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ON SOME VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract: In this paper, existence, uniqueness and numerical results for some Volterra-Fredholm integral equations are given.

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1. Introduction

The mixed type Volterra-Fredholm integral equations with or without modification of the argument have been considered in many papers (see [1], [18], [19], etc.).

Many results for Volterra-Fredholm integro-differential equations in Banach spaces were obtained in [3], [10], [15], [17], [25], etc. For numerical results for Volterra-Fredholm linear integro-differential equations, we quote here the paper [11].

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The aim of this paper is to give existence, uniqueness and numerical results for the solution of the following Volterra-Fredholm integral equation:

$$y(x) = g(x) + \int_a^x k_1(x,s)y(s)ds + \int_a^b k_2(x,u)y(u)du, \quad x \in [a,b].$$

To obtain existence and uniqueness results we apply Picard operators' technique (see [22], [23]). Moreover we propose a numerical method to obtain a solution approximation of some integral analytical theory related equations.

2. Preliminaries

Let (X,d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$P(X) = \{Y \subseteq X | Y \neq \emptyset\}$;

$F_A = \{x \in X | A(x) = x\}$ - the fixed point set of A;

$\| \cdot \|_B : C[a,b] \to \mathbb{R}_+,\ \|y\|_B = \max_{x\in[a,b]} |y(x)|e^{-\tau(x-a)},\ \tau > 0$

- a Bielecki norm on $C[a,b]$;

$d_B : C[a,b] \times C[a,b] \to \mathbb{R}_+,\ d_B(y,z) = \|y - z\|_B$

- the corresponding metric of $\| \cdot \|_B$.

We have the following main definitions and results.

Definition 2.1. (see Rus [22]) Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Theorem 2.1. (Contraction Principle) *Let (X,d) be a complete metric space and $A : X \to X$ a contraction. Then A is a Picard operator and*

$$d(x_A^*, A^n(x_0)) \leq \frac{\alpha^n}{1-\alpha} d(x_0, A(x_0)),\ \text{for all } n \in \mathbb{N}.$$

Theorem 2.2. (see Rus [21]) (Data Dependence Theorem) *Let (X,d) be a complete metric space and $A, C : X \to X$ two operators. We suppose that:*

(i) A is α-contraction and $F_A = \{x_A^*\}$;

(ii) C has fixed points and $x_C^* \in F_C$;

(iii) there exists $\eta > 0$ such that $d(A(x), C(x)) \leq \eta$, for all $x \in X$.

Then

$$d(x_A^*, x_C^*) \leq \frac{\eta}{1-\alpha}.$$

3. Volterra-Fredholm Integral Equations

Consider the following Volterra-Fredholm integral equation:

$$y(x) = g(x) + \int_a^x k_1(x,s)y(s)ds + \int_a^b k_2(x,u)y(u)du, \quad x \in [a,b], \tag{3.1}$$

where $g \in C[a,b]$, $k_1 \in C(D_1)$ and $k_2 \in C(D_2)$. Here $D_1 = \{(x,s) \in \mathbb{R}^2 |\ a \leq s \leq x \leq b\}$ and $D_2 = [a,b] \times [a,b]$.

Let $M_1 = \max_{(x,s)\in D_1} |k_1(x,s)|$ and $M_2 = \max_{(x,u)\in D_2} |k_2(x,u)|$ be.

We have:

Theorem 3.1. *In the above continuity conditions, suppose that there exists $\tau > 0$ such that*

$$\frac{1}{\tau}[M_1 + M_2 e^{\tau(b-a)}] < 1.$$

Then the equation (3.1) has a unique solution $y^* \in C[a,b]$ and this solution can be obtained by the successive approximation method, starting from any element of $C[a,b]$.

Proof. Consider the operator $A : (C[a,b], d_B) \to (C[a,b], d_B)$, defined by

$$A(y)(x) = g(x) + \int_a^x k_1(x,s)y(s)ds + \int_a^b k_2(x,u)y(u)du.$$

We obtain

$$\begin{aligned}
&|A(y)(x) - A(z)(x)| \\
&= \left| \int_a^x k_1(x,s)(y(s) - z(s))ds + \int_a^b k_2(x,u)(y(u) - z(u))du \right| \\
&\leq \int_a^x |k_1(x,s)||y(s) - z(s)|ds + \int_a^b |k_2(x,u)||y(u) - z(u)|du \\
&\leq M_1 \int_a^x |y(s) - z(s)|e^{-\tau(s-a)}e^{\tau(s-a)}ds \\
&+ M_2 \int_a^b |y(u) - z(u)|e^{-\tau(u-a)}e^{\tau(u-a)}du \\
&\leq \left[\frac{M_1}{\tau}(e^{\tau(x-a)} - 1) + \frac{M_2}{\tau}(e^{\tau(b-a)} - 1)\right] \|y - z\|_B \\
&\leq \frac{1}{\tau}(M_1 e^{\tau(x-a)} + M_2 e^{\tau(x-a+b-x)})\|y - z\|_B \\
&= \frac{e^{\tau(x-a)}}{\tau}(M_1 + M_2 e^{\tau(b-x)})\|y - z\|_B \leq \frac{e^{\tau(x-a)}}{\tau}(M_1 + M_2 e^{\tau(b-a)})\|y - z\|_B.
\end{aligned}$$

It follows that

$$|A(y)(x) - A(z)(x)|e^{-\tau(x-a)} \leq \frac{1}{\tau}(M_1 + M_2 e^{\tau(b-a)})\|y - z\|_B,$$

for all $x \in [a,b]$. Therefore,

$$\|A(y) - A(z)\|_B \leq \frac{1}{\tau}(M_1 + M_2 e^{\tau(b-a)})\|y - z\|_B\,,$$

or

$$d_B(A(y), A(z)) \leq \frac{1}{\tau}(M_1 + M_2 e^{\tau(b-a)})d_B(y, z).$$

The operator A is Lipschitz with the constant

$$L_A = \frac{1}{\tau}(M_1 + M_2 e^{\tau(b-a)}).$$

The supposed condition ensured that A is a contraction. So, we apply contraction principle. □

Remark 3.1. If M_1, M_2 and $b-a$ are such that $M_1 + M_2 e < \frac{1}{b-a}$, by choosing $\tau \in \left(M_1 + M_2 e, \frac{1}{b-a}\right)$, the condition of the previous theorem is fulfilled.

Remark 3.2. By considering

$$y(x) = g(x) + \int_0^x \frac{\sin(x+s)}{10} y(s)ds + \int_0^{\frac{1}{3}} \frac{\cos(xu)}{18} y(u)du, \quad x \in \left[0, \frac{1}{3}\right],$$

where $g \in C\left[0, \frac{1}{3}\right]$, then $M_1 = \frac{1}{10}$, $M_2 = \frac{1}{18}$. So, by choosing for example $\tau = 1$ or $\tau = 2$, the condition of Theorem 3.1 is satisfied.

Remark 3.3. If we consider the Volterra-Fredholm integral equation

$$y(x) = y(a) + \int_a^x k_1(x,s)y(s)ds + \int_a^b k_2(x,u)y(u)du, \quad x \in [a,b], \tag{3.2}$$

the corresponding operator is not a contraction.

Remark 3.4. Existence and uniqueness results for the solution of the following Volterra-Fredholm integral equation with linear modification of the argument:

$$y(x) = g(x) + \int_a^x k_1(x,s)y(\lambda s)ds + \int_a^b k_2(x,u)y(\lambda u)du,$$
$$x \in [a,b], \quad 0 < \lambda < 1\,, \tag{3.3}$$

where $a = 0$, $b > 0$ or $a < 0$, $b = 0$ or $a < 0$, $b > 0$, can be obtained in the same way as above, in the continuity conditions of g, k_1 and k_2.

Remark 3.5. In [19], by using Maia's Theorem (see [16]) and in [1], by using a generalization of a theorem of Perov and Rus (see [21]), existence and uniqueness results for the solution of a Volterra-Fredholm integral equation were obtained. In [18] the existence and uniqueness for the solution of a Volterra-Fredholm integral equation with deviating argument were studied.

4. The Numerical Model

In this section we present the numerical model suitable to (3.1) based on the global collocation using approximating splines, in particular the so called modified q.i. splines, see [4].

In the following we recall the necessary background on modified q.i. splines.

Let $X_m := \{x_{0,m} = a < x_{1,m} < ... < x_{m,m} < x_{m+1,m} = b\}$ be a partition of the interval $J := [a, b]$ with $H_m := \max_{0 \le j \le m} (x_{j+1,m} - x_{j,m}), H_m \to 0$ as $m \to \infty$ and let $\{d_j : j = 0, ..., m + 1\}$ be a vector of positive integers, where $d_0 = d_{m+1} = p$ $(p > 2)$ and $d_j \le p - 2, j = 1, ..., m$. We set $n + p := \sum_{j=0}^{m+1} d_j$ and define $\Pi_n = \{t_{in} : i = 1, 2, ..., n+p\}$ as the nondecreasing sequence obtained from X_m by repeating $x_{j,m}$ exactly d_j times, $j = 0, ..., m+1$. The t_{in} are assumed as nodes of the locally uniform spline space [20]:

$$S_{p,\Pi_n} := \{g : g|(x_{j,m}, x_{j+1,m}), \quad \in P_{p-1}, \quad j = 0, ..., m$$

(where P_{p-1} is the set of polynomials of degree less than p) and $g^{(i)}(x^+_{j,m}) = g^{(i)}(x^-_{j,m}), i = 0, 1, ..., p - d_j - 1;\ j = 1, ..., m\}$ and the end points are $p-$fold nodes.

Thus S_{p,Π_n} is the set of polynomials splines of order p with nodes at $x_{j,m}(j = 0, ..., m + 1)$ of multiplicity $d_j \le p - 2$ $(j = 1, ..., m)$, consequently every spline in S_{p,Π_n} is in $C^1(J)$.

The set of the normalized B-splines $B_{i,p}$ $(i = 1, ..., n)$ of order p defined by the following recurrence relation:

$$B_{i,p}(x) = \frac{x - t_i}{t_{i+p-1} - t_i} B_{i,p-1}(x) + \frac{t_{i+p} - x}{t_{i+p} - t_{i+1}} B_{i+1,p-1}(x)\,, \tag{4.1}$$

$$B_{i,1}(x) = \begin{cases} 1, & t_i \le x < t_{i+1}\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{4.2}$$

is considered as a basis for the spline space S_{p,Π_n}.

Let $\tau_{ij}(i=1,2,...,n; j=1,2,...,l$ with $1\le l\le p-d_k+1, \forall k=1,2,...,m)$ be a set of nodes belonging for each $i=1,2,...,n$ to $[t_i,t_{i+p}]$ and such that $\tau_{ij}\neq\tau_{ih}$ for $j\neq h$. Then the function $y(x)$ in (3.2) can be approximated by the following modified q.i. spline of polynomial precision l (see [14], [7]):

$$S_n(y(x)) := \sum_{i=1}^{n} B_{i,p}(x)\sum_{j=1}^{l} v_{ij}y(\tau_{ij})\,,$$

where

$$\begin{aligned} v_{ij} &:= \sum_{\mu=j}^{l}\frac{\alpha_{i\mu}}{\prod_{\substack{s=1\\ s\neq j}}^{\mu}(\tau_{ij}-\tau_{is})},\\ \alpha_{ij} &:= \sum_{k=1}^{j}(-1)^{j-k}\frac{(k-1)!(p-k)!}{(p-1)!}c_{i,k-1}d_{i,j-k}\,, \end{aligned} \tag{4.3}$$

with $c_{i,k-1}=\mathrm{symm}_{k-1}(t_{i+1},...,t_{i+p-1})$, $d_{i,j-k}=\mathrm{symm}_{j-k}(\tau_{i1},...,\tau_{i,j-1})$ (see [26]).

An optimal choice of the nodes $\tau_{ij}$is (see [8]):

$$T:\left\{\begin{array}{c} \tau_{i1}:=\zeta_1,\quad i=1,...,n\\ \tau_{i2}:=\zeta_{i-1},\quad \tau_{i3}:=\zeta_{i+1},...,\\ \tau_{il}:=\zeta_{i-(-1)^l[\frac{l}{2}]},\quad i=[\frac{l}{2}]+1,...,n-[\frac{l}{2}] \end{array}\right\},$$

where $\zeta_i:=\frac{t_{i+1}+...+t_{i+p-1}}{p-1}$, $i=1,...,n$ with a suitable choice (see [4]) of the remaining nodes of T.

Rewriting (3.1) as

$$y(x)=g(x)+\int_a^b\widetilde{k}_1(x,s)y(s)ds+\int_a^b k_2(x,u)y(u)du,\quad x\in[a,b], \tag{4.4}$$

where

$$\widetilde{k}_1(x,s)=\left\{\begin{array}{ll} k_1(x,s)\,, & \text{if } a\le s\le x\,,\\ 0\,, & \text{if } s>x\,, \end{array}\right.$$

we introduce suitably the approximating functions $S_n(y(x))$ in (4.4). Using the so called generalized Nystrom method [6], we obtain:

$$S_n(y(x))=g(x)+\int_a^b\widetilde{k}_1(x,s)S_n(y(s))ds+\int_a^b k_2(x,u)S_n(y(u))du,\quad x\in[a,b]\,. \tag{4.5}$$

We then choose in (a,b) a set of collocation points $\xi_k (k = 1, 2, ..., nl)$, where nl is the number of distinct elements of the set T, decoupled from the set of the τ_{ij} and we insert them in (4.5); consequently we obtain:

$$\sum_{i=1}^{n}\sum_{j=1}^{l} v_{ij} y(\tau_{ij}) B_{i,p}(\xi_k) - \sum_{i=1}^{n}\sum_{j=1}^{l} v_{ij} y(\tau_{ij}) [\int_a^b \widetilde{k}_1(\xi_k, s) B_{i,p}(s) ds + \int_a^b k_2(\xi_k, u) B_{i,p}(u) du] = g(\xi_k)\,, \quad x \in [a,b]\,. \quad (4.6)$$

The solution of linear system (4.6) gives the numerical values of y in T.

Remark 4.1. We underline that the particular choice of the τ_{ij} points in T arises from a compromise between two different constraints: maximizing the polynomial precision of the approximation and minimizing the collocation system (4.6) order (see [8]). Moreover this method allows great and advantageus overlapping of quadrature and collocation meshes.

Remark 4.2. Taking into account equation (3.3) we rewrite it

$$y(x) = g(x) + \frac{1}{\lambda}\left[\int_a^{\lambda x} \widetilde{k}_1(x, \frac{s}{\lambda}) y(s) ds + \int_a^{\lambda b} k_2(x, \frac{u}{\lambda}) y(u) du\right], \quad x \in [a,b],\ 0 < \lambda < 1\,, \quad (4.7)$$

where

$$\widetilde{k}_1(x, \frac{s}{\lambda}) = \begin{cases} k_1(x,s)\,, & \text{if } a \le s \le \lambda x\,, \\ 0\,, & \text{if } s > \lambda x\,, \end{cases}$$

and we proceed as for (4.4).

Remark 4.3. Necessary conditions for the convergence are

$$\begin{aligned} &\int_a^b \widetilde{k}_1(x,s) S_n(y(s)) ds \to \int_a^b \widetilde{k}_1(x,s) y(s) ds\,, \\ &\int_a^b k_2(x,u) S_n(y(u)) du \to \int_a^b k_2(x,u) y(u) du\,, \end{aligned} \quad (4.8)$$

with $\widetilde{k}_1(x,s)$ as in (4.4)

Under the hypotheses of existence and uniqueness of the solution of equation (3.2) and with τ_{ij} belonging to T, the conditions (4.8) are assured by Theorem 2 in [6].

5. Some Numerical Results

In the following we present some numerical results of Volterra-Fredholm integral equation using the numerical method presented in [4]. In all the examples the existence and uniqueness of the solution is guaranteed according to Theorem 3.1.

We have considered the following equation

$$f(x)+\int_0^a k_1(x,t)f(\lambda t)dt+\int_0^x k_2(x,t)f(\lambda t)dt=g(x),$$
$$x\in[0,a],\quad 0<\lambda\le 1,\quad (5.1)$$

where f is the unknown function and k_1, k_2, g are given.

In all the cases the interval $[0,a]$ has been divided by $m=11$ equispaced simple nodes $x_j=(0.1)ja$, $(j=0,1,...,10)$, except for x_0 and x_{10} of multiplicity $p=4$. The corresponding vector $\mathbf{t}$ has $n+p=17$ components.

The unknown function is approximated in $n=13$ nodes belonging to $[0,a]$. In all the examples the exactness of the method for polynomial functions till degree $p-1=3$ is tested.

In the following Tables we show the results obtained with different choices of $k_1(x,t), k_2(x,t), g(x), a$ and λ. For brevity we indicate the mean of the absolute values of the relative errors evaluated in the interval. Our computer programs are written in *MATLAB 5.3*, which has a machine precision $\varepsilon\simeq 10^{-16}$.

Table 1 is referred to the choice $k_1(x,t)=k_2(x,t)=1, a=1/4$

$g(x)$	$f(x)$	$\lambda=1$	$\lambda=0.5$
$x+\lambda/2(a^2+x^2)$	x	$2.6\cdot 10^{-15}$	$2.0\cdot 10^{-15}$
$x^3+\lambda^3/4(a^4+x^4)$	x^3	$2.5\cdot 10^{-13}$	$6.9\cdot 10^{-13}$
$\cos x+1/\lambda(\sin a\lambda+\sin\lambda x)$	$\cos x$	$1.1\cdot 10^{-08}$	$1.1\cdot 10^{-08}$

Table 1.

Tables 2a and 2b are referred to the choice $k_1(x,t)=k_2(x,t)=\cos t, a=1/4$

$g(x)$	$f(x)$
$1+\sin a+\sin x$	1
$x^2+\lambda^2((a^2-2)\sin a+2a\cos a+(x^2-2)\sin x+2x\cos x)$	x^2
$e^x+(e^{a\lambda}(\sin a+\lambda\cos a)+e^{\lambda x}(\sin x+\lambda\cos x)-2\lambda)/(1+\lambda^2)$	e^x

Table 2a.

$\lambda = 1$	$\lambda = 0.2$
$5.2 \cdot 10^{-16}$	$8.1 \cdot 10^{-16}$
$4.7 \cdot 10^{-14}$	$2.1 \cdot 10^{-14}$
$1.1 \cdot 10^{-08}$	$1.1 \cdot 10^{-08}$

Table 2b.

Tables 3a and 3b are referred to the choice $k_1(x,t) = k_2(x,t) = e^t, a = 1/6$

$g(x)$	$f(x)$
$e^a + e^x - 1$	1
$x + \lambda(e^a(a-1) + e^x(x-1) + 2)$	x
$x^2 + \lambda^2(e^a(a^2 - 2a + 2) + e^x(x^2 - 2x + 2) - 4)$	x^2
$x^3 + \lambda^3(e^a(a^3 - 3a^2 + 6a - 6) + 12 + e^x(x^3 - 3x^2 + 6x - 6))$	x^3
$e^x + 1/(1+\lambda)(e^a + e^x - 2)$	e^x

Table 3a.

$\lambda = 1$	$\lambda = 0.5$
$6.6 \cdot 10^{-16}$	$8.6 \cdot 10^{-16}$
$2.4 \cdot 10^{-14}$	$1.1 \cdot 10^{-14}$
$1.3 \cdot 10^{-11}$	$3.5 \cdot 10^{-12}$
$1.8 \cdot 10^{-09}$	$2.4 \cdot 10^{-10}$
$2.1 \cdot 10^{-09}$	$2.1 \cdot 10^{-09}$

Table 3b.

Finally, we consider the following equation

$$\begin{aligned} f(x) + \int_0^a k_1(x,t) f(\lambda t) dt + \int_0^x k_2(x,t) f(\lambda t) dt &= g(x), \\ x \in [0,a], \quad 0 < \lambda \leq 1, \end{aligned}$$

and Table 4 is referred to the choice $k_1(x,t) = 0$, $k_2(x,t) = 120 - 100e^{x+t}$.

λ	$g(x)$	$f(x)$	$[0,1]$	$[0,20]$
1	$1 + 120x - 100(1 - e^x)$	1	$4.8 \cdot 10^{-14}$	$2.8 \cdot 10^{-12}$

Table 4.

The results are comparable with those obtained in [12].

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