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# OPERATOR CALCULUS AND APPELL SYSTEMS ON CLIFFORD ALGEBRAS

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**Abstract:** Canonical Appell systems have interpretations as polynomial solutions to evolution equations. In probability they are used to obtain non-central limit theorems. Their analogues have been defined on Lie groups, the Schrödinger algebra, and quantum groups. In this paper, Appell systems are constructed on Clifford algebras of arbitrary signature using properties of the Clifford (geometric) product. These systems have natural connections with stochastic processes on fermion algebras, fermionic Fock spaces, the algebra of physical space, and quaternions. Examples of Clifford-Appell systems are computed directly using *Mathematica*.

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## 1. Introduction

Following the formalism of Feinsilver, Kocik, and Schott [3], the space of polynomials with degree not exceeding $n$ can be considered as the space of solutions, $\mathcal{Z}_n$, to the equation $D^{n+1}\psi = 0$, where $D$ is the differentiation operator. In this context, an *Appell system* is a sequence of nonzero polynomials satisfying:

— $\psi_n \in \mathcal{Z}_n$, $\forall n \geq 0$.

— $D\psi_n = \psi_{n-1}$, $\forall n \geq 1$.

A simple example of an Appell system is to define $\psi_n = x^n/n!$ with $D = d/dx$. Other examples of Appell systems include shifted moment sequences

$$\psi_n(x) = \int_{-\infty}^{\infty} (x+y)^n p(dy)\,, \tag{1}$$

where $p$ is a probability measure on $\mathbb{R}$ with all moments finite. This includes the Hermite polynomials,

$$H_n(x) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} (x+iy)^n e^{-y^2/2} dy \tag{2}$$

for the Gaussian case.

Appell systems can be interpreted as polynomial solutions of generalized heat equations. In probability theory they are also used to obtain non-central limit theorems. Their analogues have been defined on Lie groups [4], the Schrödinger algebra [3], and on quantum groups [2].

For any operator $\mathcal{A}$, set

$$\mathcal{Z}_n = \{\psi : \mathcal{A}^{n+1}\psi = 0\}$$

for $n \geq 0$. An $\mathcal{A}$-*Appell system* is a sequence of nonzero functions $\{\psi_0, \psi_1, \ldots, \psi_n, \ldots\}$ satisfying:

— $\psi_n \in \mathcal{Z}_n$, $\forall n \geq 0$.

— $\mathcal{A}\psi_n = \psi_{n-1}$, for $n \geq 1$.

The system of embeddings $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \cdots$ is a canonical $\mathcal{A}$-*Appell system decomposition.*

**Definition 1.** For fixed $n \geq 0$, let $V$ be an $n$-dimensional vector space having orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The $2^n$-dimensional *Clifford algebra* of signature $(p,q)$, where $p+q = n$, is defined as the associative algebra generated by the collection $\{\mathbf{e}_i\}$ along with the scalar $\mathbf{e}_\emptyset = 1 \in \mathbb{R}$, subject to the following multiplication rules:

$$\mathbf{e}_i\,\mathbf{e}_j + \mathbf{e}_j\,\mathbf{e}_i = 0 \quad \text{for } i \neq j, \quad \text{and} \tag{3}$$

$$\mathbf{e}_i{}^2 = \begin{cases} 1, & \text{if } 1 \le i \le p\,, \\ -1, & \text{if } p+1 \le i \le p+q = n. \end{cases} \tag{4}$$

The Clifford algebra of signature $(p, q)$ is denoted $\mathcal{C}\ell_{p,q}$.

Generally the vectors generating the algebra do not have to be orthogonal. When they are orthogonal, as in the definition above, the resulting multivectors are called *blades*.

Let $[n] = \{1, , 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$ by underlined Roman characters. The basis elements of $\mathcal{C}\ell_{p,q}$ can then be indexed by these finite subsets by writing

$$\mathbf{e}_{\underline{i}} = \prod_{k \in \underline{i}} \mathbf{e}_k. \tag{5}$$

Arbitrary elements of $\mathcal{C}\ell_{p,q}$ have the form

$$u = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}\, \mathbf{e}_{\underline{i}}, \tag{6}$$

where $u_{\underline{i}} \in \mathbb{R}$ for each $\underline{i} \in 2^{[n]}$.

The *degree* of a Clifford multi-vector is defined as the cardinality of its multi-index, i.e., $\deg\left(\mathbf{e}_{\underline{i}}\right) = |\underline{i}|$. Given an element $u \in \mathcal{C}\ell_{p,q}$, $p + q = n$, the *degree-$k$ part* of $u$ is defined by

$$\langle u \rangle_k = \sum_{|\underline{i}| = k} u_{\underline{i}}\, \mathbf{e}_{\underline{i}}. \tag{7}$$

Clifford algebras have a natural *degree decomposition*. For any $u \in \mathcal{C}\ell_{p,q}$, $u = \sum_{k=0}^{p+q} \langle u \rangle_k$. Hence, if $\mathcal{A}$ is any operator on $\mathcal{C}\ell_{p,q}$ mapping terms of degree not exceeding $\ell$ to terms of degree not exceeding $\ell - 1$ for all $1 \le \ell \le n$, then the Clifford algebra has a natural $\mathcal{A}$-Appell system decomposition of the form $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \cdots \subset \mathcal{Z}_n$, where $\psi_\ell \in \mathcal{Z}_\ell \Leftrightarrow \psi_\ell = \sum_{k=0}^{\ell} \langle \psi_\ell \rangle_k$.

Specific examples of Clifford algebras include the following: $\mathcal{C}\ell_{0,2}$ is canonically isomorphic to the algebra of quaternions; $\mathcal{C}\ell_{3,0}$ is isomorphic to the algebra of physical space (APS) spanned by the Pauli spin matrices [1]. The Clifford algebra $\mathcal{C}\ell_{n,0}$ is canonically isomorphic to the $n$-particle fermionic Fock space, and $\mathcal{C}\ell_{n,n}$ is isomorphic to the $n$-particle fermion algebra [5] via the correspondence

$$f_i \simeq \frac{1}{2}(\mathbf{e}_i - \mathbf{e}_{n+i}), \tag{8}$$

$$f_i^+ \simeq \frac{1}{2}(\mathbf{e}_i + \mathbf{e}_{n+i}), \tag{9}$$

where for each $i = 1, 2, \ldots, n$, $f_i$ and $f_i^+$ respectively denote the operators of fermion annihilation and creation at position $i$. The reader is referred to [6], [7], and [8] for more on properties of Clifford algebras.

## 2. Clifford Polynomials and Operator Calculus

Consider the linear space of polynomials with real coefficients $C_{p,q} = \mathbb{R}[x_1, \ldots, x_n]$ satisfying

$$x_i\, x_j = \begin{cases} -x_j\, x_i\,, & \text{if } i \neq j\,, \\ 1\,, & \text{if } 1 \leq i = j \leq p\,, \\ -1\,, & \text{if } p+1 \leq i = j \leq p+q = n. \end{cases} \tag{10}$$

**Remark 2.** Clearly $C_{p,q} \simeq \mathcal{C}\ell_{p,q}$ via $1 \mapsto \mathbf{e}_\emptyset$, $\prod_{j=1}^{k} x_{i_j} \mapsto \mathbf{e}_{\{i_1,\ldots,i_j\}}$.

In general, the notation $\varphi(\vec{x})$ will be used to denote anything of the form $\sum_{\underline{i}} \alpha_{\underline{i}}\, x_{\underline{i}} \in C_{p,q}$.

**Definition 3.** The *Clifford inner product* is defined by

$$\left\langle \sum_{\underline{i}} \alpha_{\underline{i}}\, x_{\underline{i}}, \sum_{\underline{j}} \beta_{\underline{j}}\, x_{\underline{j}} \right\rangle = \sum_{\underline{i}} \alpha_{\underline{i}} \beta_{\underline{i}}. \tag{11}$$

This inner product defines a sub-additive, non-multiplicative norm on $C_{p,q}$ by

$$\| \sum_{\underline{i}} \alpha_{\underline{i}}\, x_{\underline{i}} \|^2 = \sum_{\underline{i}} {\alpha_{\underline{i}}}^2. \tag{12}$$

**Lemma 4.** *The following identities in $C_{p,q}$ are easy to establish.*

$$e^{\alpha\, x_{\underline{j}}} = \begin{cases} \cosh\alpha + x_{\underline{j}}\, \sinh\alpha\,, & \textit{if } {x_{\underline{j}}}^2 = 1\,, \\ \cos\alpha + x_{\underline{j}}\, \sin\alpha\,, & \textit{if } {x_{\underline{j}}}^2 = -1. \end{cases} \tag{13}$$

$$\left(\sum_{i=1}^{n}\alpha_i\,x_i\right)^k = \begin{cases}\left(\sum_{i=1}^{p}{\alpha_i}^2-\sum_{j=p+1}^{n}{\alpha_j}^2\right)^{k/2}, & \textit{if } k\equiv 0 \pmod 2,\\ \left(\sum_{i=1}^{p}{\alpha_i}^2-\sum_{j=p+1}^{n}{\alpha_j}^2\right)^{(k-1)/2}\sum_{i=1}^{n}\alpha_i\,x_i\,, & \textit{if } k\equiv 1 \pmod 2.\end{cases} \tag{14}$$

*Proof.* The proof of (13) is by direct calculation:

$$e^{\alpha\,x_{\underline{j}}} = \sum_{k=0}^{\infty}\frac{\alpha^k {x_{\underline{j}}}^k}{k!} = \begin{cases}\sum_{k\text{ even}}\frac{\alpha^k}{k!}+\sum_{k\text{ odd}}\frac{\alpha^k}{k!}x_{\underline{j}} & \text{if } x_{\underline{j}}^2=1\\ \sum_{k\text{ even}}(-1)^{k/2}\frac{\alpha^k}{k!}+\sum_{k\text{ odd}}(-1)^{(k-1)/2}\frac{\alpha^k}{k!}x_{\underline{j}} & \text{if } x_{\underline{j}}^2=-1\end{cases}$$
$$= \begin{cases}\cosh\alpha + x_{\underline{j}}\sinh\alpha & \text{if } {x_{\underline{j}}}^2=1\\ \cos\alpha + x_{\underline{j}}\sin\alpha & \text{if } {x_{\underline{j}}}^2=-1.\end{cases} \tag{15}$$

Proof of identity (14) is by induction on $k$. When $k=1$, the identity is obvious. When $k=2$,

$$\left(\sum_{i=1}^{n}\alpha_i\,x_i\right)^2 = \sum_{(i,j)\in[n]\times[n]}\alpha_i\alpha_j\,x_i\,x_j = \sum_{i=1}^{n}{\alpha_i}^2{x_i}^2$$
$$+\sum_{1\le i<j\le n}\alpha_i\alpha_j\,(x_i\,x_j+x_j\,x_i) = \sum_{i=1}^{n}\alpha_i^2{x_i}^2 = \sum_{i=1}^{p}\alpha_i^2-\sum_{j=p+1}^{n}\alpha_j^2. \tag{16}$$

Assume the identity holds for $k$. Then, if $k+1$ is odd,

$$\left(\sum_{i=1}^{n}\alpha_i\,x_i\right)^{k+1} = \left(\sum_{i=1}^{n}\alpha_i\,x_i\right)^k\left(\sum_{i=1}^{n}\alpha_i\,x_i\right)$$
$$= \left(\sum_{i=1}^{p}\alpha_i^2-\sum_{j=p+1}^{n}\alpha_j^2\right)^{k/2}\sum_{i=1}^{n}\alpha_i\,x_i. \tag{17}$$

Otherwise, if $k+1$ is even,

$$\left(\sum_{i=1}^{n} \alpha_i\, x_i\right)^{k+1} = \left(\sum_{i=1}^{n} \alpha_i\, x_i\right)^{k} \left(\sum_{i=1}^{n} \alpha_i\, x_i\right) = \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{n} \alpha_j^2\right)^{(k-1)/2}$$
$$\times \left(\sum_{i=1}^{n} \alpha_i\, x_i\right)\left(\sum_{i=1}^{n} \alpha_i\, x_i\right) = \left(\sum_{i=1}^{p} \alpha_i^2 - \sum_{j=p+1}^{n} \alpha_j^2\right)^{(k+1)/2} . \quad (18)$$

□

**Corollary 5.**

$$\left(\sum_{i=1}^{n} x_i\right)^{k} = \begin{cases} (p-q)^{k/2}\,, & \textit{if } k \equiv 0 \pmod 2\,, \\ (p-q)^{(k-1)/2} \displaystyle\sum_{i=1}^{n} x_i\,, & \textit{if } k \equiv 1 \pmod 2. \end{cases} \quad (19)$$

Define the *ordering symbol* $\theta_{i\,j}$ by

$$\theta_{i\,j} = \begin{cases} 1\,, & \text{if } i < j\,, \\ 0\,, & \text{otherwise.} \end{cases} \quad (20)$$

Define the *multi-index product signature map* $\vartheta : 2^{[n]} \times 2^{[n]} \to \{\pm 1\}$ by

$$\vartheta(\underline{i}, \underline{j}) = \exp\left[\imath\pi \left(\sum_{k=1}^{|\underline{j}|}\left(\sum_{\ell=1}^{|\underline{i}|} \theta_{\underline{j}_k\, \underline{i}_\ell} + \theta_{p\, \underline{j}_k} \chi_{\underline{i}}(\underline{j}_k)\right)\right)\right], \quad (21)$$

where $\chi_{\underline{i}}(\underline{j}_k)$ is the *indicator function* of the multi-index $\underline{i}$, defined by

$$\chi_{\underline{i}}(\underline{j}_k) = \begin{cases} 1 & \text{if } \underline{j}_k \in \underline{i}\,, \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

For each $2 < m \le n$, the *$m$-dimensional vector product signature map* can be defined as a map $\vartheta_m : (2^{[n]})^m \to \{\pm 1\}$ by

$$\vartheta_m(i_1, \ldots, i_m)$$
$$= \exp\left[\imath\pi \left(\sum_{k=0}^{m-1}\left(\sum_{\ell=1}^{k} \theta_{i_{m-k}\, i_\ell} + \theta_{p, i_{m-k}} \chi_{\{i_\ell\}}(i_{m-k})\right)\right)\right]. \quad (23)$$

The *generalized vector product signature map* $\tilde{\vartheta} : \bigoplus_{m=2}^{n} (2^{[n]})^m \to \{\pm 1\}$ is defined as the direct sum $\tilde{\vartheta} = \sum_{m=2}^{n} \vartheta_m$.

Now define the *lowering operators* $D_j$ by

$$D_j\, x_{\underline{i}} = \vartheta(\underline{i}, \{j\}) \frac{\partial}{\partial x_j} x_{\underline{i}} = \left\langle x_{\underline{i}}\, x_j \right\rangle_{|\underline{i}|-1}. \tag{24}$$

Hence,

$$D_j\, x_{\underline{i}} = \begin{cases} \vartheta(\underline{i}, \{j\}) x_{\underline{i} \setminus \{j\}} & \text{if } j \in \underline{i}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

**Definition 6.** Define the *Clifford integrals* by

$$\{dx_i, dx_j\} = 0 \,\text{for } i \neq j\,, \tag{26}$$

$$\int dx_j = x_j\,, \tag{27}$$

$$\int\!\!\int dx_i\, dx_j = \int x_i\, dx_j = \begin{cases} x_i\, x_j & \text{if } i \neq j\,, \\ 1 & \text{if } 1 \le i = j \le p\,, \\ -1 & \text{if } p+1 \le i = j \le p+q\,, \end{cases} \tag{28}$$

so that

$$\int x_{\underline{i}}\, dx_j = \begin{cases} \vartheta(\underline{i}, \{j\}) x_{\underline{i} \cup \{j\}} & \text{if } j \notin \underline{i}\,, \\ \vartheta(\underline{i}, \{j\}) x_{\underline{i} \setminus \{j\}} & \text{if } j \in \underline{i}. \end{cases} \tag{29}$$

Clifford integrals as raising operators:

$$R_j\, x_{\underline{i}} = \left\langle \int x_{\underline{i}}\, dx_j \right\rangle_{|\underline{i}|+1} = \begin{cases} \vartheta(\underline{i}, \{j\}) x_{\underline{i} \cup \{j\}} & \text{if } j \notin \underline{i}\,, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

Observe that

$$R_j\, x_{\underline{i}} = \left\langle x_{\underline{i}}\, x_j \right\rangle_{|\underline{i}|+1}. \tag{31}$$

The raising and lowering operators $D_j$, $R_j$ satisfy the following equations:

$$D_j R_j\, x_{\underline{i}} = \begin{cases} x_{\underline{i}} & \text{if } 1 \le j \le p, j \notin \underline{i}, \\ -x_{\underline{i}} & \text{if } p+1 \le j \le p+q, j \notin \underline{i}, \\ 0 & \text{otherwise}\,, \end{cases} \tag{32}$$

$$R_j D_j\, x_{\underline{i}} = \begin{cases} x_{\underline{i}} & \text{if } 1 \leq j \leq p, j \in \underline{i}, \\ -x_{\underline{i}} & \text{if } p+1 \leq j \leq p+q, j \notin \underline{i}, \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

In other words, $D_j R_j$ and $R_j D_j$ satisfy the identity

$$D_j R_j + R_j D_j = \theta_{j\,p} - \theta_{p\,j}.$$

Now the action of the gradient operator $\nabla = \sum_{j=1}^{n} D_j$ is equivalent to multiplication via

$$\nabla x_{\underline{i}} = \left\langle \sum_{j=1}^{p+q} x_{\underline{i}}\, x_j \right\rangle_{|\underline{i}|-1}. \tag{34}$$

**Definition 7.** Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ with $\sum_{j=1}^{n} {\lambda_j}^2 = 1$. Define the *finite-dimensional lowering operator* $\nabla_{\vec{\lambda}}$ on $C_{p,q}$ by

$$\nabla_{\vec{\lambda}} = \sum_{j=1}^{n} \lambda_j D_j. \tag{35}$$

In this way,

$$\nabla_{\vec{\lambda}} \varphi(\vec{x}) = \sum_{k=1}^{n} \left\langle \sum_{j=1}^{n} \lambda_j\ \langle \varphi(\vec{x}) \rangle_k\ x_j \right\rangle_{k+1}, \tag{36}$$

for any Clifford polynomial $\varphi(\vec{x})$.

It is apparent that

$$R_j R_i = \begin{cases} -R_i R_j & \text{if } i \neq j\,, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Hence, the collection $\{R_i\}$, $1 \leq i \leq n$ generates an algebra isomorphic to the algebra of fermion creation operators.

Moreover,

$$D_i D_j = \begin{cases} -D_j D_i & \text{if } i \neq j\,, \\ 0 & \text{otherwise,} \end{cases} \tag{38}$$

so that the collection $\{D_i\}$, $1 \leq i \leq n$ generates an algebra isomorphic to the fermion annihilation operators.

**Definition 8.** For any unit vector $\vec{\lambda} \in \mathbb{R}^n$, the *finite-dimensional raising operator* $\mathcal{R}_{\vec{\lambda}}$ is defined on $C_{p,q}$ by

$$\mathcal{R}_{\vec{\lambda}} = \sum_{j=1}^{n} R_j. \tag{39}$$

The action of the finite-dimensional raising operator $\mathcal{R}_{\vec{\lambda}}$ is equivalent to multiplication via

$$\mathcal{R}_{\vec{\lambda}} x_{\underline{i}} = \left\langle \sum_{j=1}^{n} \lambda_j \, x_{\underline{i}} \, x_j \right\rangle_{|\underline{i}|+1}. \tag{40}$$

In this way,

$$\mathcal{R}_{\vec{\lambda}} \varphi(\vec{x}) = \sum_{k=1}^{n} \left\langle \sum_{j=1}^{n} \lambda_j \, \langle \varphi(\vec{x}) \rangle_k \, x_j \right\rangle_{k+1}, \tag{41}$$

for any Clifford polynomial $\varphi(\vec{x})$. Observe that $\|\mathcal{R}_{\vec{\lambda}} x_\emptyset\| = 1$.

In the infinite-dimensional case, define the *raising and lowering operators* by

$$\mathcal{R}_{\vec{\lambda}} = \sum_{j} \lambda_j \, R_j \,, \tag{42}$$

$$\nabla_{\vec{\lambda}} = \sum_{j} \lambda_j \, D_j \,, \tag{43}$$

where $\sum_j {\lambda_j}^2 = 1$. As in the finite-dimensional case, $\|\mathcal{R} x_\emptyset\| = 1$.

## 3. Clifford Appell Systems

Let $C_{p,q}$ represent a Clifford algebra for fixed $n = p+q$. Let $\{\lambda_i(k)\}$, $1 \le i \le n$, $k > 0$, denote a collection of real-valued variables satisfying

$$\sum_i \lambda_i(k)^2 < \infty, \tag{44}$$

for each $k > 0$. Moreover, assume that

$$\sum_{(i_1,\ldots,i_k) \in [n]^k} (\lambda_{i_1}(1) \cdots \lambda_{i_k}(k))^2 < \infty \,, \tag{45}$$

for $1 \le k \le n$. Let $\vec{\lambda}(k) = (\lambda_1(k), \ldots, \lambda_n(k))$, and define the canonical variable $\Lambda(k) = \sum_{i=1}^{n} \lambda_i(k)\, x_i \in C_{p,q}$.

**Definition 9.** The *dynamic lowering operator* associated with a sequence of vectors $\{\vec{\lambda}(k)\}$ is defined by $\nabla_{\vec{\lambda}(k)} = \sum_{i=1}^{n} \lambda_i(k)\, x_i$.

**Definition 10.** A *Clifford Appell system* is a system of Clifford polynomials satisfying:

1. $\psi_k$ is a Clifford polynomial of degree $k$ for each $k$, and
2. $\nabla_{\vec{\lambda}(k)}\psi_k = \psi_{k-1}$,

for a sequence of vectors $\{\vec{\lambda}(k)\}$ satisfying (45).

By definition, a Clifford Appell system $\{\psi_k\}$ in $C_{p,q}$ satisfies the recurrence relation

$$\psi_{k-1} = \sum_{m=1}^{n} \left\langle \langle \psi_k \rangle_m \Lambda(k) \right\rangle_{m-1}, \tag{46}$$

so that for $1 \le \ell \le k$,

$$\psi_{k-\ell} = \nabla_{\vec{\lambda}(k-\ell+1)} \cdots \nabla_{\vec{\lambda}(k-1)} \nabla_{\vec{\lambda}(k)} \psi_k = \sum_{m_1=1}^{n} \cdots \sum_{m_\ell=1}^{n} \left\langle \cdots \left\langle \left\langle \langle \psi_k \rangle_{m_\ell} \Lambda(k) \right\rangle_{m_\ell - 1} \Lambda(k-1) \right\rangle_{m_{\ell-1}-1} \cdots \right\rangle_{m_1 - 1}. \tag{47}$$

In the space of Clifford polynomials $C_{n,0}$, which is canonically isomorphic to the $n$-particle fermionic Fock space, the lowering operator maps monomials representing $k$-particle systems to monomials representing $(k-1)$-particle systems. I.e., the lowering operator acts as an annihilation operator.

Another interpretation involves Appell systems on Abelian unipotent-generated subalgebras of Clifford algebras of balanced signature $\mathcal{C}\ell_{n,n}$. In this case, the Appell system represents a sort of "symmetrized" fermionic Fock space used to induce random walks on the hypercube [9].

Analysis of Clifford Appell systems is simplified by considering *homogeneous* Clifford polynomials. General Clifford Appell systems can then be constructed as sums of homogeneous systems.

**Definition 11.** A *homogeneous Clifford Appell system* is a system of Clifford polynomials satisfying:

1. $\psi_k$ is a homogeneous Clifford polynomial of degree $k$ for each $k$, and
2. $\nabla_{\vec{\lambda}(k)}\psi_k = \psi_{k-1}$,

for a sequence of vectors $\{\vec{\lambda}(k)\}$ satisfying (45).

**Proposition 12.** *A homogeneous Clifford Appell system $\{\psi_k\}$ in $C_{p,q}$ satisfies the recurrence relation*

$$\psi_{k-\ell} = \nabla_{\vec{\lambda}(k-\ell+1)} \cdots \nabla_{\vec{\lambda}(k-1)} \nabla_{\vec{\lambda}(k)} \psi_k = \big\langle \cdots \big\langle \langle \psi_k \Lambda(k) \rangle_{k-1} \Lambda(k-1) \big\rangle_{k-2} \cdots \Lambda(k-\ell+1) \big\rangle_{k-\ell} = \left\langle \psi_k \prod_{j=0}^{\ell-1} \Lambda(k-j) \right\rangle_{k-\ell}, \quad (48)$$

*for $1 \leq \ell \leq k$.*

*Proof.* By definition, the system satisfies the recurrence

$$\psi_{k-1} = \langle \psi_k \Lambda(k) \rangle_{k-1} . \quad (49)$$

Recursively,

$$\psi_{k-\ell} = \Big\langle \cdots \big\langle \langle \psi_k \Lambda(k) \rangle_{k-1} \Lambda(k-1) \big\rangle_{k-2} \cdots \Lambda(k-\ell+1) \Big\rangle_{k-\ell} . \quad (50)$$

Because $\psi_k$ is homogeneous of degree $k$, $\ell$ iterations must result in homogeneous terms of degree $k-\ell$. Since $\ell$ multiplications of $k$-vectors by 1-vectors result in terms of degree $k-\ell$ only when each multiplication reduces the degree by 1, nothing is lost by removing the intermediate projections. □

Now defining the sequence of vectors $\vec{\lambda}(k) = (\lambda_1(k), \ldots, \lambda_n(k))$ as in the statement of Proposition 12, let $m_{[n]}$ denote any nonzero real scalar. The associated dynamic homogeneous Clifford Appell system is constructed by

$$\psi_n = m_{[n]}\, x_{[n]} , \quad (51)$$

$$\psi_{n-\ell} = \left\langle \psi_n \prod_{j=0}^{\ell-1} \Lambda(n-j) \right\rangle_{n-\ell} . \quad (52)$$

Appropriate conditions on the sequence $\{\vec{\lambda}(k)\}$ to ensure $\psi_k \neq 0$ for all $1 \leq k \leq n$ are addressed in the next proposition.

As Proposition 12 shows, symbolic calculations are simpler in homogeneous Clifford Appell systems than in the more general case. Another obvious simplification comes from considering the time-homogeneous rather than the dynamic case. However, assuming $\vec{\lambda}(1) = \vec{\lambda}(2) = \cdots = \vec{\lambda}$, the following proposition shows that any time-homogeneous Clifford Appell system "collapses" to zero after two applications of $\nabla_{\vec{\lambda}}$.

**Proposition 13.** *Fix a vector $\vec{\lambda}$, and assume $\vec{\lambda}(k) = \vec{\lambda}$ for all $k$. Let $\psi_j$ be any nonzero polynomial in the homogeneous Clifford Appell system $\{\psi_k\}$. Then $\psi_{j-\ell} = 0$ for all $1 < \ell \leq j$.*

*Proof.* In light of Lemma 4 and Proposition 12,

$$\psi_{k-\ell} = \begin{cases} \left\langle \left( \sum_{i=1}^{p} {\lambda_i}^2 - \sum_{j=p+1}^{p+q} {\lambda_j}^2 \right)^{\ell/2} \psi_k \right\rangle_{k-\ell}, & \text{if } \ell \equiv 0 \pmod 2, \\ \left\langle \left( \sum_{i=1}^{p} {\lambda_i}^2 - \sum_{j=p+1}^{p+q} {\lambda_j}^2 \right)^{(\ell-1)/2} \psi_k \Lambda \right\rangle_{k-\ell}, & \text{if } \ell \equiv 1 \pmod 2. \end{cases} \tag{53}$$

In either case, the degree $k - \ell$ parts are zero if $\ell > 1$. □

**Definition 14.** For $m > 1$, the *simplex* of the Cartesian product of sets $A_1 \times \cdots \times A_m$ is defined by

$$\mathcal{S}(A_1 \times \cdots \times A_m) = \{(a_i, \ldots, a_m) \in A_1 \times \cdots \times A_m \,:\, a_1 < a_2 < \cdots < a_m\}. \tag{54}$$

**Proposition 15.** *A necessary and sufficient condition for $\psi_{n-m} \neq 0$ is (i) $\psi_n \neq 0$ and (ii) there exists $(i_1, \ldots, i_m) \in \mathcal{S}([n]^m)$ such that the following holds:*

$$\sum_{\sigma \in S_m} \tilde{\vartheta}(\sigma(i_1), \ldots, \sigma(i_m)) \lambda_{\sigma(i_1)}(n-m+1) \lambda_{\sigma(i_2)}(n-m+2) \cdots \lambda_{\sigma(i_m)}(n) \neq 0, \tag{55}$$

*where the sum is taken over all permutations of the indices $i_1, \ldots, i_m$.*

*Proof.* Observe that given $\psi_n = m_{[n]}\, x_{[n]}$ for nonzero scalar $m_{[n]}$, the coefficient of $x_{\underline{i}}$ in $\psi_k$ is given by

$$\langle \psi_k, x_{\underline{i}} \rangle = m_{[n]} \sum_{\sigma \in S_{n-k}} \tilde{\vartheta}(\sigma(i_1), \ldots, \sigma(i_m)) \lambda_{\sigma(i_1)}(n-k+1) \cdots \lambda_{\sigma(i_{n-k})}(n), \tag{56}$$

where $|\underline{i}| = k$ and $\{i_1\} \cup \cdots \cup \{i_{n-k}\} = [n] \setminus \underline{i}$. Since $\psi_k \neq 0$ if and only if $\exists \underline{i} \in 2^{[n]}$ such that this coefficient is nonzero, the proof is complete. □

**Remark 16.** Constructing the $n \times n$ matrix

$$M = \begin{pmatrix} \lambda_1(1) & \lambda_1(2) & \cdots & \lambda_1(n) \\ \lambda_2(1) & & & \lambda_2(n) \\ \vdots & & & \vdots \\ \lambda_n(1) & \lambda_n(2) & \cdots & \lambda_n(n) \end{pmatrix}, \tag{57}$$

the proposition can be restated as follows: A necessary and sufficient condition for $\psi_{n-m} \neq 0$ in the dynamic Clifford Appell system is: (i) $\psi_n \neq 0$, and (ii) $\exists$ an $m$-tuple of rows in $M$ such that the submatrix formed by these $m$ rows and columns $n-m+1$ through $n$ has nonzero determinant.

**Proposition 17.** *Let $\{\psi_k\}$, $k = 0, \ldots, n$ be a homogeneous Clifford Appell system. For each $k = 0, \ldots, n$, define*

$$\Psi_k = \sum_{\ell=0}^{k} \psi_\ell \,. \tag{58}$$

*Define the lowering operator $\nabla$ on $\{\Psi_k\}$ by $\nabla = \displaystyle\sum_{k=1}^{n} \nabla_{\vec{\lambda}(k)}$, where each $\nabla_{\vec{\lambda}(k)}$ is defined by*

$$\nabla_{\vec{\lambda}(k)} \psi_j = \begin{cases} \psi_{k-1} & \text{if } k = j\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{59}$$

*Then $\{\Psi_k\}$ is a Clifford Appell system.*

*Proof.* Let $\{\psi_k\}$ be a homogeneous Clifford Appell system and let $\Psi_k$ be defined as in the proposition. Then,

$$\nabla\, \Psi_k = \nabla \left( \sum_{\ell=0}^{k} \psi_\ell \right) = \sum_{\ell=0}^{k} \nabla(\psi_\ell) = \sum_{\ell=1}^{k} \psi_{\ell-1} = \sum_{\ell=0}^{k-1} \psi_\ell = \Psi_{k-1}. \tag{60}$$

□

### 3.1. Stochastic Clifford Appell Systems

When the sequence $\{\vec{\lambda}(k)\}$ of (45) is replaced by a sequence of random vectors $\vec{\xi}(k)$, a stochastic Clifford Appell system is obtained. Let $\{\xi_i(k)\}$, $1 \leq i \leq n$, $k > 0$, denote a collection of pairwise-independent real-valued random variables satisfying

$$\sum_i \mathbb{E}(\xi_i(k))^2 < \infty, \tag{61}$$

for each $k > 0$. Moreover, assume that

$$\mathbb{E}\left(\sum_{(i_1,\ldots,i_k)\in[n]^k} (\xi_{i_1}(1)\cdots\xi_{i_k}(k))^2\right) < \infty\,, \tag{62}$$

for $1 \le k \le n$. Let $\vec{\xi}(k) = (\xi_1(k),\ldots,\xi_n(k))$, and define the random variable $\Xi(k) = \sum_{i=1}^{n} \xi_i(k)\, x_i \in C_{p,q}$.

**Definition 18.** The *stochastic lowering operator* associated with the random vector $\vec{\xi}$ is defined by $\nabla_{\vec{\xi}} = \sum_{i=1}^{n} \xi_i\, x_i$. Given a sequence of random vectors $\{\vec{\xi}(k)\}$, this determines a sequence of stochastic lowering operators, $\nabla_{\vec{\xi}(k)}$.

**Definition 19.** A *stochastic Clifford Appell system* is a system of Clifford polynomials satisfying

1. $\psi_k$ is a homogeneous Clifford polynomial of degree $k$ for each $k$, and
2. $\mathbb{E}\left(\nabla_{\vec{\xi}(k)}\psi_k\right) = \psi_{k-1}\,,$

for a system of random vectors $\{\vec{\xi}(k)\}$ satisfying (62).

By definition, a stochastic Clifford Appell system $\{\psi_k\}$ in $C_{p,q}$ satisfies the recurrence relation

$$\psi_{k-1} = \mathbb{E}\left(\sum_{m=1}^{n} \langle\langle\psi_k\rangle_m\, \Xi(k)\rangle_{m-1}\right) = \sum_{m=1}^{n} \langle\langle\psi_k\rangle_m\, \mathbb{E}\left(\Xi(k)\right)\rangle_{m-1}\,. \tag{63}$$

Because the collection $\{\Xi(k)\}$ is pairwise-independent,

$$\mathbb{E}\left[\Xi(i)\mathbb{E}(\Xi(j))\right] = \mathbb{E}(\Xi(i))\mathbb{E}(\Xi(j)) = \mathbb{E}(\Xi(i)\Xi(j)) \tag{64}$$

holds for all $i \ne j$. Hence,

$$\begin{aligned}\psi_{k-\ell} &= \mathbb{E}\left(\mathbb{E}\left(\nabla_{\vec{\xi}(k-\ell+1)}\cdots\mathbb{E}\left(\nabla_{\vec{\xi}(k-1)}\mathbb{E}\left(\nabla_{\vec{\xi}(k)}\psi_k\right)\right)\cdots\right)\right)\\ &= \mathbb{E}\left(\nabla_{\vec{\xi}(k-\ell+1)}\cdots\nabla_{\vec{\xi}(k-1)}\nabla_{\vec{\xi}(k)}\psi_k\right)\,,\end{aligned} \tag{65}$$

for $1 \le \ell \le k$.

As before, homogeneous polynomials are easier to work with and allow the construction of general Appell systems. The following results are corollaries of Proposition 12 and Proposition 15.

**Corollary 20.** *A stochastic homogeneous Clifford Appell system $\{\psi_k\}$ in $C_{p,q}$ satisfies the recurrence relation*

$$\psi_{k-1} = \langle \psi_k \mathbb{E}(\Xi(k)) \rangle_{k-1}. \tag{66}$$

*In particular, for $1 \leq \ell \leq k$,*

$$\psi_{k-\ell} = \prod_{j=0}^{\ell-1} \nabla_{\vec{\xi}(k-j)} \psi_k = \big\langle \cdots \big\langle \langle \psi_k \mathbb{E}(\Xi(k)) \rangle_{k-1} \mathbb{E}(\Xi(k-1)) \big\rangle_{k-2} \cdots \\ \mathbb{E}(\Xi(k-\ell+1)) \big\rangle_{k-\ell} = \left\langle \psi_k \prod_{j=0}^{\ell-1} \mathbb{E}(\Xi(k-j)) \right\rangle_{k-\ell}. \tag{67}$$

**Corollary 21.** *Let $\{\psi_k\}$ be a stochastic homogeneous Clifford Appell system. A necessary and sufficient condition for $\psi_{n-m} \neq 0$ is (i) $\psi_n \neq 0$ and (ii) there exists $(i_1, \ldots, i_m) \in [n]^m$ such that the following holds:*

$$\sum_{\sigma \in S_m} \tilde{\vartheta}(\sigma(i_1), \ldots, \sigma(i_m)) \mathbb{E}(\xi_{\sigma(i_1)}(n-m+1)) \cdots \mathbb{E}(\xi_{\sigma(i_m)}(n)) \neq 0, \tag{68}$$

*where the sum is taken over all permutations of the indices $i_1, \ldots, i_m$.*

As in Proposition 17, a stochastic homogeneous Clifford Appell system $\{\psi_k\}$ defines a general stochastic Clifford Appell system $\{\Psi_k\}$ with stochastic lowering operator $\nabla$ defined on $\{\Psi_k\}$ by $\nabla = \sum_{k=1}^{n} \nabla_{\vec{\xi}(k)}$, where the action of each $\nabla_{\vec{\xi}(k)}$ is defined by

$$\mathbb{E}\left(\nabla_{\vec{\xi}(k)} \psi_j\right) = \begin{cases} \psi_{k-1} & \text{if } k = j\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{69}$$

## 4. Examples

The following examples were computed using *Mathematica*.

**Example 22.** Randomly generated Appell system in $\mathcal{C}\ell_{2,2}$.

```
In[12]:= RandAppell[2, 2]
```

$$\begin{pmatrix} 0.202208 & 0.539543 & 0.732355 & 0.616101 \\ 0.644328 & 0.733072 & 0.570441 & 0.607417 \\ 0.937435 & 0.0519123 & 0.694982 & 0.0127506 \\ 0.722161 & 0.0598234 & 0.483695 & 0.955951 \end{pmatrix}$$

```
Λ[1]=0.202208 ∈1 + 0.539543 ∈2 + 0.732355 ∈3 + 0.616101 ∈4

Λ[2]=0.644328 ∈1 + 0.733072 ∈2 + 0.570441 ∈3 + 0.607417 ∈4

Λ[3]=0.937435 ∈1 + 0.0519123 ∈2 + 0.694982 ∈3 + 0.0127506 ∈4

Λ[4]=0.722161 ∈1 + 0.0598234 ∈2 + 0.483695 ∈3 + 0.955951 ∈4

The Appell system for signature (2,2) is

ψ[4] = ∈1 ∈2 ∈3 ∈4

ψ[3] = -0.955951 ∈1 ∈2 ∈3 + 0.483695 ∈1 ∈2 ∈4 + 0.0598234 ∈1 ∈3 ∈4 - 0.722161 ∈2 ∈3 ∈4

ψ[2] = 0.284999 ∈1 ∈2 - 0.00565071 ∈1 ∈3 +

    0.865575 ∈2 ∈3 + 0.244462 ∈1 ∈4 - 0.451862 ∈2 ∈4 - 0.733502 ∈3 ∈4

ψ[1] = 0.444633 ∈1 + 0.111279 ∈2 + 0.60741 ∈3 + 0.0355174 ∈4

ψ[0] = -0.316774

-------------
```

**Example 23.** Iterations of lowering operators give zero in the time-homogeneous case.

```
(* Consider constant lowering operator Λ in Cl2,2 *)
M = Table[1., {i, 1, 4}, {j, 1, 4}];
Appell[M, 2, 2]

Λ[1]=1. ∈1 + 1. ∈2 + 1. ∈3 + 1. ∈4

Λ[2]=1. ∈1 + 1. ∈2 + 1. ∈3 + 1. ∈4

Λ[3]=1. ∈1 + 1. ∈2 + 1. ∈3 + 1. ∈4

Λ[4]=1. ∈1 + 1. ∈2 + 1. ∈3 + 1. ∈4
The Appell system for signature (2,2) is

ψ[4] = ∈1 ∈2 ∈3 ∈4

ψ[3] = -1. ∈1 ∈2 ∈3 + 1. ∈1 ∈2 ∈4 + 1. ∈1 ∈3 ∈4 - 1. ∈2 ∈3 ∈4

ψ[2] = 0

ψ[1] = 0

ψ[0] = 0
```

**Example 24.** A randomly generated Appell system over $\mathcal{C}\ell_{1,2}$.

```
RandAppell[1, 2]

(0.193967  0.41318   0.98275 )
(0.575233  0.0335319 0.244682)
(0.932257  0.416867  0.215259)

Λ[1]=0.193967 ∈1 + 0.41318 ∈2 + 0.98275 ∈3

Λ[2]=0.575233 ∈1 + 0.0335319 ∈2 + 0.244682 ∈3

Λ[3]=0.932257 ∈1 + 0.416867 ∈2 + 0.215259 ∈3
```

The Appell system for signature (1,2) is

$\psi[3] = \epsilon_1 \epsilon_2 \epsilon_3$

$\psi[2] = -0.215259\,\epsilon_1 \epsilon_2 + 0.416867\,\epsilon_1 \epsilon_3 + 0.932257\,\epsilon_2 \epsilon_3$

$\psi[1] = -0.0947818\,\epsilon_1 - 0.104282\,\epsilon_2 - 0.208535\,\epsilon_3$

$\psi[0] = 0.229641$

**Example 25.** Randomly generated Appell system over $\mathcal{C}\ell_{3,3}$.

**RandAppell[3, 3]**

$$\begin{pmatrix} 0.421948 & 0.39473 & 0.840284 & 0.785509 & 0.917651 & 0.484684 \\ 0.0609862 & 0.872974 & 0.470277 & 0.0914455 & 0.732088 & 0.283176 \\ 0.814429 & 0.338121 & 0.956347 & 0.891941 & 0.160435 & 0.168498 \\ 0.0504924 & 0.158366 & 0.818273 & 0.72216 & 0.572454 & 0.408833 \\ 0.396324 & 0.32743 & 0.73217 & 0.623324 & 0.478673 & 0.842746 \\ 0.671184 & 0.75035 & 0.00839611 & 0.7513 & 0.939096 & 0.467173 \end{pmatrix}$$

$\Lambda[1] = 0.421948\,\epsilon_1 + 0.39473\,\epsilon_2 + 0.840284\,\epsilon_3 + 0.785509\,\epsilon_4 + 0.917651\,\epsilon_5 + 0.484684\,\epsilon_6$

$\Lambda[2] = 0.0609862\,\epsilon_1 + 0.872974\,\epsilon_2 + 0.470277\,\epsilon_3 + 0.0914455\,\epsilon_4 + 0.732088\,\epsilon_5 + 0.283176\,\epsilon_6$

$\Lambda[3] = 0.814429\,\epsilon_1 + 0.338121\,\epsilon_2 + 0.956347\,\epsilon_3 + 0.891941\,\epsilon_4 + 0.160435\,\epsilon_5 + 0.168498\,\epsilon_6$

$\Lambda[4] = 0.0504924\,\epsilon_1 + 0.158366\,\epsilon_2 + 0.818273\,\epsilon_3 + 0.72216\,\epsilon_4 + 0.572454\,\epsilon_5 + 0.408833\,\epsilon_6$

$\Lambda[5] = 0.396324\,\epsilon_1 + 0.32743\,\epsilon_2 + 0.73217\,\epsilon_3 + 0.623324\,\epsilon_4 + 0.478673\,\epsilon_5 + 0.842746\,\epsilon_6$

$\Lambda[6] = 0.671184\,\epsilon_1 + 0.75035\,\epsilon_2 + 0.00839611\,\epsilon_3 + 0.7513\,\epsilon_4 + 0.939096\,\epsilon_5 + 0.467173\,\epsilon_6$

The Appell system for signature (3,3) is

$\psi[6] = \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 \epsilon_6$

$\psi[5] = -0.467173\,\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 + 0.939096\,\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_6 - 0.7513\,\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_5 \epsilon_6 -$
$0.00839611\,\epsilon_1 \epsilon_2 \epsilon_4 \epsilon_5 \epsilon_6 + 0.75035\,\epsilon_1 \epsilon_3 \epsilon_4 \epsilon_5 \epsilon_6 - 0.671184\,\epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 \epsilon_6$

$\psi[4] = -0.567796\,\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 + 0.341955\,\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_5 - 0.334975\,\epsilon_1 \epsilon_2 \epsilon_4 \epsilon_5 -$
$0.479388\,\epsilon_1 \epsilon_3 \epsilon_4 \epsilon_5 + 0.380485\,\epsilon_2 \epsilon_3 \epsilon_4 \epsilon_5 + 0.225734\,\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_6 +$
$0.683559\,\epsilon_1 \epsilon_2 \epsilon_4 \epsilon_6 + 0.0516838\,\epsilon_1 \epsilon_3 \epsilon_4 \epsilon_6 + 0.050909\,\epsilon_2 \epsilon_3 \epsilon_4 \epsilon_6 -$
$0.544846\,\epsilon_1 \epsilon_2 \epsilon_5 \epsilon_6 - 0.221713\,\epsilon_1 \epsilon_3 \epsilon_5 \epsilon_6 + 0.120606\,\epsilon_2 \epsilon_3 \epsilon_5 \epsilon_6 -$
$0.546635\,\epsilon_1 \epsilon_4 \epsilon_5 \epsilon_6 + 0.488093\,\epsilon_2 \epsilon_4 \epsilon_5 \epsilon_6 + 0.077616\,\epsilon_3 \epsilon_4 \epsilon_5 \epsilon_6$

$\psi[3] = 0.121999\,\epsilon_1 \epsilon_2 \epsilon_3 + 0.376908\,\epsilon_1 \epsilon_2 \epsilon_4 + 0.163378\,\epsilon_1 \epsilon_3 \epsilon_4 - 0.209954\,\epsilon_2 \epsilon_3 \epsilon_4 -$
$0.298967\,\epsilon_1 \epsilon_2 \epsilon_5 - 0.201397\,\epsilon_1 \epsilon_3 \epsilon_5 + 0.208197\,\epsilon_2 \epsilon_3 \epsilon_5 - 0.221836\,\epsilon_1 \epsilon_4 \epsilon_5 +$
$0.128706\,\epsilon_2 \epsilon_4 \epsilon_5 - 0.0677824\,\epsilon_3 \epsilon_4 \epsilon_5 - 0.00297236\,\epsilon_1 \epsilon_2 \epsilon_6 - 0.0538478\,\epsilon_1 \epsilon_3 \epsilon_6 +$
$0.0944082\,\epsilon_2 \epsilon_3 \epsilon_6 - 0.162379\,\epsilon_1 \epsilon_4 \epsilon_6 + 0.286554\,\epsilon_2 \epsilon_4 \epsilon_6 + 0.0337598\,\epsilon_3 \epsilon_4 \epsilon_6 +$
$0.127051\,\epsilon_1 \epsilon_5 \epsilon_6 - 0.226282\,\epsilon_2 \epsilon_5 \epsilon_6 - 0.0639563\,\epsilon_3 \epsilon_5 \epsilon_6 - 0.113207\,\epsilon_4 \epsilon_5 \epsilon_6$

$\psi[2] = -0.171041\,\epsilon_1 \epsilon_2 - 0.145589\,\epsilon_1 \epsilon_3 + 0.237316\,\epsilon_2 \epsilon_3 -$
$0.220736\,\epsilon_1 \epsilon_4 + 0.438822\,\epsilon_2 \epsilon_4 + 0.067256\,\epsilon_3 \epsilon_4 + 0.0744199\,\epsilon_1 \epsilon_5 -$
$0.28967\,\epsilon_2 \epsilon_5 - 0.143309\,\epsilon_3 \epsilon_5 - 0.1829\,\epsilon_4 \epsilon_5 - 0.0719472\,\epsilon_1 \epsilon_6 +$
$0.126578\,\epsilon_2 \epsilon_6 + 0.00791704\,\epsilon_3 \epsilon_6 - 0.0212331\,\epsilon_4 \epsilon_6 + 0.0667736\,\epsilon_5 \epsilon_6$

$\psi[1] = -0.231705\,\epsilon_1 + 0.258127\,\epsilon_2 - 0.101769\,\epsilon_3 - 0.261336\,\epsilon_4 + 0.280097\,\epsilon_5 - 0.0628922\,\epsilon_6$

$\psi[0] = -0.102659$

**Example 26.** A dynamic Clifford Appell system with randomly generated $\pm\frac{1}{2}$ coefficients.

```
In[13]:= (* Random Appell system with ±1/2 coefficients in Cl4,4 *)
         M = Table[(1 / 2) * (-1) ^Floor[Random[] * 2] * 1.0, {i, 1, 8}, {j, 1, 8}];
         Appell[M, 4, 4]
```

$\Lambda[1] = -0.5\in_1 + 0.5\in_2 - 0.5\in_3 - 0.5\in_4 - 0.5\in_5 + 0.5\in_6 + 0.5\in_7 + 0.5\in_8$

$\Lambda[2] = 0.5\in_1 + 0.5\in_2 + 0.5\in_3 + 0.5\in_4 + 0.5\in_5 - 0.5\in_6 + 0.5\in_7 + 0.5\in_8$

$\Lambda[3] = 0.5\in_1 + 0.5\in_2 + 0.5\in_3 - 0.5\in_4 - 0.5\in_5 + 0.5\in_6 + 0.5\in_7 + 0.5\in_8$

$\Lambda[4] = 0.5\in_1 - 0.5\in_2 - 0.5\in_3 + 0.5\in_4 + 0.5\in_5 - 0.5\in_6 + 0.5\in_7 + 0.5\in_8$

$\Lambda[5] = 0.5\in_1 + 0.5\in_2 - 0.5\in_3 + 0.5\in_4 - 0.5\in_5 + 0.5\in_6 + 0.5\in_7 + 0.5\in_8$

$\Lambda[6] = -0.5\in_1 - 0.5\in_2 + 0.5\in_3 + 0.5\in_4 - 0.5\in_5 - 0.5\in_6 - 0.5\in_7 + 0.5\in_8$

$\Lambda[7] = -0.5\in_1 - 0.5\in_2 + 0.5\in_3 - 0.5\in_4 + 0.5\in_5 - 0.5\in_6 - 0.5\in_7 + 0.5\in_8$

$\Lambda[8] = 0.5\in_1 - 0.5\in_2 + 0.5\in_3 + 0.5\in_4 + 0.5\in_5 + 0.5\in_6 - 0.5\in_7 + 0.5\in_8$

The Appell system for signature (4,4) is

$\psi[8] = \in_1\in_2\in_3\in_4\in_5\in_6\in_7\in_8$

$\psi[7] = -0.5\in_1\in_2\in_3\in_4\in_5\in_6\in_7 - 0.5\in_1\in_2\in_3\in_4\in_5\in_6\in_8 -$
$0.5\in_1\in_2\in_3\in_4\in_5\in_7\in_8 + 0.5\in_1\in_2\in_3\in_4\in_6\in_7\in_8 + 0.5\in_1\in_2\in_3\in_5\in_6\in_7\in_8 -$
$0.5\in_1\in_2\in_4\in_5\in_6\in_7\in_8 - 0.5\in_1\in_3\in_4\in_5\in_6\in_7\in_8 - 0.5\in_2\in_3\in_4\in_5\in_6\in_7\in_8$

$\psi[6] = 0.5\in_1\in_2\in_3\in_4\in_5\in_7 - 0.5\in_1\in_2\in_3\in_5\in_6\in_7 + 0.5\in_2\in_3\in_4\in_5\in_6\in_7 +$
$0.5\in_1\in_2\in_3\in_4\in_5\in_8 - 0.5\in_1\in_2\in_3\in_5\in_6\in_8 + 0.5\in_2\in_3\in_4\in_5\in_6\in_8 + 0.5\in_1\in_2\in_3\in_4\in_7\in_8 -$
$0.5\in_1\in_2\in_4\in_5\in_7\in_8 - 0.5\in_1\in_3\in_4\in_5\in_7\in_8 + 0.5\in_1\in_2\in_3\in_6\in_7\in_8 - 0.5\in_2\in_3\in_4\in_6\in_7\in_8 +$
$0.5\in_1\in_2\in_5\in_6\in_7\in_8 + 0.5\in_1\in_3\in_5\in_6\in_7\in_8 + 0.5\in_2\in_4\in_5\in_6\in_7\in_8 + 0.5\in_3\in_4\in_5\in_6\in_7\in_8$

$\psi[5] = -0.5\in_1\in_2\in_3\in_4\in_7 + 0.5\in_1\in_2\in_3\in_5\in_7 - 0.5\in_1\in_2\in_3\in_6\in_7 + 0.5\in_2\in_3\in_4\in_6\in_7 -$
$0.5\in_2\in_3\in_5\in_6\in_7 - 0.5\in_1\in_2\in_3\in_4\in_8 + 0.5\in_1\in_2\in_3\in_5\in_8 - 0.5\in_1\in_2\in_3\in_6\in_8 + 0.5\in_2\in_3\in_4\in_6\in_8 -$
$0.5\in_2\in_3\in_5\in_6\in_8 + 0.5\in_1\in_2\in_3\in_7\in_8 - 0.5\in_1\in_2\in_4\in_7\in_8 - 0.5\in_1\in_3\in_4\in_7\in_8 +$
$0.5\in_1\in_2\in_5\in_7\in_8 + 0.5\in_1\in_3\in_5\in_7\in_8 - 0.5\in_1\in_2\in_6\in_7\in_8 - 0.5\in_1\in_3\in_6\in_7\in_8 +$
$0.5\in_2\in_3\in_6\in_7\in_8 - 0.5\in_2\in_4\in_6\in_7\in_8 - 0.5\in_3\in_4\in_6\in_7\in_8 + 0.5\in_2\in_5\in_6\in_7\in_8 + 0.5\in_3\in_5\in_6\in_7\in_8$

$\psi[4] = 0.5\in_1\in_2\in_3\in_4 - 0.5\in_1\in_2\in_3\in_5 + 0.5\in_1\in_2\in_3\in_6 -$
$0.5\in_2\in_3\in_4\in_6 + 0.5\in_2\in_3\in_5\in_6 - 0.5\in_1\in_2\in_3\in_7 + 0.5\in_1\in_2\in_4\in_7 +$
$0.5\in_1\in_3\in_4\in_7 - 0.5\in_1\in_2\in_5\in_7 - 0.5\in_1\in_3\in_5\in_7 + 0.5\in_1\in_2\in_6\in_7 + 0.5\in_1\in_3\in_6\in_7 -$
$0.5\in_2\in_3\in_6\in_7 + 0.5\in_2\in_4\in_6\in_7 + 0.5\in_3\in_4\in_6\in_7 - 0.5\in_2\in_5\in_6\in_7 - 0.5\in_3\in_5\in_6\in_7$

$\psi[3] = 1.\in_1\in_2\in_3 - 0.5\in_1\in_3\in_4 - 0.5\in_2\in_3\in_4 + 0.5\in_1\in_3\in_5 +$
$0.5\in_2\in_3\in_5 - 0.5\in_1\in_3\in_6 + 0.5\in_2\in_3\in_6 - 0.5\in_3\in_4\in_6 + 0.5\in_3\in_5\in_6 - 1.\in_1\in_2\in_7 -$
$0.5\in_1\in_3\in_7 + 0.5\in_2\in_3\in_7 - 0.5\in_1\in_4\in_7 - 0.5\in_2\in_4\in_7 - 0.5\in_3\in_4\in_7 + 0.5\in_1\in_5\in_7 +$
$0.5\in_2\in_5\in_7 + 0.5\in_3\in_5\in_7 - 0.5\in_1\in_6\in_7 + 0.5\in_2\in_6\in_7 + 0.5\in_4\in_6\in_7 - 0.5\in_5\in_6\in_7$

$\psi[2] = 1.\in_1\in_2 + 0.5\in_1\in_3 + 0.5\in_2\in_3 + 0.5\in_1\in_4 + 0.5\in_2\in_4 - 0.5\in_1\in_5 - 0.5\in_2\in_5 + 0.5\in_1\in_6 -$
$0.5\in_2\in_6 - 0.5\in_3\in_6 - 0.5\in_4\in_6 + 0.5\in_5\in_6 - 1.\in_2\in_7 - 0.5\in_3\in_7 - 0.5\in_4\in_7 + 0.5\in_5\in_7 - 0.5\in_6\in_7$

$\psi[1] = 1.5\in_1 + 0.5\in_2 - 0.5\in_3 - 0.5\in_4 + 0.5\in_5 + 1.\in_6 + 1.5\in_7$

```
ψ[0] = -1.
```

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