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ON THE PRIMARY AVOIDANCE THEOREM FOR MODULES OVER COMMUTATIVE RINGS

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Abstract: Let R be a commutative ring with identity and M an R-module. In this paper we prove the following theorem: Let M be an R-module $N_1, ..., N_n$ be submodules of M, and N is a submodule of M such that $N \subseteq N_1 \bigcup N_2 \bigcup ... \bigcup N_n$. Assume at most two of the N_k's are not primary submodule and $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for every $j \neq k$. Then $N \subseteq N_k$ for some k.

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1. Introduction

Throughout this paper, R denotes a commutative ring with identity and all related modules are unitary R-modules. Let M be an R-module. A proper submodule N of M is called primary if whenever $rm \in N$, for $r \in R$ and $m \in M$, then $m \in N$ or $r^n \in (N : M)$ for some positive integer n, where $(N : M) = \{r \in R : rM \subseteq N\}$. Let I be an ideal of the ring R. Recall that nill ideal of I, designated by $\sqrt{I}$, is the set $\sqrt{I} = \{r \in R : r^n \in I \text{ for some positive integer } n\}$.

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Let $N, N_1, ..., N_n$ be submodules of M. We call a covering

$$N \subseteq N_1 \bigcup N_2 \bigcup ... \bigcup N_n$$

efficient if no N_k is superfluous. We shall say that $N = N_1 \bigcup N_2 \bigcup ... \bigcup N_n$ efficient union if none of the N_k may be excluded. A covering of a submodule by two submodules is never efficient. Here we prove the following theorem.

Theorem. *Let M be an R-module $N_1, ..., N_n$ be submodules of M, and N is a submodule of M such that $N \subseteq N_1 \bigcup N_2 \bigcup ... \bigcup N_n$. Assume at most two of the N_k's are not primary submodule and $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for every $j \neq k$. Then $N \subseteq N_k$ for some k.*

We recall the following Lemma (see [1, Lemma 2.1]).

Lemma 1. *Let $N = N_1 \bigcup N_2 \bigcup ... \bigcup N_n$ be an efficient union of submodules of R-module M for $n > 1$. Then $\bigcap_{j \neq k} N_j = \bigcap_{j=1}^{n} N_j$ for all k.*

Proposition 1. *Let $N \subseteq N_1 \bigcup N_2 \bigcup ... \bigcup N_n$ be an efficient covering consisting of submodules of an R-module M. If $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for every $j \neq k$, then no N_k is a primary submodule of M.*

Proof. It is clear that $N = (N \bigcap N_1) \bigcup (N \bigcap N_2) \bigcup ... \bigcup (N \bigcap N_n)$ is efficient union. Then there exists an element $e_k \in N \backslash N_k$ for every $k \in \{1, 2, ..., n\}$. By Lemma 1, $\bigcap_{j \neq k} (N \bigcap N_j) \subseteq N \bigcap N_k$. If $j \neq k$, then $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ by hypothesis. Now, suppose that N_k is a primary submodule. Then $\sqrt{(N_k : M)}$ is a prime ideal. Therefore

$$\sqrt{(N_1 : M)} ... \sqrt{(N_{k-1} : M)} \sqrt{(N_{k+1} : M)} ... \sqrt{(N_n : M)} \nsubseteq \sqrt{(N_k : M)}.$$

There exists

$$s = \prod_{j \neq k} s_j \in \sqrt{(N_1 : M)} ... \sqrt{(N_{k-1} : M)} \sqrt{(N_{k+1} : M)} ... \sqrt{(N_n : M)}$$

but $s \notin \sqrt{(N_k : M)}$ where $s_1 \in \sqrt{(N_1 : M)}, ..., s_n \in \sqrt{(N_n : M)}$. Then $s_1^{n_1} \in (N_1 : M), ..., s_n^{n_n} \in (N_n : M)$ where $n_1, n_2, ..., n_n \in Z$. Let $m = \max\{n_1, n_2, ..., n_n\}$. Then $s^m \in (N_j : M)$ where $j \neq k$ but $s^m \notin (N_k : M)$. Hence $s^m e_k \in N \bigcap N_j$ for every $j \neq k$, but

$$s^m e_k \notin N \bigcap N_k \left(\text{if } s^m e_k \in N_k \text{ then } e_k \in N_k \text{ or } s \in \sqrt{(N_k : M)} \right).$$

This is contradiction to $\bigcap_{j \neq k} (N \bigcap N_j) \subseteq N \bigcap N_k$. □

Theorem 1. (The Primary Avoidance Theorem) *Let M be an R-module $N_1, ..., N_n$ be submodules of M, and N is a submodule of M such that $N \subseteq N_1 \bigcup N_2 \bigcup ... \bigcup N_n$. Assume at most two of the N_k's are not primary submodule and $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ for every $j \neq k$. Then $N \subseteq N_k$ for some k.*

Proof. We may assume that the covering is efficient since the hypothesis remains valid after reduction to an efficient covering. Then $n \neq 2$. Also $n \leq 2$ by Proposition 1. Hence $n = 1$. □

As we can see in the following Example 1, the condition that $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$ if $j \neq k$ in Theorem 1 is essential.

Example 1. *Assume that $R = Z_2$ and $M = R^3$ and let $\{x_1, x_2, x_3\}$ be standart basis of M. Then $N_1 = Rx_1, N_2 = Rx_2$ and $N_3 = R(x_1 + x_2)$ are 0-prime submodules of M (so 0-primary submodules of M), but we have $N = Rx_1 + Rx_2 \subseteq N_1 \bigcup N_2 \bigcup N_3$ and $N \nsubseteq N_i$ for every $i = 1, 2, 3$.*

Proposition 2. *Let N, K be submodules of R-module M with $K \subseteq N$.N is a primary submodule of M if and only if N/K is a primary submodule of M/K.*

Proof. Let $r(m + K) \in N/K$ with $m + K \notin N/K$, where $r \in R, m \in M$. Hence $rm \in N$ with $m \notin N$; hence N primary gives $r^k M \subseteq N$ for some k. It follows $r^k (M/K) \subseteq N/K$. Thus N/K is primary. The remaining implication is similar. □

Theorem 2. *Let M be an R-module and N a submodule of M. If Primary Avoidance Theorem holds for M, then the Primary Avoidance Theorem holds for M/N.*

Proof. This follows from Proposition 2. □

Let N be a proper submodule of a non-zero R-module M. Then the M-radical of N, denoted by rad(N), is defined to be the intersection of all prime submodules of M containing N (see [2]).

An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R.

Theorem 3. *Let M be a multiplication module and $N_1, ..., N_n$ be submodules of M, and N is a submodule of M such that $N \subseteq N_1 \bigcup N_2 \bigcup ... \bigcup N_n$. Assume at most two of the N_k's are not primary submodule* and rad $(N_j) \nsubseteq$ rad $(N_k)\,(j \neq k)$. *Then $N \subseteq N_k$ for some k.*

Proof. If rad $(N_j) \nsubseteq$ rad $(N_k)\,(j \neq k)$, then $\sqrt{(N_j : M)} \nsubseteq \sqrt{(N_k : M)}$. Indeed, if $\sqrt{(N_j : M)} \subseteq \sqrt{(N_k : M)}$, then rad $(N_j) = \sqrt{(N_j : M)}M \subseteq \sqrt{(N_k : M)}M =$ rad (N_k) [see, 2, Theorem 2.12]. Then $N \subseteq N_k$ for some k by Theorem 1. □

We call a covering $L \subseteq (L_1 + a_1) \cup (L_2 + a_2) \cup ... \cup (L_n + a_n)$ efficient if no coset is superfluous. Let $a_k = a$ for $k = 1, 2, ..., n$. We can then equivalently write $L - a \subseteq L_1 \cup L_2 \cup ... \cup L_n$ and this is a coset efficiently covered by a union of submodules of M. Let $n = 1$. Then $L \subseteq L_1 + a$. Then $a \in L_1$because otherwise $0 \notin L_1 + a$.

Proposition 3. *Let M be an R-module and N a proper submodule of M. Then N is primary if and only if $IL \subseteq N$ implies that $L \subseteq N$ or $I \subseteq \sqrt{(N : M)}$ for every ideal I of R and submodule N of M.*

Proof. $(\Rightarrow)$ Suppose $IL \subseteq N$ and $L \nsubseteq N$. Let $a \in I$. Then, there exists $x \in L \backslash N$, so $ax \in N$, hence $a \in \sqrt{(N : M)}$. Therefore, $I \subseteq \sqrt{(N : M)}$.

$(\Leftarrow)$ Let $ax \in N$ and $x \notin N$ where $a \in R, x \in M$. Therefore, $(a)\,(x) \subseteq N$ and $(x) \nsubseteq N$. Hence $(a) \subseteq \sqrt{(N : M)}$ and so $a \in \sqrt{(N : M)}$. □

The following Lemma is known (see [1, Lemma 2.4]).

Lemma 2. *Let $L \subseteq (L_1 + a_1) \cup (L_2 + a_2) \cup ... \cup (L_n + a_n)$ be an efficient covering of L by cosets, where $n \geq 2$. Then $L \cap \left(\bigcap_{j \neq k} L_j\right) \subseteq L_k$, but $L \nsubseteq L_k$ for all k.*

Proposition 4. *Let M be an R-module. Let $L + a \subseteq L_1 \cup L_2 \cup ... \cup L_n$ be an efficient covering with $n \geq 2$.If $\sqrt{(L_j : M)} \nsubseteq \sqrt{(L_k : M)}$ for every $j \neq k$, then no L_k is primary submodule of M.*

Proof. By Lemma 2, $L \cap \left(\bigcap_{j \neq k} L_j\right) \subseteq L_k$, but $L \nsubseteq L_k$ for all k. Put $I = \left(\bigcap_{j \neq k} L_j : M\right)$. Then $IL \subseteq L \bigcap \left(\bigcap_{j \neq k} L_j\right) \subseteq L_k$. Let L_k be primary. Then $I = \left(\bigcap_{j \neq k} L_j : M\right) = \bigcap_{j \neq k} (L_j : M) \subseteq \sqrt{(L_k : M)}$ by Proposition 3. Since $\sqrt{(L_k : M)}$ is prime ideal, $\sqrt{(L_j : M)} \subseteq \sqrt{(L_k : M)}\,(j \neq k)$. This is a contradiction. □

Theorem 4. *Let M be an R-module. Let $L + a \subseteq L_1 \cup L_2 \cup ... \cup L_n$ be a covering such that at least $n - 1$ of the submodules $L_1, L_2, ..., L_n$ are primary and $\sqrt{(L_j : M)} \nsubseteq \sqrt{(L_k : M)}$ for every $j \neq k$. Then $L + Ra \subseteq L_k$ for some k.*

Proof. If we reduce the covering to an efficient one it is immediate from Proposition 4 that only one term will remain, say $L + a \subseteq L_k$, and hence $L + Ra \subseteq L_k$. □

References

[1] C.P. Lu, Unions of prime submodules, *Houston J. Math.*, **23**, No. 2 (1997), 203-213.

[2] Zeinab Abd El-Bast, Patric F. Smith, Multiplication modules, *Commun. Algebra*, **16**, No. 4 (1988), 755-779.