Let R_i, $i = 1, ..., m$, be the indicator function of the event $\{\exists$ no N^α loop free point in $K_i^{'}$ or in $K_{i+1}\}$. Note that $R_i = 0$ if and only if $\exists N^\alpha$ loop free point in $K_i^{'}$ and in K_{i+1}. Thus if $j_i + N^{1-2\alpha\zeta-\epsilon_1} \leq k \leq j_{i+1} - N^{1-2\alpha\zeta-\epsilon_1}$ and $R_i = 0$, then $Y_k = \tilde{Y}_k$. Therefore for a sufficiently large N,

$$|\sum_k Y_k - \tilde{Y}_k| \leq 2N^{1-\alpha\zeta-\epsilon_1+\delta} + 2N^{1-\alpha\zeta-\delta} \sum_i R_i. \tag{3.1}$$

Let $\lambda = 1 - 2\alpha\zeta - \epsilon_1 - \alpha$. Then,

$$P\{\sum_i R_i \geq \frac{1}{4} N^\gamma\} \leq 4c_1 e^{-c_2 N^\lambda} N^{\alpha\zeta+\delta-\gamma}. \tag{3.2}$$

Since ϵ_1 is arbitrary, for $\alpha < \frac{1}{1+2\zeta}$, $\lambda > 0$ and the right side of (3.2) goes to 0 as $N \to \infty$. Let now $\epsilon_2 \ll \min\{\epsilon_1 - \delta; \frac{\delta}{2}\}$. Then using (3.1) we get

$$P\{\sum_k Y_k - \tilde{Y}_k \geq N^{1-\alpha\zeta-\epsilon_2}\} \leq P\{\sum_i R_i \geq \frac{1}{4} N^{\delta-\epsilon_2}\}. \tag{3.3}$$

Put $\delta - \epsilon_2 = \gamma$. Then (3.3) goes to 0 by (3.2). From (3.3) it follows that $\frac{1}{Na_N} \sum_k Y_k - \tilde{Y}_k \to 0$ in probability. We can write $\sum_k \tilde{Y}_k = 1 + \sum_i X_i$, where X_i are the independent random variables, $X_i = \sum_{k=j_{i-1}}^{j_i - 1} \tilde{Y}_k$. Then, using (3.2) and Chebyshev's Inequality, we can show,

$$\frac{1}{E(\sum_k \tilde{Y}_k)} \sum_k \tilde{Y}_k \to 1 \text{ in probability.}$$

By (3.2) and Lemma 3.1, it follows that $E(\sum_{k=0}^{N} \tilde{Y}_k) \sim Na_N$, completing the proof of the theorem. □

Proposition 3.1. *Let $0 < \alpha < \frac{1}{1+2\zeta}$. Let $\sigma(N) = \sigma_\alpha(N)$ be defined as in Section 2. Then:*

(a) $\frac{\sigma(N)a_{\sigma(N)}}{N} \to 1$ *in probability as* $N \to \infty$.

(b) Assume $a_N \sim \frac{1}{N^q}$ *for some* $q > 0$ *and let* $\tau_M \sim M^{-q/(1-q)}$. *Then* $\frac{\sigma(M)\tau_M}{M} \to 1$, *in probability as* $M \to \infty$.

Remark. The following argument helps explain why it is reasonable to assume $a_N \sim \frac{1}{N^q}$ for some $q > 0$.

Let S^1 and S^2 be two independent simple random walks with killing rate $1-\lambda, \lambda \in (0,1]$. Let

$$g(\lambda) = P\{S^1_i \neq S^2_j, (0,0) \prec (i,j) \prec (N^\alpha, N^\alpha)\}.$$

Here $(a_1, a_2) \preceq (b_1, b_2)$ means $a_1 \leq b_1$, $a_2 \leq b_2$ and $(a_1, a_2) \prec (b_1, b_2)$ if $(a_1, a_2) \preceq (b_1, b_2)$ but $(a_1, a_2) \neq (b_1, b_2)$. We let R_λ be the number of intersection times, i.e.

$$R_\lambda = \sum_{i=0}^{N^\alpha} \sum_{i=0}^{N^\alpha} 1_{\{S^1_i = S^2_j, (i,j) \preceq (N^\alpha, N^\alpha)\}}.$$

Our goal is to estimate the growth rate of $E(R_\lambda)$. Let $p_n(0)$ be the probability a simple random walk starting from 0 returns to 0 in n steps. For $x \in Z^3$, let $\overline{p}_0(x) = \delta(x)$, where $\delta(x) = 1$ if $x = 0$, and 0 otherwise. For $n > 0$, let

$$\overline{p}_n(x) = 2(\frac{3}{2\pi n})^{3/2} e^{\frac{-3|x|^2}{2n}}.$$

We define the error term $E(n,x)$ by $E(n,x) = p_n(x) - \overline{p}_n(x)$ if $n + x_1 + x_2 + x_3$ is even and 0 otherwise. Here x_1, x_2, x_3 are the coordinates of x. In the calculation below, we are going to use the bound for $E(n,x)$ formulated in [9], namely $|E(n,x)| \leq O(n^{-(d+2)/2})$, where d is the dimension of the lattice space. Then

$$\begin{aligned} E(R_\lambda) &= \sum_{i=0}^{N^\alpha} \sum_{i=0}^{N^\alpha} \lambda^{i+j} p_{i+j}(0) = \sum_{j=0}^{N^\alpha} \lambda^j (j+1) p_j(0) \\ &+ \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j p_j(0)(2N^\alpha + 1 - j) = \sum_{j=0}^{N^\alpha} \lambda^j j p_j(0) + \sum_{j=0}^{N^\alpha} \lambda^j p_j(0) \\ &+ \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j p_j(0)(2N^\alpha + 1) - \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j p_j(0) j = \sum_{j=0}^{N^\alpha} \lambda^j j p_j(0) + \sum_{j=0}^{2N^\alpha} \lambda^j p_j(0) \end{aligned}$$

$$+ 2N^\alpha \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j p_j(0) - \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j p_j(0) j.$$

We will now show that each summand of $E(R_\lambda)$ is of order $N^{\alpha/2}$. For the first summand we get the following estimate

$$\begin{aligned}\sum_{j=0}^{N^\alpha} \lambda^j(j) p_j(0) &= \sum_{j=0}^{N^\alpha} \lambda^j(j)(\overline{p}_j(0) + E_j(0)) \\ &\leq C \sum_{j=0}^{N^\alpha} \lambda^j(j) j^{-3/2} + O(j^{-5/2}) = C \sum_{j=0}^{N^\alpha} \lambda^j(j) j^{1/2} + O(j^{-5/2}) \sim CN^{\alpha/2}.\end{aligned}$$

The second summand is bounded by a constant since $\sum_{j=0}^{2N^\alpha} \lambda^j p_j(0) \leq G_\alpha(0) \leq C$, where $G_\alpha(0)$ is the Green function for random walks.

For summand three we get the following estimate

$$\begin{aligned}2N^\alpha \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j p_j(0) &= 2N^\alpha \sum_{j=N^\alpha+1}^{2N^\alpha} (\overline{p}_j(0) + E_j(0)) \\ &\leq C 2N^\alpha \sum_{j=N^\alpha+1}^{2N^\alpha} \frac{1}{j^{3/2}} + O(j^{-5/2}) \leq C \frac{N^{2\alpha}}{(N^\alpha + 1)^{3/2}} \sim N^{\alpha/2}.\end{aligned}$$

For summand four we get

$$\begin{aligned}\sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j p_j(0) j &= \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j j(\overline{p}_j(0) + E_j(0)) \\ &\leq C \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j j j^{-3/2} + O(j^{-5/2}) = \sum_{j=N^\alpha+1}^{2N^\alpha} \lambda^j j^{-1/2} + O(j^{-5/2}) \sim CN^{\alpha/2}.\end{aligned}$$

We conclude then $E(R_\lambda) \sim CN^{\alpha/2}$ and since $g(\lambda) \geq [E(R_\lambda)]^{-1}$, it is reasonable to assume $a_N \sim N^{-q}$, for some $q > 0$.

We now proceed with the proof of Proposition 3.1.

Proof of (a). Let $s > 0$ be a constant. It suffices to prove that $\frac{\sigma(M_t) a_{\sigma(M_t)}}{M_t}$ converges to 1 a.s. for any sequence $M_t \geq t^s$. By Theorem 2.1 there exists $\Omega' \subset \Omega$ such that $P(\Omega') = 1$ and $\frac{\rho_\alpha(N_t)}{N_t a_{N_t}} - 1 \to 0$, for all $\omega \in \Omega'$. Let N_t' be a sequence such that $N_t^{'} \geq t^s$. Then for a fixed t there exists a sequence ξ_t such that $\xi_t^s \leq N_t^{'} < (\xi_t+1)^s$. Note that $t \leq \xi_t$. For $\omega \in \Omega^{'}$,

$$\rho_\alpha(\xi_t^s)((\xi_t+1)^s a_{N_t^{'}})^{-1} \leq \rho_\alpha(N_t^{'})(N_t^{'} a_{N_t^{'}})^{-1} \leq \rho_\alpha(\xi_t+1)^s(\xi_t^s a_{N_t^{'}})^{-1}. \quad (3.4)$$

By Theorem 2.1 and (3.2) the upper and lower bounds of this inequality converge to 1 in probability. Substituting $\sigma(M_t)$ for $N_t^{'}$ gives $M_t(\sigma(M_t)a_{\sigma(M_t)})^{-1} \rightarrow 1$. □

Proof of (b). From (a) we have $\sigma(M_t)a_{\sigma(M_t)}(M_t)^{-1} \rightarrow 1$. By assumption, $\frac{\sigma(M_t)\sigma(M_t)^{-q}}{M_t} \rightarrow 1$. Therefore, $\frac{\sigma(M_t)(\omega)}{M_t^{1/(1-q)}} \rightarrow 1$, as $t \rightarrow \infty$. Since this holds for all $M_t \geq t^s$, $\sigma(N)(N^{1/(1-q)})^{-1} \rightarrow 1$ in probability. By Proposition 3.1a, $\frac{N^{1/(1-q)}}{\sigma(N)} \cdot \frac{\sigma(N)a_{\sigma(N)}}{N} \rightarrow 1$ in probability. This and Proposition 3.1a imply $a_{\sigma(N)}(\tau_N)^{-1} \rightarrow 1$ in probability. Using Proposition 3.1a again, we get, $\sigma(N)\tau_N(N)^{-1} \rightarrow 1$ in probability. Hence $[\sigma_\alpha(N)a_{\sigma_\alpha(N)}](N)^{-1} \rightarrow 1$ in probability. □

Proof of Theorem 2.2. The proof of Theorem 2.2 follows from the following Central Limit Theorem (Theorem 3.1) and Proposition 3.1. □

Theorem 3.1. *Let X_j be i.i.d. random variables with $E(X_j) = 0$ and $Var(X_j) = 1$. Let ν_n be positive integer valued random variables such that $\frac{\nu_n}{n} \rightarrow c$ in probability. Then $\frac{S_{\nu_n}}{\sqrt{cn^p}}$ converges in distribution to a standard normal random variable $\mathcal{N}$.*

Proof of Theorem 2.3. For each N, choose $0 \leq j_1 < j_2 ... < j_m$, satisfying $j_i - j_{i-1} \sim N^\alpha, N^s - N^t \leq j_i \leq N^s$. Let $X = \sum_{i=1}^{m} 1_{\{j_i\}}$. Then,

$$E(Z(N^s - N^t, N^s)) \leq c_1 e^{-c_2 N^{t-\alpha}}.$$

Consider the interval $[0, N^s]$ divided into subintervals of length $N^t, t < s$. Then,

$$\begin{aligned} P\{\rho_\infty(N^s) < N^{s-t}\} &\leq N^{s-t} P\{[N^s - N^t, N^s] \text{ has no loop free point}\} \\ &\leq c_1 N^{s-t} e^{-c_2 N^{t-\alpha}} \leq N^{s-2t+3\alpha/2}, \end{aligned}$$

and

$$\begin{aligned} P\{N^s < \sigma_\infty(N^{s-t})\} &\leq P\{\rho_\infty(N^s) < \rho_\infty(\sigma_\infty(N^{s-t}))\} \\ &= P\{\rho_\infty(N^s) < N^{s-t}\} \leq c_1 N^{s-t} e^{-c_2 N^{t-\alpha}}. \quad (3.5) \end{aligned}$$

Let $M = N^{s-t}$. Then, $P\{\sigma_\infty(M) > M^{\frac{s}{s-t}}\} \leq (c_1 M)e^{-c_2 M^{\frac{t-\alpha}{s-t}}}$. We show the L^2 norm of the difference of the N^α LEW and the LEW goes to 0. Let $e_N = \max\{c_N, d_N\}$. Then,

$$||\frac{S_{\sigma_\alpha(N)}}{c_N} - \frac{S_{\sigma_\infty(N)}}{d_N}||_2 \leq \frac{2 \cdot ||S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}||_2}{e_N}.$$

Let

$$\mathcal{S} = \{\omega \in \Omega : \exists \text{a loop between } i \text{ and } j,\ 0 \leq i \leq \sigma_{\alpha(N)}, |i-j| > N^\alpha\}.$$

Let $\mathcal{T}$ be the set of all $\omega \in \Omega$ such that there exists a loop between i and j, $0 \leq i \leq \sigma_{\alpha(N)}$, $\sigma_\infty(N) > N^{\frac{s}{s-t}}$, and $|i-j| > N^\alpha$. We also let

$$G = \{\omega \in \Omega : |\frac{S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}}{e_N}| > M\}.$$

Let $\mathcal{I}$ be the indicator function defined on $\mathcal{S}$. Then:

$$\begin{aligned}\frac{||S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}||_2^2}{e_N^2} &= E(\frac{|S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}|^2}{e_N^2} \cdot \mathcal{I}) \\ &= \int_G \frac{|S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}|^2}{e_N^2} \cdot \mathcal{I} dP + \int_{G^c} \frac{|S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}|^2}{e_N^2} \cdot \mathcal{I} dP. \quad (3.6)\end{aligned}$$

Then $\forall \epsilon > 0\ \exists\ M_0$ such that $\forall M \geq M_0$,

$$\begin{aligned}(\int_\Omega \frac{|S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}|^2}{e_N^2})^{1/2} &\leq \frac{1}{e_N}(||S_{\sigma_\infty(N)}||_2 + ||S_{\sigma_\alpha(N)}||_2) \\ &\leq \frac{||S_{\sigma_\infty(N)}||_2}{d_N} + \frac{||S_{\sigma_\alpha(N)}||_2}{c_N} = 2 < \infty. \quad (3.7)\end{aligned}$$

By the Dominated Convergence Theorem, for all $\epsilon > 0$ there exists M_0 such that for all $M \geq M_0$,

$$\int_G \frac{|S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}|^2}{e_N^2} \cdot \mathcal{I} dP \leq \int_G \frac{|S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}|^2}{e_N^2} dP < \ \epsilon, \text{ if } M \geq M_0.$$

Consider now the second summand with $M = M_0$, we have

$$\int_{G^c} \frac{|S_{\sigma_\infty(N)} - S_{\sigma_\alpha(N)}|^2}{e_N^2} \cdot \mathcal{I} dP \leq M_0^2 \int_\Omega \mathcal{I} dp$$

$$= M_0^2 \cdot (E1_{\mathcal{T}^c} + E1_{\mathcal{T}}) \le M_0^2 \cdot \{E1_{\mathcal{T}^c} + c_1 N^{s-t} e^{-c_2 N^{t-\alpha}}\}$$

$$\sim M_0^2 \cdot \sum_{i=0}^{N^{\frac{s}{s-t}}} \sum_{j=N^\alpha}^{\infty} \frac{1}{|j-i|^{3/2}} + N^{\frac{s-2t+3\alpha/2}{s-t}} \le M_0^2 \Big(\frac{N^{\frac{s}{s-t}}}{N^{\alpha/2}} + c_1 N^{s-t} e^{-c_2 N^{t-\alpha}}\Big).$$

For $\alpha > 2$ there exist s and t such that the last term goes to 0. □

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