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COMMON FIXED POINT THEOREMS FOR A FAMALY OF MAPPINGS

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Abstract: Common fixed point theorems for a family $H_{f,x}$ of mappings involving a densifying mapping in metric spaces are proved. The results presented in this paper generalize a few results in the literature.

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1. Introduction and Preliminaries

Let (X, d) be a metric space and $f, g : X \to X$ be mappings. For $x, y \in X$ and $A \subseteq X$, put:

$O_f(x) = \{f^n x : n \geq 0\}$, $O_f(x, y) = O_f(x) \cup O_f(y)$,
$\delta(A) = \sup\{d(a, b) : a, b \in A\}$,
$H_f = \{g : g : X \to X \text{ and } g(\cap_{n\geq 1} f^n X) \subseteq \cap_{n\geq 1} f^n X\}$,
$H_{f,x} = \{g : g : X \to X \text{ and } g(\cap_{n\geq 1} f^n(\overline{O_f(x)})) \subseteq \cap_{n\geq 1} f^n(\overline{O_f(x)})\}$

and $\overline{A}$ denotes the closure of A. It is well known that the family H_f of self

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mappings in a metric space (X,d) includes the family of commuting self mappings with f in the metric space (X,d) as proper subset [4]. Liu [4], [5], [6], [7], [8], [9], [10], [11], [12] proved the existence of fixed and common fixed points for several families H_f and H_g of contractive type mappings in metric spaces, compact metric spaces and compact Hausdorff spaces, respectively, which generalize some results due to Jungck [2], Fisher [1], Pande [13], etc.

Inspired by the results in [1], [2], [4]-[12], in this paper, we establish some new common fixed point theorems for two classes of contractive type mappings involving the family $H_{f,x}$, which is a local version of the family H_f.

Recall that a self mapping f in a metric apace X is said to be densifying if $\alpha(fA) < \alpha(A)$ for every bounded subset A of X with $\alpha(A) > 0$, where α denotes the measure of non-compactness in the sense of Kurotowski [3].

2. Common Fixed Point Theorems

The following lemma plays a key role in this paper.

Lemma 2.1. *Let f be a continuous self mapping of a complete metric space (X,d). Assume that there exist $x_0 \in X$ and a positive integer m such that f^m is densifying. Then:*

(a) $\cap_{n\geq 1} f^n(\overline{O_f(x_0)})$ *is nonempty and compact;*

(b) $f(\cap_{n\geq 1} f^n(\overline{O_f(x_0)})) = \cap_{n\geq 1} f^n(\overline{O_f(x_0)})$;

(c) $\lim_{n\to\infty} \delta(f^n(\overline{O_f(x_0)})) = \delta(\cap_{n\geq 1} f^n(\overline{O_f(x_0)}))$.

Proof. Let x_0 be a point of X. Note that

$$\begin{aligned}\alpha(O_f(x_0)) = \max\{\alpha(\{x_0, fx_0, \ldots, f^{m-1}x_0\}), \alpha(O_f(f^m(x_0)))\} \\ = \alpha(O_f(f^m(x_0))) = \alpha(f^m(O_f(x_0)))\,,\end{aligned}$$

and f^m is densifying. Consequently, we infer easily that $\overline{O_f(x_0)}$ is compact. Since f is continuous and $\overline{O_f(x_0)}$ is compact, it follows that

$$f(\overline{O_f(x_0)}) \subseteq \overline{f(O_f(x_0))} \subseteq \overline{O_f(x_0)},$$

$f^n(\overline{O_f(x_0)})$ is compact and $f^{n+1}(\overline{O_f(x_0)}) \subseteq f^n(\overline{O_f(x_0)})$ for $n \geq 1$. The compactness of $\overline{O_f(x_0)}$ and the continuity of f ensure that $\{f^n(\overline{O_f(x_0)}) : n \geq 1\}$ has the finite intersection property. Thus $\cap_{n\geq 1} f^n(\overline{O_f(x_0)})$ is a nonempty compact subset of $\overline{O_f(x_0)}$. Let $D = \cap_{n\geq 1} f^n(\overline{O_f(x_0)})$. We now assert that $f(D) = D$. For any $v \in D$, there exists $x_n \in f^{n-1}(\overline{O_f(x_0)})$ such that $fx_n = v$ for $n \geq 1$.

From the compactness of $\overline{O_f(x_0)}$ we may (by selecting a subsequence, if necessary) assume that $\{x_n\}_{n\geq 1}$ converges to some point $u \in \overline{O_f(x_0)}$. Since $\{x_k : k \geq n+1\} \subseteq f^n(\overline{O_f(x_0)})$ and $f^n(\overline{O_f(x_0)})$ is compact, it follows that $u \in f^n(\overline{O_f(x_0)})$ for $n \geq 1$, which yields that $u \in D$. Notice that $fu = v$. Consequently we deduce that $D \subseteq f(D)$. It is easy to see that $f(D) \subseteq D$. Therefore $f(D) = D$.

Note that $f^n(\overline{O_f(x_0)})$ is compact and $f^n(\overline{O_f(x_0)}) \supseteq f^{n+1}(\overline{O_f(x_0)})$ for $n \geq 1$. It follows that $\{\delta(f^n(\overline{O_f(x_0)}))\}_{n\geq 1}$ is a nonincreasing sequence and there exist $r \geq 0$ and $a_n, b_n \in f^n(\overline{O_f(x_0)})$ such that

$$d(a_n, b_n) = \delta(f^n(\overline{O_f(x_0)})) \downarrow r \geq \delta(D) \quad \text{as } n \to \infty. \tag{2.1}$$

From the compactness of $f^n(\overline{O_f(x_0)})$ it follows that there exist $\{a_{n_k}\}_{k\geq 1} \subseteq \{a_n\}_{n\geq 1}$, $\{b_{n_k}\}_{k\geq 1} \subseteq \{b_n\}_{n\geq 1}$ and $\{a, b\} \subseteq \overline{O_f(x_0)}$ such that

$$\lim_{k\to\infty} a_{n_k} = a, \quad \lim_{k\to\infty} b_{n_k} = b. \tag{2.2}$$

For any $n \geq 1$ and $k \geq 1$, we get that

$$\{a_{n_i}, b_{n_i} : i \geq k\} \subseteq f^{n_k}(\overline{O_f(x_0)}) \subseteq f^n(\overline{O_f(x_0)}).$$

This leads to $\{a, b\} \subseteq f^n(\overline{O_f(x_0)})$ for any $n \geq 1$ by (2.2). That is,

$$\{a, b\} \subseteq \cap_{k\geq 1} f^{n_k}(\overline{O_f(x_0)}) = D. \tag{2.3}$$

Combining (2.1)-(2.3), we infer that

$$d(a, b) = \lim_{k\to\infty} d(a_{n_k}, b_{n_k}) = \lim_{k\to\infty} \delta(f^{n_k}(\overline{O_f(x_0)})) = r \leq \delta(D).$$

That is, $r = \delta(D)$ by (2.1). This completes the proof. □

Our main results are as follows.

Theorem 2.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a continuous mapping. Assume that there exist $x_0 \in X$ and three positive integers m, i, j such that f^m is densifying and*

$$d(f^i x, f^j y) < \delta(H_{f,x_0}(\cap_{n\geq 1} f^n(\overline{O_f(x_0)}))) \tag{2.4}$$

for all $x, y \in \overline{O_f(x_0)}$ with $f^i x \neq f^j y$. Then f has a unique fixed point $c \in \overline{O_f(x_0)}$ such that $gc = c$ for all $g \in H_{f,x_0}$ and $\lim_{n\to\infty} f^n x = c$ for every $x \in \overline{O_f(x_0)}$.

Proof. Let $D = \cap_{n\geq 1} f^n(\overline{O_f(x_0)})$. From Lemma 2.1, we have that $fD = D \neq \emptyset$. Now we assert that D consists of a singleton. Otherwise, $\delta(D) > 0$. It follows from the compactness of D that there exist $p, q \in D$ with $d(p,q) = \delta(D)$. Notice that $f(D) = D$. Therefore there exist $a, b \in D$ such that $f^i a = p$ and $f^j b = q$. Obviously, $H_{f,x_0}(D) \subseteq D$. Using (2.4), we deduce that

$$0 < \delta(D) = d(p,q) = d(f^i a, f^j b) < \delta(H_{f,x_0}(\cap_{n\geq 1} f^n(\overline{O_f(x_0)}))) \leq \delta(D),$$

which is a contradiction. Thus D is a singleton. Let $D = \{c\}$. That is, c is a fixed point of f. Clearly $gc = c$ for all $g \in H_{f,x_0}$.

Suppose that f has another fixed point $d \in \overline{O_f(x_0)}$ with $d \neq c$. It is easy to see that $d \in \cap_{n\geq 1} f^n(\overline{O_f(x_0)}) = D = \{c\}$. Hence $d = c$. Thus c is the unique fixed point of f in $\overline{O_f(x_0)}$.

Note that $\cap_{n\geq 1} f^n(\overline{O_f(x_0)}) = D = \{c\}$. For each $x \in \overline{O_f(x_0)}$ and $n \geq 0$, we have that $\{c, f^n x\} \subseteq f^n(\overline{O_f(x_0)})$. Thus Lemma 2.1 yields that

$$d(f^n x, c) \leq \delta(f^n(\overline{O_f(x_0)})) \to \delta(D) = 0 \quad \text{as } n \to \infty,$$

that is, $\lim_{n\to\infty} f^n x = c$. This completes the proof. □

Theorem 2.2. *Let (X,d) be a complete metric space and $f : X \to X$ be a continuous mapping. Assume that there exist $x_0 \in X$ and two positive integers m and i such that f^m is densifying and*

$$d(f^i x, f^i y) < \delta(H_{f,x_0}(\overline{O_f(x,y)})) \tag{2.5}$$

for any $x, y \in \overline{O_f(x_0)}$ with $x \neq y$. Then f has a unique fixed point $u \in \overline{O_f(x_0)}$ such that $gu = u$ for all $g \in H_{f,x_0}$ and $\lim_{n\to\infty} f^n x = u$ for every $x \in \overline{O_f(x_0)}$.

Proof. Put $D = \cap_{n\geq 1} f^n(\overline{O_f(x_0)})$. Lemma 2.1 ensures that D is nonempty and compact and $fD = D$. Suppose that $\delta(D) > 0$. Thus there exist $c, d, p, q \in D$ such that $d(p,q) = \delta(D)$, $f^i c = p$, $f^i d = q$ and $c \neq d$. Clearly, $H_{f,x_0}(\overline{O_f(c,d)}) \subseteq D$. In view of (2.5), we deduce that

$$0 < \delta(D) = d(p,q) = d(f^i c, f^i d) < \delta(H_{f,x_0}(\overline{O_f(c,d)})) \leq \delta(D),$$

which is a contradiction. Therefore D is a singleton. Put $D = \{u\}$. Consequently, u is a fixed point of f and $gu = u$ for each $g \in H_{f,x_0}$.

Suppose that $v \neq u$ is another fixed point of f in $\overline{O_f(x_0)}$. It is easy to see that $v \in \cap_{n\geq 1} f^n(\overline{O_f(x_0)}) = D = \{u\}$. Hence $v = u$. Thus u is the unique fixed point of f in $\overline{O_f(x_0)}$. Notice that for any $x \in \overline{O_f(x_0)}$

$$d(f^n x, u) \leq \delta(f^n(\overline{O_f(x_0)})) \to \delta(D) = 0 \quad \text{as } n \to \infty,$$

that is, $\lim_{n\to\infty} f^n x = a$. This completes the proof. □

Remark 2.1. Lemma 2.1, Theorem 2.1 and Theorem 2.2 extend, improve and unify the corresponding results in [1], [2], [4]-[12].

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