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ON EXACT SOLUTIONS OF A QUASILINEAR WAVE EQUATION WITH SOURCE TERM

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Abstract: In this paper, we construct exact solutions of the Cauchy problem of degenerate quasilinear wave equation of Kirchhoff type. We observe the effect of the nonlinear terms on the existence of solutions. We construct solutions which exist globally for some initial data, or blow up in a finite time for other initial data.

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1. Introduction

In this paper, we construct exact solutions of the following initial value problem of a quasilinear wave equation with nonlinear source term and we study their asymptotic behaviour concerning the time variable:

$$\begin{cases} (|u'|^{l-2}u')' - M(\|u_x\|_2^2)u_{xx} = \gamma|u|^{p-1}u, \text{ in } \mathbb{R}\times[0,+\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \text{ in } \mathbb{R}, \end{cases} \tag{P}$$

where $M(r) = r^{\frac{p-1}{2}}$, $\|u_x\|_2^2 = \int_{-\infty}^{\infty} |u_x(x,t)|^2\,dx, l > 2, p > l-1$ and $\gamma > 0$ are constants.

For the problem (P), when $l > 2$ and $M \equiv 1$ without source term and with nonlinear dissipation, Benaissa and Mimouni [3] determined suitable relations between l and p, for which the energy decays exponentially or alternatively

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polynomially. More precisely: they showed that the energy of the solutions decays with exponential type if $l+1 \geq p$ and decays polynomially if $l+1 < p$.

For the problem (P) (in the case of bounded domain), when $l > 2$ and M is not a constant function with nonlinear dissipation, Benaissa and Messaoudi [2] have investigated the blowup of solutions. They have shown that, for suitably chosen initial data and a relation between l and p, any classical solution blows up in finite time.

When $l = 2$, the equation without a source term is often called the wave equation of Kirchhoff type which has been introduced in order to study the nonlinear vibrations of an elastic string by Kirchhoff [9] and the existence of local and global solutions in Sobolev and Gevrey classes was investigated by many authors (see [7], [6] and [8]).

When $l = 2$, in [4] Ebihara, Hoshino and Kurokiba have constructed one of the solutions for (P) and studied their behavior in t. They have shown that there exists a solution which blows up at a finite time under some initial condition and in the other case there exists a global solution which decays with the order $\mathcal{O}(t^{-\frac{2}{p-1}})(t \to \infty)$. We think that the interaction of the term $(|u'|^{l-2}u')'$ $(l > 2)$ with the source term $|u|^{p-1}u$ and velocity $M(r)$ have an effect on the result of [4]. We construct the solution of the form $u(x,t) = v(x)\varphi(t)$ for (P) when $M(r) = r^{\frac{p-1}{2}}, p > 1, l > 2$ and $p + 3 = 2l$ and we study the behavior of the solutions as the time t increases.

2. Separating the Variables

Let $u(x,t) = v(x)\varphi(t)$ and substitute in (P). Then we see that (P) is changed to (P'):

$$\begin{cases} |v|^{l-2}v(|\varphi'|^{l-2}\varphi')' - \left(\displaystyle\int_{-\infty}^{\infty} |v_x(x,t)|^2\,dx\right)^{\frac{p-1}{2}} |\varphi|^{p-1}\varphi v_{xx} \\ -\gamma|v|^{p-1}v|\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = v(x)\varphi(0) = \varphi_0 v(x), \qquad x \in \mathbb{R}, \\ u_t(x,0) = v(x)\varphi_t(0) = \varphi_1 v(x), \qquad x \in \mathbb{R}. \end{cases} \tag{P'}$$

Here we consider the case $\varphi(t) > 0$, so that the first equation of (P') is equivalent to

$$\frac{\alpha^{p-1}v_{xx} + \gamma|v|^{p-1}v}{|v|^{l-2}v} = \frac{(|\varphi'|^{l-2}\varphi')'}{\varphi^p} = \lambda, \tag{1}$$

where $\alpha^2 = \int_{-\infty}^{\infty} |v_x(x)|^2\,dx$ and λ is a positive constant. Therefore we can obtain the following two problems from (1):

$$\begin{cases} \alpha^{p-1} v_{xx} - \lambda |v|^{l-2} v + \gamma |v|^{p-1} v = 0, \quad x \in \mathbb{R}, \\ \alpha^2 = \displaystyle\int_{-\infty}^{\infty} |v_x|^2\,dx < +\infty, \end{cases} \tag{$P1$}$$

$$\begin{cases} (|\varphi'|^{l-2}\varphi')' = \lambda \varphi^p, \quad t \geq 0, \\ \varphi(0) = \varphi_0,\ \varphi_t(0) = \varphi_1, \qquad \varphi(t) \geq 0, \quad t \geq 0. \end{cases} \tag{$P2$}$$

3. Problem ($P1$)

In this section, we construct the positive solution $v(x)$ of $(P1)$ with $\lim_{|x|\to\infty} v(x) = 0$. For this purpose we deal with the following problem:

$$\begin{cases} \alpha^{p-1} v_{xx} - \lambda |v|^{l-2} v + \gamma |v|^{p-1} v = 0, \quad x \in \mathbb{R}, \\ v(0) = A\ (A > 0 : \text{constant}), \quad v_x(0) = 0, \quad \lim\limits_{|x|\to\infty} u(x) = 0, \\ \frac{\alpha^2}{2} = \displaystyle\int_0^{\infty} |v_x(x)|^2\,dx < +\infty, \end{cases} \tag{$P1$}'$$

where α is a fixed positive number. If $v(x)$ is a solution of $(P1)'$, then we can solve $(P1)$ by setting $v(-x) = v(x)$ for $x > 0$ because of $(P1)'_2$.

Multiplying the first equation of $(P1)'$ by $2v_x$ and integrating from 0 to x, we obtain

$$\alpha^{p-1} v_x^2 = \frac{2\lambda}{l} v^l - \frac{2\gamma}{p+1} v^{p+1} - \frac{2\lambda}{l} A^l - \frac{2\gamma}{p+1} A^{p+1}. \tag{2}$$

If we choose $c > 0$ such that

$$A = \left(\frac{(p+1)\lambda}{\gamma\, l} \right)^{\frac{1}{p+1-l}}, \tag{3}$$

then (2) implies

$$\alpha^{\frac{p-1}{2}} v_x = \mp \sqrt{\frac{2\lambda}{l} v^l - \frac{2\gamma}{p+1} v^{p+1}}. \tag{4}$$

Here we consider the case where v is positive and $v_x < 0$, so that we treat the following equation which is derived from (4):

$$\frac{v_x}{v^{\frac{l}{2}} \sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1} v^{p+1-l}}} = -\alpha^{-\frac{p-1}{2}}. \tag{5}$$

If we integrate (5) from c to v, then we obtain

$$\int_v^c \frac{dz}{z^{\frac{l}{2}} \sqrt{\frac{2\lambda}{l} - \frac{2\gamma}{p+1} z^{p+1-l}}} = \alpha^{-\frac{p-1}{2}} x. \tag{6}$$

If there exists $x^* \in (0,\infty)$ such that $v(x^*)=0$, then we get
$$\int_0^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}z^{p+1-l}}}=\alpha^{-\frac{p-1}{2}}x^*.$$
But one can easily show $\int_0^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}z^{p+1-l}}}=\infty$ with use of (3) and $\alpha^{-\frac{p-1}{2}}x^*<+\infty$, thus $v(x)$ is monotone decreasing and $v(x)>0$. And it $\lim\limits_{x\to\infty} v(x)=k>0$, then we get from (6)
$$\int_k^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}z^{p+1-l}}}=\lim_{x\to\infty}\alpha^{-\frac{p-1}{2}}x\ (=\infty).$$
But $\int_k^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}z^{p+1-l}}}$ is finite, so we deduce that $\lim\limits_{x\to\infty} v(x)=0$.

By putting $y=\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}z^{p+1-l}}$ in order to calculate the left hand side of (6), we see
$$I=\int_v^c \frac{dz}{z^{\frac{l}{2}}\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}z^{p+1-l}}}$$
$$=\int_0^{\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}z^{p+1-l}}} \frac{2^{\frac{2p-l}{2(p+1-l)}}\left(\frac{\gamma}{(p+1)}\right)^{\frac{l-2}{2(p+1-l)}}}{(p+1-l)(\frac{2\lambda}{l}-y^2)^{\frac{2p-l}{2(p+1-l)}}}\,dy.$$
We suppose that $\frac{2p-l}{2(p+1-l)}=\frac{3}{2}$, then we obtain
$$I=\int_0^{\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}z^{l-2}}} \frac{2^{\frac{3}{2}}\left(\frac{\gamma}{(p+1)}\right)^{\frac{1}{2}}}{(l-2)(\frac{2\lambda}{l}-y^2)^{\frac{3}{2}}}\,dy=\frac{l}{\lambda(l-2)}\frac{\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}v^{l-2}}}{v^{\frac{l-2}{2}}}.$$
Then (6) becomes
$$\frac{l}{\lambda(l-2)}\frac{\sqrt{\frac{2\lambda}{l}-\frac{2\gamma}{p+1}v^{l-2}}}{v^{\frac{l-2}{2}}}=\alpha^{-\frac{p-1}{2}}x,$$
which implies
$$v(x)=\left(\frac{\frac{l}{2\lambda}}{\left(\frac{\lambda(l-2)}{l}\right)^2\alpha^{-(p-1)}x^2+\frac{2\gamma}{p+1}}\right)^{\frac{1}{l-2}}. \tag{7}$$

If we put $\mu = \left(\frac{\lambda(m-2)}{m}\right)^2 \alpha^{-(p-1)}$ and differentiate (7), then we get

$$v_x(x) = -\frac{2}{m-2}\left(\frac{l}{2\lambda}\right)^{\frac{1}{l-2}} \mu \frac{x}{\left(\mu\ x^2 + \frac{2\gamma}{p+1}\right)^{1+\frac{1}{l-2}}}.$$

Thus, we have

$$\int_0^\infty |v_x(x)|^2\,dx = \left(\frac{2}{l-2}\left(\frac{m}{2\lambda}\right)^{\frac{1}{l-2}}\right)^2 \mu^2 \int_0^\infty \frac{x^2}{\left(\mu\ x^2 + \frac{2\gamma}{p+1}\right)^{2+\frac{2}{l-2}}}\,dx.$$

When we put $H = \int_0^\infty \frac{x^2}{\left(\mu\ x^2 + \frac{2\gamma}{p+1}\right)^{2+\frac{2}{l-2}}}\,dx$ and let $z = \sqrt{\mu/a}\ x$, where $a = \frac{2\gamma}{p+1}$, we have

$$H = \frac{1}{a^{\frac{1}{2}+\frac{2}{l-2}}\mu^{\frac{3}{2}}}\int_0^\infty \frac{z^2}{(1+z^2)^{2+\frac{2}{l-2}}}\,dz$$
$$= \frac{l-2}{2l}\frac{1}{a^{\frac{1}{2}+\frac{2}{l-2}}\mu^{\frac{3}{2}}}\int_0^\infty \frac{1}{(1+z^2)^{1+\frac{2}{l-2}}}\,dz \quad = \frac{l-2}{2l}\frac{1}{a^{\frac{1}{2}+\frac{2}{l-2}}\mu^{\frac{3}{2}}}K \quad (K < +\infty).$$

Therefore we obtain

$$\int_0^\infty |v_x(x)|^2\,dx = \left(\frac{2}{l-2}\left(\frac{m}{2\lambda}\right)^{\frac{1}{l-2}}\right)^2 \sqrt{\mu}\ \frac{l-2}{2l}\frac{1}{a^{\frac{1}{2}+\frac{2}{l-2}}}K\,.$$

If we take α such as

$$\alpha = \left\{\frac{4\sqrt{2}}{\sqrt{l}}\left(\frac{l(l-1)}{2}\frac{\gamma}{\lambda}\right)^{\frac{1}{2}+\frac{2}{l-2}} K\right\}^{\frac{1}{l}} \lambda^{\frac{3}{2l}},$$

then $(P1)'_4$ is satisfied. Therefore we can get the following theorem.

Theorem. *If $l > 2$ and $p = 2l-3$, then the solution $v(x)$ of $(P1)'$ such that $v(0) = \left(\frac{2(l-2)}{l}\frac{\lambda}{\gamma}\right)^{\frac{1}{l-2}}$ and $v_x(0) = 0$, is given by*

$$v(x) = \left(\frac{\frac{l}{2\lambda}}{\left(\frac{\lambda(l-2)}{l}\right)^2 \alpha^{-(p-1)}x^2 + \frac{2\gamma}{p+1}}\right)^{\frac{1}{l-2}},$$

where $\alpha = \left\{\frac{4\sqrt{2}}{\sqrt{l}}\left(\frac{l(l-1)}{2}\frac{\gamma}{\lambda}\right)^{\frac{1}{2}+\frac{2}{l-2}} K\right\}^{\frac{1}{l}} \lambda^{\frac{3}{2l}}$ with $K = \int_0^\infty \frac{dz}{(1+z^2)^{1+\frac{2}{l-2}}}$.

This solution can be extended to one of the solutions of $(P1)$.

4. Problem $(P2)$

In this section, we consider the following problem:

$$\begin{cases} (|\varphi'|^{l-2}\varphi')' = \lambda\varphi^p, \quad t \geq 0, \\ \varphi(0) = \varphi_0, \quad \varphi_t(0) = \varphi_1, \\ \varphi(t) \geq 0, \quad t \geq 0, \end{cases} \tag{$P2$}$$

where λ is the constant appeared in the previous section. If we multiply the equation $(P2)_1$ by $\varphi_t(t)$ and integrate from 0 to t as in Section 2, then we have

$$\varphi_t(t) = \pm\left(\frac{l\lambda}{(l-1)(p+1)}v^{p+1} + |\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1}\right)^{\frac{1}{l}}, \tag{8}$$

because of $(P2)_2$ and $(P2)_3$. In order that we construct the solution of $(P2)$ from (8), the following condition has to be satisfied

$$G(\varphi) \equiv \frac{l\lambda}{(l-1)(p+1)}v^{p+1} + |\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} \geq 0.$$

We consider the following three cases $(i), (ii)$ and (iii).

(i) If $\varphi_0 \geq 0$ and $\varphi_1 > 0$, then $G(\varphi(0)) = G(\varphi_0) > 0$. Hence $\varphi_t(t) = (G(\varphi(t)))^{\frac{1}{l}} > 0$ for sufficiently small $t > 0$ since $(G(\varphi(t)))^{\frac{1}{l}}$ is monotone increasing function, we see that $\varphi_t > 0$‘ for all $t > 0$ where $\varphi(t)$ exist. Since one can show $\int_{\varphi_0}^{\infty}(1\backslash(G(\varphi))^{\frac{1}{l}})\,d\varphi < +\infty$, we see that $\varphi \to +\infty$ as $t \to T^*$, where $T^* = \int_{\varphi_0}^{\infty}(1\backslash(G(\varphi))^{\frac{1}{l}})\,d\varphi$.

(ii) In the case of $\varphi_0 \geq 0$ and $|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} = 0$, by solving (8) we obtain

$$\varphi_\pm(t) = \left(\varphi_0^{-\frac{p+1-l}{l}} \mp \frac{(p+1-l)\left(\frac{\lambda l}{(l-1)(p+1)}\right)^{\frac{1}{l}}}{l}t\right)^{-\frac{l}{p+1-l}},$$

with double signs in same order. Obviously φ_+ decays with the order $\mathcal{O}(t)^{-\frac{l}{p+1-l}}$ as $t \to +\infty$ and φ_- blows up at $\varphi_0^{-\frac{p+1-l}{l}}\frac{l}{p+1-l}\left(\frac{(l-1)(p+1)}{\lambda l}\right)^{\frac{1}{l}}$.

(iii) If $\varphi_0 \geq 0, \varphi_1 < 0$ and $|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} > 0$, then $\varphi_t(t) = -(G(\varphi(t)))^{\frac{1}{l}} < 0$ locally. Then, since $G(\varphi(t))$ is monotone increasing function, we see that $\varphi_t(t) < 0$ for all $t > 0$ where $\varphi(t)$ exist. From (8),

$$\int_{\varphi_0}^{0} -\frac{d\varphi}{(G(\varphi))^{\frac{1}{l}}} \equiv T^{**} < +\infty.$$

Therefore T^{**} exists such that $\varphi(t) \to 0$ as $t \to T^{**}$. From (8), we can get

$$\int_{\varphi(t)}^{0} -\frac{1}{T^{**}-t}\frac{d\varphi}{(G(\varphi))^{\frac{1}{l}}} = 1. \tag{9}$$

Let $s = (T^{**}-t)r$ in (9) and let $t \to T^{**}$, we have

$$\lim_{t\to T^{**}} \varphi(t)(T^{**}-t)^{-1} = \left(|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1}\right)^{\frac{1}{l}}. \tag{10}$$

From (8) and (10), we have

$$\lim_{t\to T^{**}} \varphi(t)\varphi'(t)(T^{**}-t)^{-1} = -\left(|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1}\right)^{\frac{2}{l}}. \tag{11}$$

Then we have the following theorem.

Theorem. *(1) If $\varphi_0 \geq 0$ and $\varphi_1 > 0$, then by putting*

$$T^* = \int_{\varphi_0}^{\infty} (1\backslash(G(\varphi))^{\frac{1}{l}})\, d\varphi,$$

the solution $\varphi(t)$ of $(P2)$ blows up at $t = T^*$, that is $\varphi(t) \to +\infty$ as $t \to T^*$.

(2) If $\varphi_0 \geq 0$ and $|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} = 0$, then the solution $\varphi(t)$ of $(P2)$ is given by

$$\varphi_{\pm}(t) = \left(\varphi_0^{-\frac{p+1-l}{l}} \mp \frac{(p+1-l)\left(\frac{\lambda l}{(l-1)(p+1)}\right)^{\frac{1}{l}}}{l} t\right)^{-\frac{l}{p+1-l}},$$

with double signs in same order.

(3) If $\varphi_0 \geq 0, \varphi_1 < 0$ and $|\varphi_1|^l - \frac{l\lambda}{(l-1)(p+1)}\varphi_0^{p+1} > 0$, then by putting

$$\int_{\varphi_0}^{0} -\frac{d\varphi}{(G(\varphi))^{\frac{1}{l}}} \equiv T^{**},$$

the solution $\varphi(t)$ vanishes at $t = T^{**}$ and satisfies (10) and (11).

Depending on the choice of the constants φ_i $(i = 0, 1)$ we obtain a solution which blows up in a finite time and another case we have a solution which decays with order $\mathcal{O}(t)^{-\frac{l}{p+1-l}}$ as $t \to \infty$.

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