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THE $BMAP/PH/\infty$ QUEUE AS AN INFINITE SERVER NETWORK

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Abstract: We consider the $BMAP/PH/\infty$ queue in its most general form as an open network of w infinite-server stations with exponential service time distributions and Markov routing. Customers arrive from outside the network according to a spatial BMAP as defined in [5]. As a consequence, the starting vector of any phase type service time distribution in the $BMAP/PH/\infty$ station is customer specific and can depend on the batch size as well as the phase transition of the underlying BMAP phase process. We present results for the transient as well as the stationary behavior of the network by deriving expressions for the joint distributions and generating functions of customer numbers at arbitrary network stations. The approach is based on former studies on the $BMAP/G/\infty$ queue [2, 5]. It supplements the detailed analysis of the $BMAP/PH/\infty$ queue given by Masuyama and Takine [12]. The network oriented analysis may be relevant for telecommunications performance analyses.

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1. Introduction

The $BMAP/PH/\infty$ queue can be regarded as an open network of infinite-server (IS) stations with exponentially distributed service times and batch Markovian

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arrival process (BMAP) — and vice versa. Each network node corresponds to a transient state associated with the *PH*-service time distribution. This *network point of view* features some advantages in comparison with other approaches since it provides a more detailed insight in the course of events. We shall show that variants of the $BMAP/PH/\infty$ queue can be treated by similar analysis techniques.

Traditional investigations proceed from the more general $GI/G/\infty$ queue as studied in several papers [2, 5, 6, 9, 10, 11, 13, 14]. The most detailed analysis (with emphasis on numerically tractable solutions) is given by Masuyama and Takine [12]. It is based on a matrix differential equation for the time-dependent matrix joint generating function of the number of customers of different classes in the system. Classes are distinguished by different PH - service time distributions. In contrast to that in [5] a convolutional differential equation for sequences of transient state probability matrices was set up for $BMAP/G/\infty$ systems which describes the dynamics of the random variable of customer numbers in the system.

This convolutionary differential equation — basically an aggregate formulation of Chapman-Kolmogorov forward equations — provides an elegant representation of transient and stationary state probability distributions. Its main drawback has to be seen in the fact that its solution is numerically extremely expensive, although the corresponding generating functions are somewhat easier to handle. By use of a *spatial* BMAP representing the random environment of the infinite server network we achieve expressions for the marginal distributions of customer numbers at separate network nodes as well as joint distributions — results which, on the whole, comprise those of [12]. In fact, the obtained generating functions differ from those in [12] in that not only class specific aspects are included (a class being determined by the starting phase of service) but also the respective actual service phases at observation epochs are given. Spatial BMAPs (SMAPs) have been defined in [5] and [7, 8]. Their use corresponds to the assumption that different BMAP streams are allowed to enter the $BMAP/PH/\infty$ station.

The article is organized as follows. In Section 2 we describe the characteristics of the network model and its external arrival process. In Section 3 the convolutionary differential equation is set up and its solution is deduced. Section 4 is devoted to generating functions and results in terms of $BMAP/PH/\infty$ analysis. In Section 5 we outline a method to treat numerically the complex computation problem, and in Section 6 a short summary is given.

Throughout we designate the set of all non-negative integers by $\mathbb{N}_0$, the set of all positive integers by $\mathbb{N}$, the set of all reals by $\mathbb{R}$, and the set of all non-negative reals by $\mathbb{R}_0$. Further we use the following notations and definitions: Vectors and matrices are marked by bold face lower case letters and capital letters, respectively, with I the identity matrix and O the null matrix. For the family $\boldsymbol{M}(w,k)$ of all $(w \times k)$-matrices with non-negative integer entries we assume that a Gödel-numbering $\boldsymbol{M}(w,k) \to \mathbb{N}_0$ is performed (where, in particular, O is mapped on 0) such that matrix-indexed sequences $\mathcal{A} = \{A_M\}_{M\in\boldsymbol{M}(\mathrm{w,k})}$, $\mathcal{B} = \{B_M\}_{M\in\boldsymbol{M}(\mathrm{w,k})}$ of $(m\times m)$-matrices A_M, B_M can be considered for which a discrete convolution is defined by[1] $(\mathcal{A} * \mathcal{B})_M = \sum_{L\le M} A_L B_{M-L}$.

The family of all sequences of that type forms a semi-group $\mathcal{F}_{(m\times m)}$ with respect to the operator "$*$", with unit element $\mathbf{1} = \{I, O, O, \ldots\}$ and null element $\mathbf{0} = \{O, O, O, \ldots\}$. Over this semi-group a norm $\|\cdot\|^* : \mathcal{F}_{(m\times m)} \to \mathbb{R}_0$ is declared by $\|\mathcal{A}\|^* = \sum_{M\in\boldsymbol{M}(w,k)} \|A_M\|$, where $\|A_M\|$ is some proper matrix norm. We choose $\|A_M\| = \sum_{i,j\in E} |A_{M;ij}|$.

2. The Network of Phases

Consider an open network of w infinite-server (IS) stations that performs Markov routing with matrix $Q = (q_{h\ell})_{h,\ell\in\{1,\ldots,w,w+1\}}$, defining the transition kernel of some discrete time Markov chain with w transient states $1,...,w$ and one absorbing state $w+1$, the latter indicating the exterior of the network. q_{hw+1} is the probability to leave the network after service completion at station h with $q_{hw+1} = 1 - \sum_{\ell=1}^{w} q_{h\ell}$ for $h \in \{1,...,w\} =: V$. Service times at any IS station are assumed to be independent and exponentially distributed with mean intensity μ_h. As a consequence, a customer's travel through the network resembles a path of a continuous time Markov process with w transient states, one absorbing state $w+1$, and transition kernel $p_{h\ell}(t) = \left(1 - e^{\mu_h t}\right) q_{h\ell}$, $h \in V,\ \ell \in V \cup \{w+1\}$.

Let $\hat{G} = (g_{h\ell})_{h,\ell\in V\cup\{w+1\}}$ denote the generator of this Markov process, and set $G = (g_{h\ell})_{h,\ell\in V}$, such that

$$\hat{G} = \begin{pmatrix} G & \boldsymbol{a} \\ \mathbf{o} & 0 \end{pmatrix},$$

where $\mathbf{o}$ is the zero row vector $(0,\ldots,0)$, and $\boldsymbol{a} = (g_{1\,w+1}, \ldots, g_{w\,w+1})^T$ the column vector of instantaneous absorption rates. G is a regular $w \times w$-matrix due to the transience of states $1,\ldots,w$. Let further H_v denote the time that a customer who arrives from outside the network at node v spends in the network.

[1] $L \le M$ means $\ell_{\ell k} \le m_{\ell k}$ for all entries $\ell_{\ell k}$, $m_{\ell k}$ of the matrices L and M, respectively.

Then H_v is $PH(\boldsymbol{e}_v, G)$ phase type distributed with[2]

$$H_v(t) = \mathbb{P}(H_v \le t) = 1 - \boldsymbol{e}_v \exp(G\,t)\,\mathit{1}. \tag{1}$$

We propose that arrivals from outside the network occur according to a spatial Markovian arrival process (SMAP). In its most general form an SMAP is defined as a MAP whose arrival events represent the occurrences of random point fields in some Polish space $\mathbb{X}$ [5]. In our case $\mathbb{X}$ is made up by the node set V of the network graph, and a point field in $\mathbb{X}$ corresponds to some vector $\mathbf{n} = (n_1, \dots, n_w)$. The definition of the SMAP is performed in a more simpler form as follows.

Let $(N_t, J_t)_{t\ge0}$ be a common BMAP with additive component N_t, phase variable J_t, and state space $\mathbb{N}_0 \times E$, $E = \{1, ..., m\}$. Its generator $\mathcal{G}$ is given in block matrix form with entries $\mathcal{G}_{k\ell} = D_{\ell-k}$ for $\ell \ge k$, and $\mathcal{G}_{k\ell} = O$ for $\ell < k$, where any non-zero entry D_n is an $(m \times m)$-matrix $(D_{n;ij})_{i,j\in E}$ with $D_{n;ij} = \gamma_i\, p_i(n, j)$ if $(n, j) \ne (0, i)$, $D_{0;ii} = -\gamma_i$. Here $p_i(n, j)$, for $(n, j) \ne (0, i)$, is the probability for a batch arrival of size n together with a phase transition $i \to j$, and γ_i is the total transition rate of the Markov process (N_t, J_t) in phase i. The $p_i(n, j)$ satisfy $\sum_{\substack{j\in E\\ j\ne i}} \sum_{n=0}^{\infty} p_i(n, j) + \sum_{n=1}^{\infty} p_i(n, i) = 1$. The BMAP is called regular, if $\gamma_i > 0\ \ \forall i \in E$, and stable, if $\gamma = \max_{i\in E} \gamma_i < \infty$; we assume regularity and stability throughout. Moreover, we assume that the mean batch size is finite for each phase i, which implies $\| \sum_{n=1}^{\infty} nD_n\| \le \sum_{n=1}^{\infty} n \sum_{i,j\in E} D_{n;ij} < \infty$. In order to construct the spatial version of this BMAP let U_n, for any $n \in \mathbb{N}_0$, denote the set of all non-negative integer vectors $\mathbf{n} = (n_1, \dots, n_w) \in \mathbb{N}_0^w$ whose components sum to n. We write $\sigma(\mathbf{n}) = \sigma(n_1, \dots, n_w)$ for $\sum_{\nu=1}^{w} n_\nu$, i.e. $U_n = \{\mathbf{n} : \sigma(\mathbf{n}) = n\}$. On U_n define a family $\{\psi_{n;ij} : n \in \mathbb{N}_0,\ i, j \in E\}$ of probability measures such that $\psi_{n;ij}(\mathbf{n})$ is to be interpreted as the probability for the event that an arriving batch of size n in coincidence with a phase transition $i \to j$ is located in $S = \{k_1, \dots, k_\ell\} \subset V$ iff $n_{k_\nu} > 0$ for $\nu \in \{1, \dots, \ell\}$, and $n_k = 0$ else, i.e. exactly n_i among all n arriving customers reach node i, $i = 1, \dots, w$. Consider a block matrix $\tilde{\mathcal{G}}$ with entries $\tilde{\mathcal{G}}_{\mathbf{n},\mathbf{m}} = D_{\mathbf{m}-\mathbf{n}}$ for $\mathbf{n} \le \mathbf{m}$, and $\tilde{\mathcal{G}}_{\mathbf{n},\mathbf{m}} = O$ if $\mathbf{n} \not\le \mathbf{m}$, where any matrix $D_{\mathbf{n}}$ is defined through $D_{\mathbf{n};ij} = D_{\sigma(\mathbf{n});ij}\psi_{\sigma(\mathbf{n});ij}(\mathbf{n})\ \ \forall\ i, j \in E,\ \mathbf{n} \in \mathbb{N}_0^w$. Then the two-dimensional Markov process $(\mathbf{n}_t, J_t)_{t\ge0}$ with generator $\tilde{\mathcal{G}}$ is an SMAP with state space $\mathbb{N}_0^w \times E$ and rate matrices $D_{\mathbf{n}}$ for $\mathbf{n} \in \mathbb{N}_0^w$.[3] The vector $\mathbf{n}_t = (n_{t1}, \dots, n_{tw})$ marks the number of arrivals $n_{t\nu}$ at node $\nu \in V$ up to time t, $\nu = 1, \dots, w$. Let $P_{\mathbf{n}}(t)$

[2] Here $\boldsymbol{e}_v$, for $v \in V$, denotes the row w-vector that has a one as v-th entry and zeros everywhere else, and $\mathit{1}$ is the column w-vector with all entries equal to 1.

[3] Notice, that $\sum_{\mathbf{n}\in\mathbb{N}_0^w} D_{\mathbf{n}} = \sum_{n\in\mathbb{N}_0} D_n = D$, the generator of the phase process $(J_t)_{t\ge0}$.

denote the transition matrix of this SMAP, i.e. $P_{\mathbf{n}}(t) = (P_{\mathbf{n};ij}(t))_{i,j\in E}$ with $P_{\mathbf{n};ij}(t) = \mathbb{P}(\mathbf{n}_t = \mathbf{n}, J_t = j \mid \mathbf{n}_0 = \mathbf{0}, J_0 = i)$.

Setting $\Pi(t) := \{P_{\mathbf{n}}(t)\}_{\mathbf{n}\in\mathbb{N}_0^w}$ and $\Delta := \{D_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^w}$, it is well known (see [3]) that

$$\Pi(t) = e^{*\Delta\, t} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!}\Delta^{*\nu}, \tag{2}$$

where a ν-fold convolution $\Delta^{*\nu}$ of the sequence Δ with itself is defined through $\Delta^{*0} = \{I, O, O, \ldots\}$, $\Delta^{*\nu+1} = \Delta^{*\nu} * \Delta$. Expression (2) implies the validity of the important relation

$$P_{\mathbf{n}}(\tau) = \delta_{\mathbf{0n}}\, I + \tau D_{\mathbf{n}} + o(\tau) \ \text{ for small } \ \tau. \tag{3}$$

3. The Basic Differential Equation

Let $\mathbf{u} = (u_1, \ldots, u_k)$ be a vector of k different network nodes $(1 \le k \le w)$. For $K = \{1, \ldots, k\} \subset V$ let $X = ((x_{\nu\mu}))_{\nu\in V,\mu\in K}$ the $(w \times k)$-matrix whose entry $x_{\nu\mu}$ describes the number of customers who started in node ν at some time $\tau \le t$ and are resident in node u_μ at time t. Let further $q_{x_{\nu\mu};ij}(\tau, t; \mathbf{u})$ denote the probability to observe, at time $\tau \le t$, exactly these numbers $x_{\nu\mu}$, given that the process starts with an empty network and with the arrival process in phase i, and is in phase j at time τ, and let $Q_{X;ij}(\tau, t; \mathbf{u})$ be the respective joint probability. The corresponding $(m \times m)$-matrix is called $Q_X(\tau, t; \mathbf{u})$.

For any $(w \times k)$-matrix $Y = ((y_{\nu\mu}))$ with non-negative integer entries let $\beta_Y(\mathbf{n}; t-\tau; \mathbf{u})$ be the probability for the event that among n_ν customers who arrived in node ν at time $\tau \le t$, $y_{\nu\mu}$ will stay in node u_μ at time $t > 0$, $1 \le \nu \le w$, $1 \le \mu \le k$, $(n_1, \ldots, n_w) = \mathbf{n}$. Here $\beta_O(\mathbf{o}; t-\tau, \mathbf{u}) = 1$. We shall present exact expressions for the $\beta_Y(\mathbf{n}; t-\tau; \mathbf{u})$ at the end of this section.

Let us set

$$R_Y(t-\tau; \mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}_0^w} D_{\mathbf{n}} \cdot \beta_Y(\mathbf{n}; t-\tau; \mathbf{u}). \tag{4}$$

The matrices $R_Y(t-\tau; \mathbf{u})$, for obvious reasons, are called *t-resident rate matrices.* We further define the sequences $\mathcal{Q}(\tau, t; \mathbf{u}) = (Q_X(\tau, t; \mathbf{u}))_{X\ge O}$ and $\mathcal{R}(t-\tau; \mathbf{u}) = (R_Y(t-\tau; \mathbf{u}))_{Y\ge O}$ of $(m \times m)$-matrices.[4] Assuming that the network process starts with an empty network we have $Q_O(0, 0; \mathbf{u}) = I$ and $\mathcal{Q}(0, 0; \mathbf{u}) = \mathbf{1} = \{I, O, O, \ldots\}$.

[4] X, Y are elements of the family $\boldsymbol{M}(w, k)$ of $(w \times k)$-integer matrices with non-negative entries.

Theorem 3.1. *The probability matrices $Q_X(\tau, t; \mathbf{u})$ are obtained as component solutions of the convolutional differential equation*

$$\frac{\partial}{\partial \tau} \mathcal{Q}(\tau, t; \mathbf{u}) = \mathcal{Q}(\tau, t; \mathbf{u}) * \mathcal{R}(t - \tau; \mathbf{u}). \tag{5}$$

Proof. Observing

$$Q_X(\tau + \Delta\tau, t; \mathbf{u}) = \sum_{O \le L \le X} Q_L(\tau, t; \mathbf{u}) \sum_{\mathbf{n} \in \mathbb{N}_0^w} P_{\mathbf{n}}(\Delta\tau) \beta_{X-L}(\mathbf{n}; t - \tau; \mathbf{u})$$

and $\beta_O(\mathbf{0}; t - \tau, \mathbf{u}) = 1$ and exployting (3) we arrive at[5]

$$\begin{aligned} Q_X(\tau + \Delta\tau, t; \mathbf{u}) = \sum_{O \le L \le X} Q_L(\tau, t; \mathbf{u}) \sum_{\mathbf{n} \in \mathbb{N}_0^w} \Big(\delta_{\mathbf{0n}} I + \Delta\tau D_{\mathbf{n}} + o(\Delta\tau) \Big) \\ \cdot \beta_{X-L}(\mathbf{n}; t - \tau; \mathbf{u}) = Q_X(\tau, t; \mathbf{u}) + \Delta\tau \\ \cdot \sum_{O \le L \le X} Q_L(\tau, t; \mathbf{u}) R_{X-L}(t - \tau; \mathbf{u}) + o(\Delta\tau)\, M. \end{aligned}$$

$\Delta\tau \to 0$ yields $\frac{\partial}{\partial \tau} \mathcal{Q}(\tau, t; \mathbf{u}) = \sum_{O \le L \le X} Q_L(\tau, t; \mathbf{u}) R_{X-L}(t - \tau; \mathbf{u})$, which proves the assertion. □

The basic differential equation (5) has first been derived in [2] for some simpler cases and has been deduced in [5] from the Markov property of the underlying process. In fact, this Markov property guarantees differentiability of the $Q_X(\tau, t; \mathbf{u})$, and it turns out that (5) represents a component version of the respective Kolmogorov differential equation. Unfortunately, closed form solutions in pure matrix form are not achievable in general. Only for systems with mutually commuting rate matrices D_n or deterministic service analytic expressions are known [1, 2, 5].

We prove a preparational lemma next.

Lemma 3.1. *Let, for $0 \le \tau \le t$, the sequences $\mathcal{R}^{[\ell]}(\tau, t; \mathbf{u})$ of $(m \times m)$-matrices $R_M^{[\ell]}(\tau, t; \mathbf{u})$ be defined by $\mathcal{R}^{[0]}(\tau, t; \mathbf{u}) = \mathbf{1}$, and $\mathcal{R}^{[\ell+1]}(\tau, t; \mathbf{u}) = \int_0^\tau \mathcal{R}^{[\ell]}(\xi, t; \mathbf{u}) * \mathcal{R}(t - \xi; \mathbf{u})\, d\xi$ for $\ell \in \mathbb{N}_0$. Then, with $C = \|D_0\| + \sum_{n=1}^\infty n\|D_n\|$,*

$$\|\mathcal{R}^{[\ell]}(\tau, t; \mathbf{u})\|^* = \sum_{M \in \boldsymbol{M}(w,k)} \|R_M^{[\ell]}(\tau, t; \mathbf{u})\| \le \frac{C^\ell t^\ell}{\ell!}$$

for any $\ell \in \mathbb{N}_0$.

[5] M denotes the constant $(m \times m)$-matrix with all entries equal to 1.

Proof. According to $\beta_Y(\mathbf{n}; t-\tau; \mathbf{u}) = 0$ for $\sigma(\mathbf{n}) < \|Y\| = \sum_{i=1}^{w}\sum_{j=1}^{k} y_{ij}$,

$$\begin{aligned}
\|\mathcal{R}(t-\xi;\mathbf{u})\|^* &= \sum_{Y\geq O}\left\|\sum_{\mathbf{n}\in\mathbb{N}_0^w} D_{\mathbf{n}}\beta_Y(\mathbf{n};t-\xi;\mathbf{u})\right\| \\
&= \sum_{Y\geq O}\left\|\sum_{\sigma(\mathbf{n})\geq\|Y\|} D_{\mathbf{n}}\cdot\, \beta_Y(\mathbf{n};t-\xi;\mathbf{u})\right\| \leq \sum_{Y\geq O}\left\|\sum_{\sigma(\mathbf{n})\geq\|Y\|} D_{\mathbf{n}}\right\| \\
&= \|D_0\| + \sum_{\substack{Y\geq O\\ Y\neq O}}\sum_{\sigma(\mathbf{n})\geq\|Y\|}\|D_{\mathbf{n}}\| = \|D_0\| + \sum_{r=1}^{\infty}\sum_{n\geq r}\|D_n\| = \|D_0\| \\
&\qquad + \sum_{n=1}^{\infty}\sum_{r=1}^{n}\|D_n\| = \|D_0\| + \sum_{n=1}^{\infty} n\|D_n\|.
\end{aligned}$$

So, $\|\mathcal{R}(t-\xi;\mathbf{u})\|^* \leq C$ for any $\xi\in[0,t]$, and therefore

$$\begin{aligned}
\|\mathcal{R}^{[\ell+1]}(\tau,t;\mathbf{u})\|^* &\leq \int_0^\tau \left\|\mathcal{R}^{[\ell]}(\xi,t;\mathbf{u}) * \mathcal{R}(t-\xi;\mathbf{u})\right\|^* d\xi \\
&\leq \int_0^\tau \|\mathcal{R}^{[\ell]}(\xi,t;\mathbf{u})\|^*\cdot\|\mathcal{R}(t-\xi;\mathbf{u})\|^* d\xi \leq C\cdot\int_0^\tau \|\mathcal{R}^{[\ell]}(\xi,t;\mathbf{u})\|^* d\xi.
\end{aligned}$$

Observing $\int_0^\tau \mathcal{R}(t-\xi;\mathbf{u})d\xi = \mathcal{R}^{[1]}(\tau,t;\mathbf{u})$, the assertion follows by induction on ℓ. □

Lemma 3.1 implies that the series $\|\sum_{i=0}^{\infty}\mathcal{R}^{[\ell]}(\tau,t;\mathbf{u})\|^*$ converges uniformly for all $0\leq\tau\leq t$.

We are interested in a solution of (5) for the case $\tau = t$ in order to describe the state probabilities at time t. Set $Q_X(t,t;\mathbf{u}) = ((Q_{X;ij}(t,t;\mathbf{u})))_{i,j\in E}$ $\forall\ X\in \boldsymbol{M}(w,k)$, and $\mathcal{Q}(t,t;\mathbf{u}) = \mathcal{Q}(t;\mathbf{u})$. Recall that $Q_{X;ij}(t;\mathbf{u})$ describes the transient joint probabilities to observe, at time t, the arrival process in phase j and $x_{\nu\mu}$ customers in node u_μ among those who arrived before t in node ν, $1\leq\nu\leq w$, $1\leq\mu\leq k$, given that the process started with an empty system and with the arrival process in phase i. We say that the matrix sequence $\mathcal{Q}(t;\mathbf{u})$ describes the transient distribution of starting node dependent customer numbers in node vector $\mathbf{u}$. The stationary distribution is obtained by letting $t\to\infty$ in $\mathcal{Q}(t;\mathbf{u})$, i.e. $\lim_{t\to\infty}\mathcal{Q}(t;\mathbf{u}) = \mathcal{Q}(\mathbf{u}) = (Q_X(\mathbf{u}))_{X\in\boldsymbol{M}(w,k)}$, where each $(m\times m)$-matrix $Q_X(\mathbf{u})$ is made up of identical rows since the starting phase of the Markov additive arrival process has no impact on the steady state probabilities. The equilibrium distribution exists for each stable SMAP and each node subset $\{u_1,\ldots,u_k\}\subset V$.

The general solution of (5) is established in the next theorem.

Theorem 3.2. *The sequence of time-dependent probability matrices for the starting node dependent numbers of customers who are resident at time t in nodes $u_1, \dots, u_k$ is given by*

$$\mathcal{Q}(t,t;\mathbf{u}) = \sum_{\ell=0}^{\infty} \mathcal{R}^{[\ell]}(t,t;\mathbf{u}), \tag{6}$$

where $\mathcal{R}^{[0]}(t,t;\mathbf{u}) = \mathbf{1}$, $\mathcal{R}^{[\ell+1]}(t,t;\mathbf{u}) = \int_0^t \mathcal{R}^{[\ell]}(\xi;\mathbf{u}) * \mathcal{R}(t-\xi;\mathbf{u})d\xi$ *for* $\ell \geq 0$.

Proof. Equation (6) is well defined, since for any fixed $t \geq 0$ the sum $\| \sum_{\ell=0}^{\infty} \mathcal{R}^{[\ell]}(t,t;\mathbf{u}))\|^*$ converges uniformly according to Lemma 3.1. For any fixed $t \geq \tau$, $\sum_{\ell=0}^{\infty} \mathcal{R}^{[\ell]}(\tau,t;\mathbf{u})$ is differentiable with respect to τ and satisfies the basic differential equation (5) for $0 \leq \tau \leq t$, as is easily seen. So, (6) is a solution of (5). Assume, that $\mathcal{W}(\tau,t;\mathbf{u})$ is another solution of (5). Because of $\mathcal{Q}(0,0;\mathbf{u}) = \mathbf{1}$ any solution of the integral equation $\mathcal{Q}(\tau,t;\mathbf{u}) = \mathbf{1} + \int_0^\tau \mathcal{Q}(\xi,t;\mathbf{u}) * \mathcal{R}(t-\xi;\mathbf{u})d\xi$ is a solution of (5), and vice versa. Consequently,

$$\begin{aligned} &\|\mathcal{Q}(\tau,t;\mathbf{u}) - \mathcal{W}(\tau,t;\mathbf{u})\|^* \\ &\qquad \leq \int_0^\tau \|\mathcal{Q}(\xi,t;\mathbf{u}) - \mathcal{W}(\xi,t;\mathbf{u})\|^* \|\mathcal{R}(t-\xi;\mathbf{u})\|^* d\xi. \end{aligned} \tag{7}$$

Since $\mathcal{W}(\tau,t;\mathbf{u})$ is a differentiable sequence, it is a continuous sequence, and hence the maximum $\max_{\tau\in[0,t]} \|\mathcal{Q}(\tau,t;\mathbf{u}) - \mathcal{W}(\tau,t;\mathbf{u})\|^* =: c$ exists. Inserting repeatedly this constant in (7) and observing $\|\mathcal{R}(t-\xi\|^* \leq C$ (see proof of Lemma 3.1) one obtains, after ℓ steps, $\|\mathcal{Q}(\tau,t;\mathbf{u}) - \mathcal{W}(\tau,t;\mathbf{u})\|^* \leq c \cdot \frac{C^\ell \tau^\ell}{\ell!}$, which proves that $\mathcal{W}(\tau,t;\mathbf{u}) = \mathcal{Q}(\tau,t;\mathbf{u})$. This is true for any $0 \leq \tau \leq t$, i.e. the solution $\mathcal{Q}(\tau,t;\mathbf{u})$ is unique. □

The sequence of stationary probability matrices for the number of customers in nodes $u_1, \dots, u_k$ is now obtained as

$$\mathcal{Q}(\mathbf{u}) = \lim_{t\to\infty} \sum_{\ell=0}^{\infty} \mathcal{R}^{[\ell]}(t;\mathbf{u}). \tag{8}$$

What has been left open so far is the calculation of the probabilities $\beta_Y(\mathbf{n}; t-\tau;\mathbf{u})$ for $\mathbf{n} = (n_1, \dots, n_w) \in \mathbb{N}_0^w$, $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{N}_0^k$, $1 \leq k \leq w$.

According to (1) a customer, who arrived at node ν at time τ, will be under service in node u_μ at time $t \geq \tau$ with probability $(\boldsymbol{e}_\nu \exp(G\,(t-\tau)))_{u_\mu}$. Consider a batch arrival onto $\mathbf{n} = (n_1, \dots, n_w)$ and let $x_{\nu\mu}$ represent the number of customers who start in node ν and stay in node u_μ at time $t \geq \tau$, $1 \leq \mu \leq k$. Let further $\sum_{\mu=1}^k x_{\nu\mu} = \eta_\nu$ be the ν-th row sum of $X = ((x_{\nu\mu}))_{\nu\in V,\mu\in K}$, and set

$\Phi_\nu(t-\tau;\bar{\mathbf{u}}) = \sum_{h\in V\setminus K}[\boldsymbol{e}_\nu \exp(G\,(t-\tau))]_h$ for the probability that a customer who arrived at node ν at time τ is still under service at time $t \geq \tau$, but not in any of the nodes $u_1, \ldots, u_k$ (for $k = w$ this sum equals zero).

Lemma 3.2. *The probability $P_{y_{\nu 1},\ldots,y_{\nu k}}(n_\nu; t-\tau, \mathbf{u})$ for the event that $y_{\nu\mu}$ among n_ν customers who arrived at node ν at time $\tau \leq t$ are under service in node u_μ at time t ($\mu = 1, \ldots, k$) is given as*

$$P_{y_{\nu 1},\ldots,y_{\nu k}}(n_\nu; t-\tau, \mathbf{u}) = \sum_{\ell=\eta_\nu}^{n_\nu} \frac{n_\nu!}{\left(\prod_{\mu=1}^{k} y_{\nu\mu}!\right)(n_\nu-\ell)!(\ell-\eta_\nu)!} H_\nu(t-\tau)^{n_\nu-\ell} \cdot \prod_{\mu=1}^{k}\Big(\boldsymbol{e}_\nu \exp(G\,(t-\tau))\Big)_{u_\mu}^{y_{\nu\mu}} \cdot \Phi_\nu(t-\tau;\bar{\mathbf{u}})^{\ell-\eta_\nu}. \quad (9)$$

Proof. Let ℓ be the total number of t-resident customers arriving in node ν at time τ, $\eta_\nu \leq \ell \leq n_\nu$. The set of all n_ν customers arriving in node ν at time τ can be partitioned into $k+2$ subsets as follows: There are k sets with cardinalities $y_{\nu\mu}$ of t-resident customers staying in nodes u_μ ($\mu = 1, \ldots, k$), respectively, one set of cardinality $\ell - \eta_\nu$ of t-resident customers staying somewhere in node set $V \setminus K$ at time t, and one set of cardinality $n_\nu - \ell$ of customers who have left the system before t. Inserting the probabilities for these events into an expression that counts the possibilities to partition a population of n_ν elements into $\ell + 2$ disjoint subpopulations with the corresponding properties, and summing over all possibilities for ℓ yields (9). □

As an immediate consequence from the fact that customers, after their arrival in a network of IS stations, behave independently we finally have

$$\beta_Y(\mathbf{n}; t-\tau; \mathbf{u}) = \prod_{\nu=1}^{w} P_{y_{\nu 1},\ldots,y_{\nu k}}(n_\nu; t-\tau, \mathbf{u}). \quad (10)$$

4. Generating Functions and Interpretations

For any sequence $\mathcal{A}(t) = \{A_X(t)\}_{X\in \boldsymbol{M}(w,k)}$ of $(m \times m)$-matrices a joint Z-transform $A(t,Z)$, for $Z = ((z_{\nu\mu}))_{\nu\in V,\mu\in K}$, can be defined as

$$A^\star(t,Z) = \sum_{X\geq O}\prod_{\nu=1}^{w}\prod_{\mu=1}^{k} z_{\nu\mu}^{x_{\nu\mu}} A_X(t),$$

where $X = ((x_{\nu\mu}))_{\nu\in V,\mu\in K}$. For norm bounded sequences $\mathcal{A}(t)$ and $\mathcal{B}(t)$ it is easy to show that $(\mathcal{A}(t) * \mathcal{B}(t))^\star(Z) = A^\star(t, Z)B^\star(t, Z)$.

As a consequence, equation (5) gives $\frac{\partial Q^\star(\tau,t,Z;\mathbf{u})}{\partial\tau} = Q^\star(\tau, t, Z; \mathbf{u})R^\star(t - \tau, Z; \mathbf{u})$.

The solution of this equation, for $\tau = t$, reads

$$Q^\star(t, Z; \mathbf{u}) = \sum_{n=0}^{\infty} R^{\star[\ell]}(t, Z; \mathbf{u}), \tag{11}$$

where $Q^\star(t, t, Z; \mathbf{u}) =: Q^\star(t, Z; \mathbf{u})$, $R^{\star[0]}(t, Z; \mathbf{u}) = I$, and $R^{\star[\ell+1]}(t, Z; \mathbf{u}) = \int_0^\tau R^{\star[\ell]}(\xi, Z; \mathbf{u})R^\star(t - \xi, Z; \mathbf{u}) \ \forall\, \ell \in \mathbb{N}_0$.

Joint factorial moments of numbers $x_{\nu\mu}$ of customers who entered the network at node ν and are resident in node u_μ at time t are now obtained as

$$M(s_{11}, \ldots, s_{wk}; t) = \lim_{z_{11}\downarrow 1} \ldots \lim_{z_{wk}\downarrow 1} \frac{\partial^{s_{11}}}{\partial {z_{\nu 1}}^{s_{11}}} \cdots \frac{\partial^{s_{wk}}}{\partial {z_{\nu k}}^{s_{wk}}}\ Q^\star(t, Z; \mathbf{u}). \tag{12}$$

Network nodes correspond to service phases in a $BMAP/PH/\infty$ station. In this context a customer's "start node" defines his class [12]. In order to calculate class specific joint distributions as well as joint generating functions of customer numbers in the system — irrespective of their service phase at time t — we have to consider the case $k = w$ together with the probabilities

$$\beta_{\mathbf{r}}(\mathbf{n}; t - \tau) = \prod_{\nu=1}^{w} \binom{n_\nu}{r_\nu} H_\nu(t - \tau)^{n_\nu - r_\nu}(1 - H_\nu(t - \tau))^{r_\nu} \tag{13}$$

which have to replace the $\beta_Y(\mathbf{n}; t - \tau; \mathbf{u})$ in equation (4). The t-resident rate matrices then take the form $R_{\mathbf{r}}(t-\tau) = \sum_{\mathbf{n}\in\mathbb{N}_0^w} D_{\mathbf{n}}\beta_{\mathbf{r}}(\mathbf{n}; t-\tau)$, $\mathbf{r} = (r_1, \ldots, r_w)$, and the basic convolutional differential reads

$$\frac{\partial}{\partial\tau}\mathcal{Q}(\tau, t) = \mathcal{Q}(\tau, t) * \mathcal{R}(t - \tau).$$

Here $\mathcal{Q}(\tau, t) = (Q_{\mathbf{r}}(\tau, t))_{\mathbf{r}\in\mathbb{N}_0^w}$ and $\mathcal{R}(t - \tau) = (R_{\mathbf{r}}(t - \tau))_{\mathbf{r}\in\mathbb{N}_0^w}$.

Setting $\mathcal{Q}(t, t) = \mathcal{Q}(t)$, and $\mathcal{R}^{[0]}(t) = \mathbf{1}$, $\mathcal{R}^{[\ell+1]}(t) = \int_0^t \mathcal{R}^{[\ell]}(\xi) * \mathcal{R}(t - \xi)d\xi$ for $\ell \geq 0$, the solution for $\tau \to t$ is given as

$$\mathcal{Q}(t) = \sum_{\ell=0}^{\infty} \mathcal{R}^{[\ell]}(t). \tag{14}$$

Letting $\mathbf{z} = (z_1, \ldots, z_w)$, $0 \leq z_\nu \leq 1 \ \ \forall\, \nu \in V$, we obtain the generating function of this sequence of probability matrices as $Q(t, \mathbf{z}) = \sum_{\mathbf{r}\geq\mathbf{0}} \prod_{\nu=1}^{w} z_\nu^{r_\nu}$ $Q_{\mathbf{r}}(t)$.

Accordingly, the expression for the time-dependent joint factorial moments of class specific numbers of customers who are resident in the system at time t is

$$M(s_1, \ldots, s_w; t) \quad = \quad \lim_{z_1 \downarrow 1} \ldots \lim_{z_w \downarrow 1} \frac{\partial^{s_1}}{\partial {z_1}^{s_1}} \ldots \frac{\partial^{s_w}}{\partial {z_w}^{s_k}} \; Q^{\star}(t, \mathbf{z}). \tag{15}$$

Equation (15) is the pendant to corresponding results presented in [12]. By letting $t \to \infty$ in (14) we finally obtain the joint steady state probability matrices for class (phase) specific customer numbers in a $BMAP/PH/\infty$ - station:

$$Q_{\mathbf{r}} = \sum_{n=0}^{\infty} R_{\mathbf{r}}^{[n]}. \tag{16}$$

The joint steady state probability to observe r_ν class ν customers in the system $(1 \le \nu \le w)$ and the arrival process in phase j reads $q_{r_1,\ldots,r_w;j} = \sum_{n=0}^{\infty} R_{(\mathbf{r};ij)}^{[n]}$, $\forall i \in E$. In principle, expressions (6), (8) and (12) as well as (14), (15) and (16) represent the complete solution of the general $BMAP/PH/\infty$ problem, offering results for transient as well as steady state probabilities and moments. Nevertheless, actual computations are numerically very complicated and often beyond the limits of what is possible.

5. Algorithmic Treatment

In this section we provide an algorithmic preparation of results in order to compute transient state probabilities with respect to class specific numbers of customer in the system. This is exemplified for both, the case of an arbitrary BMAP, and the case of a BMAP with mutually commuting rate matrices D_n.

5.1. General Case

W.l.o.g. we assume a maximum value n_{max} for the batch size. Let $\mathbf{n}_{max} = (n_{1,max}, \ldots, n_{w,max})$ such that $n_{max} = \sum_{i=1}^{w} n_{i,max} =: \sigma(\mathbf{n}_{max})$, and set $D_{\mathbf{n}} = O$ for $\mathbf{n} > \mathbf{n}_{max}$.

The basic feature is discretization: Given t, the interval $(0, t]$ has to be divided into $\hat{N}$ compartments $(\tau_j, \tau_{j+1}]$ with $\tau_j = j \cdot t/\hat{N}$ for $0 \le j \le \hat{N}$.

The functions (13) are differentiable w.r.t. τ in $(0, t]$ so that linear interpolation yields good approximations if $\hat{N}$ is large enough. Let $\mathbf{r} = (r_1, \ldots, r_w)$, and $\mathbf{n} = (n_1, \ldots, n_w) \le \mathbf{n}_{max}$. Set $\triangle := \frac{t}{\hat{N}}$ and replace, for $\xi \in (\tau_j, \tau_{j+1}]$, the function $\beta_{\mathbf{h}}(\mathbf{n}; t-\xi)$ by $\hat{\beta}_{\mathbf{h}}(\mathbf{n}; t-\xi) = a_j(\mathbf{h},\mathbf{n})(\xi-\tau_j) + b_j(\mathbf{h},\mathbf{n})$, where $a_j(\mathbf{h},\mathbf{n}) = \frac{\beta_{\mathbf{h}}(\mathbf{n};t-\tau_{j+1})-\beta_{\mathbf{h}}(\mathbf{n};t-\tau_j)}{\triangle}$, and $b_j(\mathbf{h},\mathbf{n}) = \beta_{\mathbf{h}}(\mathbf{n}; t-\tau_j)$. Here $\tau_0 \approx 0$ for large t.

Then, as is easily seen, the approximated function $\hat{R}_{\mathbf{h}}(t-\xi)$ takes the form $\hat{R}_{\mathbf{h}}(t-\xi) = R_{\mathbf{h}}(t-\tau_j) + \Big(R_{\mathbf{h}}(t-\tau_{j+1}) - R_{\mathbf{h}}(t-\tau_j)\Big)\cdot\frac{\xi-\tau_j}{\triangle}$ for $\tau_j \le \xi \le \tau_{j+1}$ and $0 \le j \le \hat{N}-1$.

We set $A_0(j,\mathbf{h}) := R_{\mathbf{h}}(t-\tau_j)$, $A_1(j,\mathbf{h}) := \frac{R_{\mathbf{h}}(t-\tau_{j+1})-R_{\mathbf{h}}(t-\tau_j)}{\triangle}$, and $A_k(j,\mathbf{h}) = O$ for $k \ge 2$, and refer to $\mathcal{A}(j,\mathbf{h}) = \{A_k(j,\mathbf{h})\}_{k\in\mathbb{N}_0}$ as the corresponding sequence of $w\times w$-matrices.

The value of $\hat{R}^{[1]}_{\mathbf{h}}(\xi) = \int_0^\xi \hat{R}_{\mathbf{h}}(t-\eta)d\eta$ reads

$$\hat{R}^{[1]}_{\mathbf{h}}(\xi) = B^{[1]}_0(j,\mathbf{h}) + B^{[1]}_1(j,\mathbf{h})\cdot(\xi-\tau_j) + B^{[1]}_2(j,\mathbf{h})\cdot(\xi-\tau_j)^2$$

for $\tau_j \le \xi \le \tau_{j+1}$, by use of $(m\times m)$-matrices

$$B^{[1]}_0(j,\mathbf{h}) = \sum_{i=0}^{j-1}\int_{\tau_i}^{\tau_{i+1}}[A_0(i,\mathbf{h}) + A_1(i,\mathbf{h})(\eta-\tau_i)]d\eta$$

$$= \triangle\left\{\frac{1}{2}[R_{\mathbf{h}}(t) - R_{\mathbf{h}}(t-\tau_j)] + \sum_{i=1}^{j} R_{\mathbf{h}}(t-\tau_i)\right\},$$

$$B^{[1]}_1(j,\mathbf{h}) = A_0(j,\mathbf{h}) = R_{\mathbf{h}}(t-\tau_j),$$

$$B^{[1]}_2(j,\mathbf{h}) = A_1(j,\mathbf{h})/2 = \tfrac{1}{2\triangle}[R_{\mathbf{h}}(t-\tau_{j+1}) - R_{\mathbf{h}}(t-\tau_j)] \approx \tfrac{1}{2}\tfrac{\partial}{\partial\tau}R_{\mathbf{h}}(t-\tau)\big|_{\tau=\tau_j}.$$

The corresponding sequence is $\mathcal{B}^{[1]}(j,\mathbf{h}) = \{B^{[1]}_k(j,\mathbf{h})\}_{k\in\mathbb{N}_0}$ with $B^{[1]}_k(j,\mathbf{h}) = O$ for $k \ge 3$.

For sequences $\mathcal{U} = \{U_k(\mathbf{h})\}_{k\in\mathbb{N}_0,\mathbf{h}\in\mathbb{N}_0^w}$ and $\mathcal{V} = \{V_k(\mathbf{h})\}_{k\in\mathbb{N}_0,\mathbf{h}\in\mathbb{N}_0^w}$ of $(m\times m)$-matrices $U_k(\mathbf{h})$ and $V_k(\mathbf{h})$, respectively, we mark the double convolution with respect to both, the integer index k and the vectorial index $\mathbf{h}$, by $\Big(\mathcal{U}\bullet\mathcal{V}\Big)_k(\mathbf{r}) = \sum_{\mathbf{h}\le\mathbf{r}}\sum_{\ell=0}^{k} U_\ell(\mathbf{h})V_{k-\ell}(\mathbf{r}-\mathbf{h})$.

As is easily seen, $\hat{R}^{[2]}_{\mathbf{h}}(\xi) = \sum_{\nu=0}^{4} B^{[2]}_\nu(j,\mathbf{h})\cdot(\xi-\tau_j)^\nu$ for $\tau_j \le \xi \le \tau_{j+1}$, $0 \le j \le \hat{N}-1$, with

$$B^{[2]}_0(j,\mathbf{h}) = \sum_{i=0}^{j-1}\sum_{\nu=1}^{4}\frac{\triangle^\nu}{\nu}\cdot\Big(\mathcal{B}^{[1]}(i)\bullet\mathcal{A}(i)\Big)_{\nu-1}(\mathbf{h})$$

and

$$B_k^{[2]}(j,\mathbf{h}) = \frac{\Big(\mathcal{B}^{[1]}(j) \bullet \mathcal{A}(j)\Big)_{k-1}(\mathbf{h})}{k},$$

for $1 \le k \le 4$.

Setting $B_k^{[2]}(j,\mathbf{h}) = O$ for $k \ge 5$ and $\mathcal{B}^{[2]}(j) = \{B_k^{[2]}(j,\mathbf{h})\}_{k\in\mathbb{N}_0,\mathbf{h}\in\mathbb{N}_0^w}$, and performing stepwise integration, we immediately obtain the following result by induction.

Lemma 5.1. *For $\xi \in [0,t]$ and $n \ge 1$ the approximated functions $\hat{R}_{\mathbf{h}}^{[n+1]}(\xi)$ are given by $\hat{R}_{\mathbf{h}}^{[n+1]}(\xi)$ are given by $\hat{R}_{\mathbf{h}}^{[n+1]}(\xi) = \sum_{\nu=0}^{2n+2} B_\nu^{[n+1]}(j,\mathbf{h}) \cdot (\xi - \tau_j)^\nu$ for $\tau_j \le \xi \le \tau_{j+1}$, $0 \le j \le \hat{N} - 1$. The coefficients $B_\nu^{[n+1]}(j,\mathbf{h})$ form the matrix sequence $\mathcal{B}^{[n+1]}(j) = \{B_k^{[n+1]}(j,\mathbf{h})\}_{k\in\mathbb{N}_0,\mathbf{h}\in\mathbb{N}_0^w}$ with $B_k^{[n+1]}(j,\mathbf{h}) = O$ for $k > 2n+2$,*

$$B_0^{[n+1]}(j,\mathbf{h}) = \sum_{i=0}^{j-1}\sum_{\nu=1}^{2n+2} \frac{\triangle^\nu}{\nu} \cdot [\mathcal{B}^{[n]}(i) \bullet \mathcal{A}(i)]_{\nu-1}(\mathbf{h}),$$

$$B_k^{[n+1]}(j,\mathbf{h}) = \frac{1}{k}[\mathcal{B}^{[n]}(j) \bullet \mathcal{A}(j)]_{k-1}(\mathbf{h})$$

for $1 \le k \le 2n+2$.

This representation leads to $\hat{R}_{\mathbf{r}}^{[n]}(t) = \sum_{\nu=0}^{2n} B_\nu^{[n]}(\hat{N}-1,\mathbf{r})\triangle^\nu$, such that the transient state probabilities are approximately obtained as

$$\hat{Q}_{\mathbf{r}}(t) = \sum_{n=0}^{n_{stop}} \hat{R}_{\mathbf{r}}^{[n]}(t) = \sum_{n=0}^{n_{stop}}\sum_{\nu=0}^{2n} B_\nu^{[n]}(\hat{N}-1,\mathbf{r})\triangle^\nu, \quad \mathbf{r} \ge \mathbf{0}.$$

Here, according to Lemma 3.1, n_{stop} may be selected as to be the first integer n satisfying $\frac{C^{n+1}t^{n+1}}{(n+1)!} < \varepsilon$ for some given $\varepsilon > 0$.

5.2. Commuting Rate Matrices

Assume $D_k D_\ell = D_\ell D_k$ for $k, \ell \in \mathbb{N}_0$ implying that this is true also for all matrices $D_{\mathbf{n}} = D_{\sigma(\mathbf{n})}\psi_{\sigma(\mathbf{n})}(\mathbf{n})$ and all t-resident rate matrices $R_{\mathbf{r}}(t-\tau) = \sum_{\mathbf{n}\ge\mathbf{r}} D_{\mathbf{n}}\beta_{\mathbf{r}}(\mathbf{n}; t-\tau)$. Then

$$\sum_{\mathbf{n}\ge\mathbf{h}} D_{\mathbf{n}}\beta_{\mathbf{h}}(\mathbf{n}; t-\tau) \sum_{\mathbf{m}\ge\mathbf{r}-\mathbf{h}} D_{\mathbf{m}} \int_0^\tau \beta_{\mathbf{r}-\mathbf{h}}(\mathbf{m}; t-\xi)\, d\xi$$
$$= \sum_{\mathbf{m}\ge\mathbf{r}-\mathbf{h}} D_{\mathbf{m}} \int_0^\tau \beta_{\mathbf{r}-\mathbf{h}}(\mathbf{m}; t-\xi)\, d\xi \sum_{\mathbf{n}\ge\mathbf{h}} D_{\mathbf{n}}\beta_{\mathbf{h}}(\mathbf{n}; t-\tau),$$

and so by induction, $\mathcal{R}(t-\tau)*\mathcal{R}^{[1]*\nu}(\tau,t)=\mathcal{R}^{[1]*\nu}(\tau,t)*\mathcal{R}(t-\tau)$ for $0\le\tau\le t$, $\nu\in\mathbb{N}_0$. Due to $\mathcal{Q}(0,t)=\mathbf{1}$ and $\mathcal{R}^{[1]}(0.t)=\mathbf{0}$ a closed form solution of (5) is achievable as

$$Q(\tau,t)=e^{*\,\mathcal{R}^{[1]}(\tau,t)}\quad\text{for }\ 0\le\tau\le t,$$

where $\tau\to t$ yields $Q_{\mathbf{r}}(t)=\sum_{\nu=0}^{\infty}(R^{[1]*\nu}(t))_{\mathbf{r}}\,\frac{1}{\nu!}$. Thus, the determination of the transient state probability matrices $Q_{\mathbf{r}}(t)$ is put down to a numerical computation of sums of matrices of the form $R_{\mathbf{h}}^{[1]}(t)=\int_0^t R_{\mathbf{h}}(t-\xi)d\xi$. Using discretization we have

$$\hat{R}_{\mathbf{h}}^{[1]}(t)=\triangle\left\{\frac{1}{2}[\hat{R}_{\mathbf{h}}(t)-\hat{R}_{\mathbf{h}}(0)]+\sum_{i=1}^{\hat{N}}\hat{R}_{\mathbf{h}}(t-\tau_i)\right\},$$

where $\triangle=\frac{t}{\hat{N}}$.

6. Summary

In this paper we have analyzed the $BMAP/PH/\infty$ system in its most general form by interpreting the conglomerate of exponential stages of the service process as an open "queueing" network feeded by some spatial batch Markovian arrival process. The matrices that describe the probabilities to observe, at time $\tau\le t$, a certain number of customers who started in some node (stage) v and are resident, at time t, in some node (stage) u form a sequence which satisfies a convolutionary differential equation. We have presented the general solution of this differential equation and have shown how to perform a numerical computation.

The network approach allows us to consider systems in which any arriving customer may receive his own specific phase type distributed service. Transient and steady state probabilities for the number of observable customers in some phase u are computable with respect to the very node (stage) in which their service has started. The same is true for joint generating functions and factorial moments. Concerning this generality the results seem to comprise, to the author's best knowledge, all previous results on the $BMAP/PH/\infty$ system as known so far from literature. Our remarks on numerical treatment show that the computational complexity can be controlled by adequate discretization, and is fairly limited in case of mutually commuting rate matrices.

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