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ON A FAMILY OF PROJECTIVE TORIC VARIETIES

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Abstract: By means of suitable sequences of graphs, we have studied in [2] the reduced lexicographic Gröbner bases of a family of homogeneous toric ideals. In this paper, we deepen the analysis of those bases and derive some geometric properties of the corresponding projective toric varieties.

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1. Introduction

Let K be any field and r be any integer ≥ 2. Let $\Pi_{\underline{r}\times\underline{3}\times\underline{3}}$ be the following map between polynomial rings:

$$K[\underline{x}] := K[x_{ijk}] \to K[u_{ij}, v_{ik}, w_{jk}]\,,$$
$$x_{ijk} \mapsto u_{ij}v_{ik}w_{jk}\,,$$

with $i \in \underline{r} := \{1, 2, \ldots, r\}$, $j, k \in \underline{3} := \{1, 2, 3\}$.

The prime homogeneous ideal

$$I_{\underline{r}\times\underline{3}\times\underline{3}} := \mathrm{Ker}(\Pi_{\underline{r}\times\underline{3}\times\underline{3}})$$

defines a projective toric variety $Y_{\underline{r}\times\underline{3}\times\underline{3}}$ in $\mathbb{P}_K^{9r-1}$.

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It is well known (cf. e.g. [6, Chapter 4]) that a family of generators of $I_{\underline{r}\times\underline{3}\times\underline{3}}$ can be described as follows.

Let $\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$ indicate the $(6r+9)\times 9r$ matrix having columns

$$\underline{a}^{(ijk)}_{\underline{r}\times\underline{3}\times\underline{3}} := \underline{e}_{ij} \oplus \underline{e}_{ik} \oplus \underline{e}'_{jk}\ ,\ i \in \underline{r},\quad j,k \in \underline{3},$$

where $\{\underline{e}_{ij}\} = \{\underline{e}_{ik}\}$ is the canonical basis of the $\mathbb{Z}$-module of $r \times 3$ integer matrices (denoted by $\mathbb{Z}^{\underline{r}\times\underline{3}}$) and $\{\underline{e}'_{jk}\}$ is the canonical basis of $\mathbb{Z}^{\underline{3}\times\underline{3}}$.

We think of $\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$ as of the matrix of the $\mathbb{Z}$-morphism

$$\mathbb{Z}^{\underline{r}\times\underline{3}\times\underline{3}} \to \mathbb{Z}^{\underline{r}\times\underline{3}} \oplus \mathbb{Z}^{\underline{r}\times\underline{3}} \oplus \mathbb{Z}^{\underline{3}\times\underline{3}},$$
$$\underline{u} \mapsto \mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}\ \underline{u}\,,$$

where $\mathbb{Z}^{\underline{r}\times\underline{3}\times\underline{3}}$ denotes the $\mathbb{Z}$-module of 3-dimensional integer matrices of format $r \times 3 \times 3$.

Notice that given any integer vector $\underline{u}$, there is a unique way of writing it as the difference of two vectors with non negative entries: $\underline{u} = \underline{u}^+ - \underline{u}^-$.

Then one proves that the set:

$$\mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}} := \{\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} \mid \underline{u} \in \mathrm{Ker}_{\mathbb{Z}}(\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}})\}$$

is a set of generators of $I_{\underline{r}\times\underline{3}\times\underline{3}}$. Moreover, if $<$ is any term order on $K[\underline{x}]$, then the reduced Gröbner basis of $I_{\underline{r}\times\underline{3}\times\underline{3}}$ w.r.t. $<$ consists of a suitable finite subset of $\mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$.

From now on, we write $\Pi_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$, $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ and $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ instead of $\Pi_{\underline{r}\times\underline{3}\times\underline{3}}$, $I_{\underline{r}\times\underline{3}\times\underline{3}}$ and $Y_{\underline{r}\times\underline{3}\times\underline{3}}$, respectively.

We have proved in [1] that the elements of $\mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$ are in one to one correspondence with certain r-tuples of closed paths of a suitable graph $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ to be described later (cf. Section 2).

We use this result to find the dimension of $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ in Corollary 3.

Let $<_{Lex}$ denote the pure lexicographic term order induced on $K[\underline{x}]$ by

$$x_{ijk} <_{Lex} x_{i'j'k'} \Leftrightarrow (i,j,k) <_{lex} (i',j',k'),$$

where $(i,j,k) <_{lex} (i',j',k')$ if and only if the first nonzero component of the difference vector is negative.

We have proved in [2, Theorem 7.1], that the reduced lexicographic Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ for each $r \geq 6$ only depends on the reduced lexicographic Gröbner bases for $r = 5,4,3,2$.

The proof of [2, Theorem 7.1] has not required a complete description of the bases for $r = 5,4,3,2$. It is the main purpose of this note to give such a complete description (cf. Section 3) and derive from it some properties enjoyed by every variety $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$, $r \geq 2$ (cf. Section 4). The same complete description will probably allow a thorough investigation of the Hilbert functions of the varieties $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$.

One should remark that our complete description of the bases for $r = 5, 4, 3, 2$ is not the result of computer calculations. It is obtained by means of considerations on the graphs which can occur. In fact our description:

(1) makes it possible to discern a pattern in the thousands of polynomials one finds in the outputs of computer calculations, and

(2) suggests the new algorithm of Remark 11, aimed at computing the maximal simplices of the lexicographic triangulation of $\mathrm{conv}(\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}})$.

There are three cases, though, in degree 8, 9 and 10, respectively, where we use [3] to scan the outputs and count how many binomials with certain properties exist, in order to confirm the completeness of some lists of ours.

One should also remark that by a method first suggested by [4], the description of the reduced lexicographic Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ amounts to solving the integer programming problem associated with the matrix $\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$. Our investigation of such a problem (in [1] and [2]) has been the source of our interest in the varieties $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$.

2. The Graph $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$

We begin with some definitions and results contained in [1] and [2].

The graph $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ mentioned in the Introduction is defined to be the bipartite graph having $V_1 := \underline{r} \times \underline{3}$ and $V_2 := \underline{r} \times \underline{3}$ as vertex classes, and $E := \{e_{ijk} \mid i \in \underline{r}\ ,\ j \in \underline{3}\ ,\ k \in \underline{3}\}$ as set of edges, where

$$e_{ijk} = \{(i,j) \in V_1,\ \ (i,k) \in V_2\}.$$

$\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ is the disjoint union of r copies of the complete bipartite graph $K_{3,3}$; we denote them by $K_{3,3}^{(1)}, K_{3,3}^{(2)}, \ldots, K_{3,3}^{(r)}$.

For every choice of i, i' in $\underline{r}$, j in $\underline{3}$ and k in $\underline{3}$, we say that the edges e_{ijk} and $e_{i'jk}$ are parallel.

An $r-$tuple $S := (C_1, C_2, \ldots, C_r)$ is called an *admissible $r-$tuple of closed paths* of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ if the following properties hold:

(1) For every $i \in \underline{r}$, either $C_i = \emptyset$ or C_i is a closed path of the subgraph $K_{3,3}^{(i)}$ of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$.

(2) For every edge e occurring in S, there are in S as many edges parallel to e that occur in even position as edges parallel to e in odd position.

(3) For every $i \in \underline{r}$ such that $C_i \neq \emptyset$, every edge of C_i either is always in odd position ("odd edge"), or is always in even position ("even edge").

The closed paths above have to be considered as cyclic structures, with no definite starting point (but still with a division of edges into even and odd).

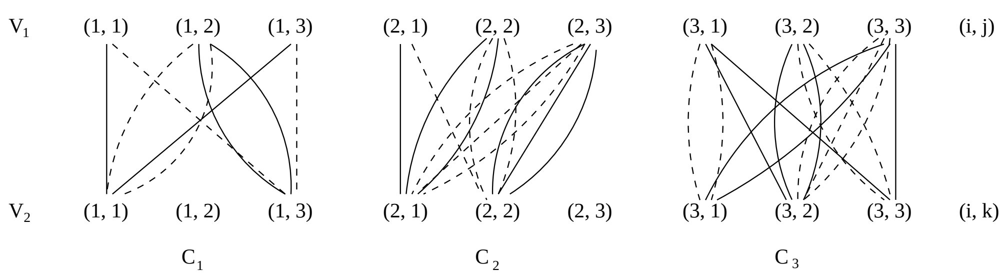
In [1] we have proved the following result, reformulated in this way in [2, Theorem 2.4].

Theorem 1. *Let us associate the variable x_{ijk} with the edge e_{ijk} of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$, and viceversa. With every admissible $r-$tuple $S := (C_1, \ldots, C_r)$ of closed paths of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$, we can associate the binomial $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-}$, where the nonzero entries of $\underline{u}^+$ are given by the multiplicities of all odd edges of S, the nonzero entries of $\underline{u}^-$ by the multiplicities of all even edges, and the multiplicity of an edge e of C_i is the number of times e occurs in C_i. It turns out that $\underline{x}^{\underline{u}^+} - \underline{x}^{\underline{u}^-} \in \mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$ and that the application*

$$\{\textit{admissible r-tuples of closed paths of } \mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}\} \to \mathcal{B}_{\underline{r}\times\underline{3}\times\underline{3}}$$

defined in this way is a bijection.

Example 2. Let us consider the graph $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ and the following admissible triplet $S := (C_1, C_2, C_3)$ of closed paths of $\mathcal{G}_{\underline{3}\times\underline{3}\times\underline{3}}$ (the dotted edges being in even position):



The binomial associated with S is

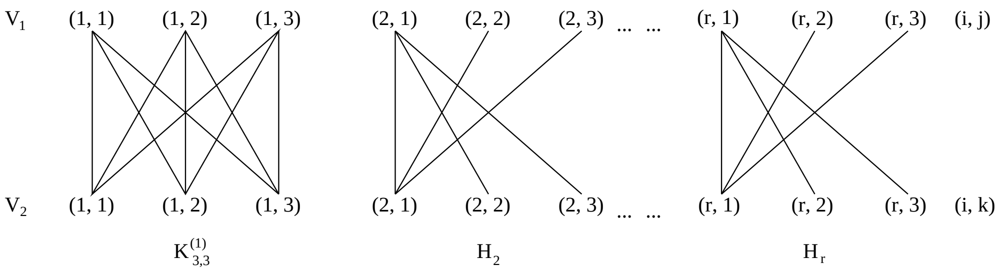
$$x_{111} x_{123}^2 x_{131} x_{211} x_{221}^2 x_{232}^3 x_{312} x_{313} x_{322}^2 x_{331}^2 x_{333} \\ - x_{113} x_{121}^2 x_{133} x_{212} x_{222}^2 x_{231}^3 x_{311}^2 x_{323}^2 x_{332}^3 .$$

In the next section we shall see which admissible $r-$tuples of closed paths correspond to the elements of the reduced lexicographic Gröbner basis of the ideal $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$.

Here we use the previous theorem for computing the dimension of the variety $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$.

Corollary 3. *For every $r \geq 2$, $\dim(Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}) = 5r+3$.*

Proof. We must prove that the Krull dimension of $K[\underline{x}]/I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ is $5r+4$. Consider the following subgraph, $\mathcal{H}$, of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$:



Let $\underline{y}$ be the subset of $\underline{x}$ corresponding to $\mathcal{H}$. Since $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ is a prime ideal, it is enough to prove that $\underline{y}$ is maximally independent modulo $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$. Assume for a contradiction that it is not independent. Then there exists $f \in I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}} \cap K[\underline{y}]$, $f \neq 0$. Hence the reduced Gröbner basis (w.r.t. $<_{Lex}$) of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ contains a binomial g with $in_{<_{Lex}}(g)$ dividing $in_{<_{Lex}}(f)$. But g corresponds to an $r-$tuple $S := (C_1, \dots, C_r)$ having its maximum edges (w.r.t. $<_{Lex}$) occurring among the edges of the subgraph $\mathcal{H}$ (a maximum edge is just an edge corresponding to an indeterminate occurring in the maximum monomial of g: cf. [1, Definition 4.1]). Let C_m be the rightmost nonempty path of S; $2 \leq m \leq r$. Then the maximum edges of C_m occur among the edges of H_m, in $\mathcal{H}$. But this contradicts the fact that C_m must contain at least two nonintersecting maximum edges (cf. [1, Lemma 4.11]). This shows that $\underline{y}$ is independent; it remains to be proved that it is maximally independent. Suppose not. If x_{ijk} is not in $\underline{y}$, then $2 \leq i \leq r$ and the corresponding e_{ijk} occurs in $K_{3,3}^{(i)}$, but does not belong to H_i. Hence $j, k \in \{2, 3\}$, so that a contradiction arises from the existence in $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ of the following (nonzero) binomial: $x_{ijk}x_{i11}x_{1j1}x_{11k} - x_{ij1}x_{i1k}x_{1jk}x_{111}$. □

Remark 4. The dimension of $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ can be also obtained as a special case of a result recorded in the introduction of [5]. Namely, $\dim(Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}) + 1$ equals the rank of the matrix $\mathcal{A}^{(r)}$ and such a rank is $9r - (r-1)(n-d)$, with $n = 9$ and $d = 5$. In the above, $\mathcal{A}^{(r)}$ (which turns out to coincide with our $\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$, up to a permutation of rows) stands for the $r-$th Lawrence lifting of $\mathcal{A}$, and $\mathcal{A}$ is the 6×9 matrix whose columns are $\{\underline{e}_j \oplus \underline{e}_k | j\ , k \in \underline{3}\}$, $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ being the canonical basis.

3. *RG*−Sequences

By definition, an *RG−sequence of* $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ *w.r.t.*$<_{Lex}$ is an admissible $r-$ tuple of closed paths, S, such that the bijection of Theorem 1 sends S to an element of the reduced lexicographic Gröbner basis of the ideal $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$.

The degree of an $RG-$sequence is the degree of the corresponding binomial.

Let us refer to [2, Theorem 7.1]:

Theorem 5. *Let* $r' \in \{2,3,4,5\}$ *and* $S' := (D_1, \dots, D_{r'})$ *an RG−sequence of* $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$ *(w.r.t.* $<_{Lex}$*) such that* $D_{i'} \neq \emptyset$ *for every* $i' \in \underline{r'}$*. For every* $r \geq 6$ *and for every choice of indices* $i_1, i_2, \dots, i_{r'}$ *such that* $1 \leq i_1 < i_2 < \cdots < i_{r'} \leq r$*, consider the r−tuple* $S_{i_1,\dots,\, i_{r'}} := (C_1, \dots, C_r)$ *such that*

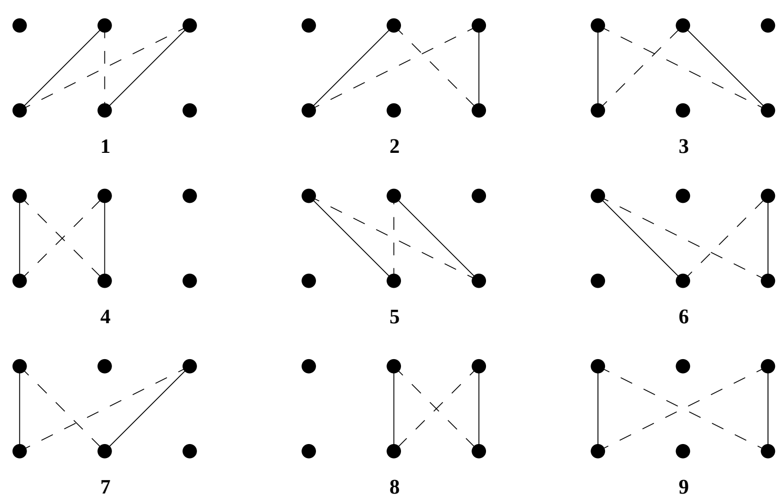
$$C_h := \begin{cases} D_{i'} & \text{if } h = i_{i'} \\ \\ \emptyset & \text{else,} \end{cases}$$

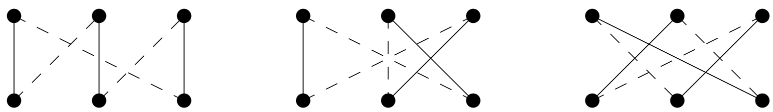
clearly an admissible r−tuple of closed paths of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$*. Then all and only the RG−sequences of* $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ *(w.r.t.* $<_{Lex}$*) are obtained in this way, when* r' *ranges over* $\{2,3,4,5\}$ *and* S' *ranges over the set of all the RG−sequences of* $\mathcal{G}_{\underline{r'}\times\underline{3}\times\underline{3}}$ *with no empty path.*

We are now going to describe all $RG-$sequences of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ (w.r.t. $<_{Lex}$) having formats $2\times3\times3$, $3\times3\times3$, $4\times3\times3$, $5\times3\times3$.

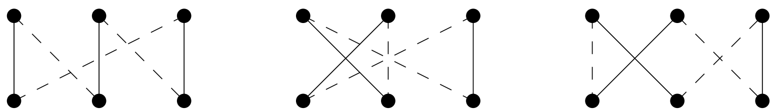
Consistently with [2, Section 5], a section in which the reader may want to browse, we stipulate the following.

(a) $\mathbf{n} \in \{\mathbf{1}, \dots, \mathbf{15}\}$ denotes one of the following fifteen cycles:





10 **11** **12**

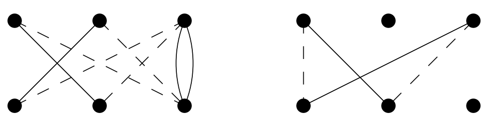


13 **14** **15**

(b) In the above list, the maximum edges w.r.t. $<_{Lex}$ are the continuous ones, and are assumed to be odd.

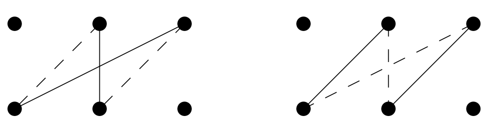
(c) $\widehat{\mathbf{n}}$ denotes the cycle obtained from $\mathbf{n}$ by turning odd (resp., even) edges into even (resp., odd) ones. $\widehat{\mathbf{n}}$ is called the anti-isomorphic cycle of $\mathbf{n}$.

(d) Given two cycles $\mathbf{n_1}$ and $\mathbf{n_2}$, $\mathbf{n_1}+\mathbf{n_2}$ denotes the closed path obtained by patching together $\mathbf{n_1}$ and $\mathbf{n_2}$ and deleting the edges that in so doing happen to be even and odd at the same time. Similarly for $\mathbf{n_1}+\widehat{\mathbf{n_2}}$ and $\widehat{\mathbf{n_1}}+\widehat{\mathbf{n_2}}$. For instance:



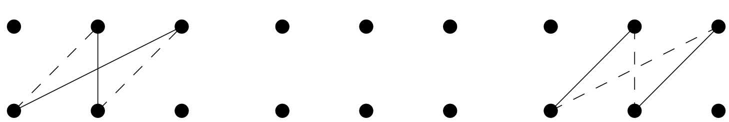
2+6 $\mathbf{6+\hat{9}}$

We are going to list the $RG-$sequences occurring in the formats $r\times 3\times 3$, $2\leq r\leq 5$, according to their degrees. Moreover, whenever two different $RG-$sequences only differ by some empty paths, we just describe the one with no empty paths. For instance, the $RG-$sequence of format $2\times 3\times 3$:



$\hat{\mathbf{1}}$ **1**

will be described (in degree 4), while the $RG-$sequence of format $3\times 3\times 3$:



$\hat{\mathbf{1}}$ $\emptyset$ **1**

will not.

3.1. $RG-$Sequences of Degree 4

They first occur in format $2 \times 3 \times 3$. As shown in [1], they consist of all the pairs of anti-isomorphic cycles of length 4. In format $2 \times 3 \times 3$ there are 9 of them, one for each of the cycles $\mathbf{1}, \ldots, \mathbf{9}$.

3.2. $RG-$Sequences of Degree 6

As seen in [1], we have all the pairs of anti-isomorphic cycles of length 6 occurring in $2 \times 3 \times 3$, and all the triplets of length 4 cycles occurring in $3 \times 3 \times 3$ (cf. [1, Proposition 4.10]). The number of the above pairs is 6 (one for each of the cycles $\mathbf{10}, \ldots, \mathbf{15}$). As for the number of the above triplets, we deduce from [1, Proposition 4.10] that, for every vertex of $K_{3,3}$, one can construct $3! = 6$ degree 6 $RG-$sequences, so that one finds in format $3 \times 3 \times 3$ exactly 36 degree 6 $RG-$sequences (without empty paths).

3.3. $RG-$Sequences of Degree 7

As seen in [1], they first occur in format $3 \times 3 \times 3$. They are all the non-divisible triplets (non-divisible in the sense of [1, Definition 4.15]) consisting of one length 6 cycle and two length 4 cycles, built starting from a length 6 "raw" cycle plus one of its chords (cf. [1, Proposition 4.13 and Proposition 4.16]). By a raw cycle we simply mean a cycle whose edges are not divided into two classes of even and odd edges.

We list all the $RG-$sequences of degree 7 (28 of them altogether) divided into six groups, according to the length 6 raw cycles involved. For simplicity, the three upper vertices of $K_{3,3}$ are denoted by a, b, c (left to right). The three lower vertices by d, e, f (left to right).

(1) Length 6 cycle: the raw cycle supporting both $\mathbf{10}$ and $\widehat{\mathbf{10}}$

(1.1) Chord $\{a, e\}$: $(\mathbf{6}, \widehat{\mathbf{10}}, \widehat{\mathbf{4}})$.

(1.2) Chord $\{b, f\}$: $(\mathbf{3}, \widehat{\mathbf{10}}, \widehat{\mathbf{8}})$.

(1.3) Chord $\{c, d\}$: $(\mathbf{10}, \widehat{\mathbf{9}}, \mathbf{1})$, $(\widehat{\mathbf{10}}, \widehat{\mathbf{1}}, \mathbf{9})$.

(2) Length 6 cycle: the raw cycle supporting both $\mathbf{11}$ and $\widehat{\mathbf{11}}$.

(2.1) Chord $\{a, e\}$: $(\mathbf{5}, \widehat{\mathbf{11}}, \mathbf{7})$.

(2.2) Chord $\{b, d\}$: $(\mathbf{1}, \widehat{\mathbf{11}}, \mathbf{3})$.

(2.3) Chord $\{c, f\}$: Both triplets obtained by a permutation of the first two cycles of $(\mathbf{8}, \widehat{\mathbf{9}}, \mathbf{11})$.

(3) Length 6 cycle: the raw cycle supporting both $\mathbf{12}$ and $\widehat{\mathbf{12}}$

(3.1) Chord $\{a, d\}$: Both triplets obtained by a permutation of the first two cycles of $(\mathbf{3}, \widehat{\mathbf{7}}, \mathbf{12})$.

(3.2) Chord $\{b, e\}$: $(\widehat{\mathbf{12}}, \widehat{\mathbf{5}}, \mathbf{1})$, $(\mathbf{12}, \widehat{\mathbf{1}}, \mathbf{5})$.

(3.3) Chord $\{c, f\}$: Both triplets obtained by a permutation of the first two cycles of $(\mathbf{6}, \widehat{\mathbf{2}}, \mathbf{12})$.

(4) Length 6 cycle: the raw cycle supporting both $\mathbf{13}$ and $\widehat{\mathbf{13}}$

(4.1) Chord $\{b, d\}$: $(\mathbf{2}, \widehat{\mathbf{13}}, \mathbf{4})$.

(4.2) Chord $\{c, e\}$: $(\mathbf{7}, \widehat{\mathbf{13}}, \mathbf{8})$.

(4.3) Chord $\{a, f\}$: $(\mathbf{13}, \widehat{\mathbf{9}}, \mathbf{5})$, $(\widehat{\mathbf{13}}, \widehat{\mathbf{5}}, \mathbf{9})$.

(5) Length 6 cycle: the raw cycle supporting both $\mathbf{14}$ and $\widehat{\mathbf{14}}$

(5.1) Chord $\{c, e\}$: $(\mathbf{1}, \widehat{\mathbf{14}}, \mathbf{6})$.

(5.2) Chord $\{b, f\}$: $(\mathbf{5}, \widehat{\mathbf{14}}, \mathbf{2})$.

(5.3) Chord $\{a, d\}$: Both triplets obtained by a permutation of the first two cycles of $(\mathbf{4}, \widehat{\mathbf{9}}, \mathbf{14})$.

(6) Length 6 cycle: the raw cycle supporting both $\mathbf{15}$ and $\widehat{\mathbf{15}}$

(6.1) Chord $\{a, f\}$: $(\mathbf{15}, \widehat{\mathbf{6}}, \mathbf{3})$, $(\widehat{\mathbf{15}}, \widehat{\mathbf{3}}, \mathbf{6})$.

(6.2) Chord $\{b, e\}$: Both triplets obtained by a permutation of the first two cycles of $(\mathbf{4}, \widehat{\mathbf{8}}, \mathbf{15})$.

(6.3) Chord $\{c, d\}$: $(\widehat{\mathbf{15}}, \widehat{\mathbf{7}}, \mathbf{2})$, $(\mathbf{15}, \widehat{\mathbf{2}}, \mathbf{7})$.

3.4. *RG*−Sequences of Degree 8

By the results in [1], they can first occur in format $4 \times 3 \times 3$, and each one of them must consist of four length 4 cycles.

All the admissible 4−tuples of length 4 cycles constructed in [2, Proposition 4.1] are *RG*−sequences. We claim that there are no others.

Since, thanks to [2, Proposition 4.1], for every edge of $K_{3,3}$ we can construct $4! = 24$ admissible 4−tuples of length 4 cycles not involving that edge, we have in this way $9 \cdot 24 = 216$ *RG*−sequences of degree 8. But 216 is precisely the number given us by a computer, if we ask it to count how many degree 8 binomials occur in the reduced lexicographic Gröbner basis of format $4 \times 3 \times 3$.

3.5. *RG*−Sequences of Degree 9

One of them already occurs in format $3 \times 3 \times 3$ (cf. [2, Remark 3.3]), namely: $(\mathbf{3} + \mathbf{6}, \widehat{\mathbf{9}}, \mathbf{12})$.

Each *RG*−sequence first occurring in format $4 \times 3 \times 3$ must consist of a single length 6 cycle and three length 4 cycles.

[2, Proposition 4.3] shows a way to construct admissible 4−tuples consisting of a single length 6 cycle and three length 4 cycles, starting from a "raw" length 6 cycle and a selected edge on it (as before, a raw cycle is simply a cycle whose edges are not divided into two classes of even and odd edges). The admissible sequences so obtained are not always $RG-$sequences, but the following 40 are. We list them divided into six groups, according to the length 6 raw cycles involved. We have identified the listed 40 $RG-$sequences by means of a technique close to that used in the proof of [2, Theorem 5.2]. One checks by computer that there are no other $RG-$sequences of format $4 \times 3 \times 3$ with no empty paths. Such a check is performed by finding in the reduced lexicographic Gröbner basis precisely 40 binomials which have at least one indeterminate x_{ijk} for every choice of i in $\{1, 2, 3, 4\}$.

(1) Length 6 cycle: the raw cycle supporting both $\mathbf{10}$ and $\widehat{\mathbf{10}}$

(1.1) Edge $\{a, d\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{10}, \widehat{\mathbf{9}}, \widehat{\mathbf{4}}, \mathbf{7})$.

(1.2) Edge $\{b, d\}$: $(\mathbf{3}, \widehat{\mathbf{10}}, \widehat{\mathbf{1}}, \mathbf{2})$.

(1.3) Edge $\{c, e\}$: $(\mathbf{6}, \widehat{\mathbf{10}}, \widehat{\mathbf{1}}, \mathbf{7})$.

(1.4) Edge $\{c, f\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{10}, \widehat{\mathbf{8}}, \widehat{\mathbf{9}}, \mathbf{2})$.

(2) Length 6 cycle: the raw cycle supporting both $\mathbf{11}$ and $\widehat{\mathbf{11}}$

(2.1) Edge $\{a, d\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{11}, \widehat{\mathbf{3}}, \widehat{\mathbf{7}}, \mathbf{4})$.

(2.2) Edge $\{b, e\}$: Both 4−tuples obtained by a permutation of the first and second cycles of $(\mathbf{1}, \mathbf{5}, \widehat{\mathbf{11}}, \mathbf{4})$.

(3) Length 6 cycle: the raw cycle supporting both $\mathbf{12}$ and $\widehat{\mathbf{12}}$

(3.1) Edge $\{c, d\}$: The six 4−tuples obtained by a permutation of the first three cycles of $(\widehat{\mathbf{2}}, \widehat{\mathbf{7}}, \mathbf{9}, \mathbf{12})$.

(3.2) Edge $\{a, f\}$: The six 4−tuples obtained by a permutation of the first three cycles of $(\mathbf{3}, \mathbf{6}, \widehat{\mathbf{9}}, \mathbf{12})$.

(4) Length 6 cycle: the raw cycle supporting both $\mathbf{13}$ and $\widehat{\mathbf{13}}$

(4.1) Edge $\{a, d\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{13}, \widehat{\mathbf{9}}, \widehat{\mathbf{4}}, \mathbf{3})$.

(4.2) Edge $\{a, e\}$: $(\mathbf{7}, \widehat{\mathbf{13}}, \widehat{\mathbf{5}}, \mathbf{6})$.

(4.3) Edge $\{b, f\}$: $(\mathbf{2}, \widehat{\mathbf{13}}, \widehat{\mathbf{5}}, \mathbf{3})$.

(4.4) Edge $\{c, f\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{13}, \widehat{\mathbf{8}}, \widehat{\mathbf{9}}, \mathbf{6})$.

(5) Length 6 cycle: the raw cycle supporting both $\mathbf{14}$ and $\widehat{\mathbf{14}}$

(5.1) Edge $\{c, f\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{14}, \widehat{\mathbf{2}}, \widehat{\mathbf{6}}, \mathbf{8})$.

(5.2) Edge $\{b, e\}$: Both 4−tuples obtained by a permutation of the first and second cycles of $(\mathbf{5}, \mathbf{1}, \widehat{\mathbf{14}}, \mathbf{8})$.

(6) Length 6 cycle: the raw cycle supporting both $\mathbf{15}$ and $\widehat{\mathbf{15}}$

(6.1) Edge $\{a, d\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\widehat{\mathbf{15}}, \widehat{\mathbf{3}}, \widehat{\mathbf{7}}, \mathbf{9})$.

(6.2) Edge $\{a, e\}$: $(\mathbf{4}, \mathbf{15}, \widehat{\mathbf{6}}, \mathbf{5})$.

(6.3) Edge $\{b, d\}$: $(\mathbf{4}, \mathbf{15}, \widehat{\mathbf{2}}, \mathbf{1})$.

(6.4) Edge $\{b, f\}$: $(\mathbf{8}, \widehat{\mathbf{15}}, \widehat{\mathbf{3}}, \mathbf{5})$.

(6.5) Edge $\{c, e\}$: $(\mathbf{8}, \widehat{\mathbf{15}}, \widehat{\mathbf{7}}, \mathbf{1})$.

(6.6) Edge $\{c, f\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{15}, \widehat{\mathbf{2}}, \widehat{\mathbf{6}}, \mathbf{9})$.

3.6. *RG*−Sequences of Degree 10

Some of them first occur in format $4 \times 3 \times 3$, some others first occur in format $5 \times 3 \times 3$.

3.6.1. Format $4 \times 3 \times 3$

The RG−sequences of degree 10 first occurring in format $4 \times 3 \times 3$ consist of three length 4 cycles and a length 8 closed path having one (and only one) double edge among the minimum edges (= dotted = even). There are 20 such RG−sequences and we list them according to the double edges occurring in them.

(1) Double edge $\{a, d\}$: Both 4−tuples obtained by a permutation of the first and second cycles of $(\mathbf{3}, \mathbf{7}, \widehat{\mathbf{4}} + \widehat{\mathbf{9}}, \mathbf{8})$.

(2) Double edge $\{a, e\}$: $(\mathbf{5}, \mathbf{4} + \widehat{\mathbf{6}}, \widehat{\mathbf{7}}, \mathbf{2})$.

(3) Double edge $\{a, f\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{3} + \mathbf{6}, \widehat{\mathbf{5}}, \widehat{\mathbf{9}}, \mathbf{1})$.

(4) Double edge $\{b, d\}$: $(\mathbf{1}, \mathbf{4} + \widehat{\mathbf{2}}, \widehat{\mathbf{3}}, \mathbf{6})$.

(5) Double edge $\{b, e\}$:

(5.1) Both 4−tuples obtained by a permutation of the second and third cycles of $(\widehat{\mathbf{4}} + \widehat{\mathbf{8}}, \widehat{\mathbf{1}}, \widehat{\mathbf{5}}, \mathbf{9})$.

(5.2) The six 4−tuples obtained by a permutation of the first three cycles of $(\mathbf{4}, \widehat{\mathbf{9}}, \mathbf{8}, \mathbf{1} + \mathbf{5})$.

(6) Double edge $\{b, f\}$: $(\mathbf{5}, \mathbf{8} + \widehat{\mathbf{3}}, \widehat{\mathbf{2}}, \mathbf{7})$.

(7) Double edge $\{c, d\}$: Both 4−tuples obtained by a permutation of the second and third cycles of $(\mathbf{2} + \mathbf{7}, \widehat{\mathbf{1}}, \widehat{\mathbf{9}}, \mathbf{5})$.

(8) Double edge $\{c, e\}$: $(\mathbf{1}, \mathbf{8} + \widehat{\mathbf{7}}, \widehat{\mathbf{6}}, \mathbf{3})$.

(9) Double edge $\{c, f\}$: Both 4−tuples obtained by a permutation of the first and second cycles of $(\mathbf{2}, \mathbf{6}, \widehat{\mathbf{8}} + \widehat{\mathbf{9}}, \mathbf{4})$.

3.6.2. Format $5 \times 3 \times 3$

All the admissible 5−tuples of length 4 cycles (hence of degree 10) constructed in [2, Proposition 4.5] are RG−sequences. We claim that there are no others.

Since, thanks to [2, Proposition 4.5], for every edge of $K_{3,3}$ we can construct $5! = 120$ admissible 5−tuples of length 4 cycles, we have in this way $9 \cdot 120 = 1080$ RG−sequences of degree 10. But 1080 is precisely the number given us by a computer, if we ask it to count how many degree 10 binomials occur in the reduced lexicographic Gröbner basis of format $5 \times 3 \times 3$, which have at least one indeterminate x_{ijk} for every choice of i in $\{1, 2, 3, 4, 5\}$.

This completes the description of all the RG−sequences of format $r \times 3 \times 3$, $2 \leq r \leq 5$. For $r \geq 6$, every RG−sequence is obtained from one of these by means of a suitable insertion of empty paths.

4. Geometric Consequences

In this section we derive some consequences of the material in Section 3.

Proposition 6. *The initial ideal* $in_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$ *is square-free for every* $r \geq 2$. *In particular, it is a radical ideal.*

Proof. Close inspection of the description of all RG−sequences of format $r \times 3 \times 3$, $2 \leq r \leq 5$, shows that $in_{<_{Lex}}(g)$ is square-free for every g occurring in the reduced lexicographic Gröbner basis of the corresponding ideal $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$. Since, for $r \geq 6$, every RG−sequence is obtained from one of the previous RG−sequences by means of a suitable insertion of empty paths, it follows that $in_{<_{Lex}}(g)$ is square-free for every g occurring in the reduced lexicographic Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ for every $r \geq 2$. □

Remark 7. The square-freeness of the previous proposition may not extend to other types of format. For instance, the reduced lexicographic Gröbner basis of $I_{\mathcal{A}_{\underline{3}\times\underline{4}\times\underline{4}}}$ contains the following binomial:

$$x_{111}x_{134}x_{142}x_{212}x_{213}x_{224}x_{244}x_{314}^2x_{322}x_{331}x_{343} -$$
$$x_{112}x_{131}x_{144}x_{214}^2x_{222}x_{243}x_{311}x_{313}x_{324}x_{334}x_{342}.$$

Following [6, Chapter 8], we let $\Delta_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$ denote the initial complex of the ideal $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ w.r.t. $<_{Lex}$: it is a simplicial complex on the vertex set $\{(ijk) \mid i \in \underline{r},\ j\ ,k \in \underline{3}\}$ (whose Stanley-Reisner ideal is precisely the radical ideal $in_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$).

Let $Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}} := \mathrm{conv}\,(\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}})$ denote the polytope which is the convex hull of the points of $\mathbb{R}^{6r+9}$ corresponding to the columns $\underline{a}^{(ijk)}_{\underline{r}\times\underline{3}\times\underline{3}}$ of the matrix $\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$.

We can consider $\Delta_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$ as a regular triangulation of $Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ (cf. [6, Theorem 8.3]), whose maximal simplices are the $(5r+3)-$simplices. Such a regular triangulation is called lexicographic triangulation.

[6, Theorem 4.16] says that the degree of the variety $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ is given by the normalized volume of $Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$, hence by the sum of the normalized volumes of the maximal simplices of its lexicographic triangulation.

Corollary 8. *For every $r \geq 2$, the lexicographic triangulation of the polytope $Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}} \subseteq \mathbb{R}^{6r+9}$ is unimodular (= every maximal simplex has normalized volume 1). In particular, the degree of $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ equals the number of the maximal simplices of $Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ relative to the lexicographic triangulation.*

Proof. All follows from Proposition 6, since [6, Corollary 8.9] says that the lexicographic triangulation is unimodular if and only if the lexicographic initial ideal $in_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$ is square-free. □

Remark 9. Again recalling [6, Chapter 8], a subset σ of $\{(ijk) \mid i \in \underline{r},\ j\ ,k \in \underline{3}\}$ is a $(5r+3)-$simplex of $\Delta_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$ if, and only if, $|\sigma| = 5r+4$ and the reduced Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ w.r.t. $<_{Lex}$ contains no polynomial having initial term with support enclosed in σ (here the support of a monomial in the indeterminates x_{ijk} is the set of triplets (ijk) associated with the indeterminates actually occurring in that monomial).

Since $in_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$ is square-free, we can state in our language: σ is a $(5r+3)-$simplex if, and only if, $|\sigma| = 5r+4$ and no $RG-$sequence of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$ (w.r.t. $<_{Lex}$) exists such that its maximum edges occur among the edges of the subgraph $\{e_{ijk} \mid (ijk) \in \sigma\}$ of $\mathcal{G}_{\underline{r}\times\underline{3}\times\underline{3}}$.

In principle, Remark 9 gives us a procedure to list all maximal simplices of $Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$, thereby getting the degree of the variety $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$. We illustrate this point by the following example, dealing with format $2\times 3\times 3$.

Example 10. We claim that $Y_{\mathcal{A}_{\underline{2}\times\underline{3}\times\underline{3}}}$ has degree 81, i.e., there exist 81 $13-$simplices of $\Delta_{<_{Lex}}(I_{\mathcal{A}_{\underline{2}\times\underline{3}\times\underline{3}}})$.

Let us identify every cardinality 14 subset σ of $\{(ijk) \mid i \in \underline{2},\ j\ ,k \in \underline{3}\}$ with the subgraph

$$(\sigma_1, \sigma_2) := \{e_{ijk} \mid (ijk) \in \sigma\}$$

of $\mathcal{G}_{\underline{2}\times\underline{3}\times\underline{3}}$. The notation stresses the fact that the written subgraph is the disjoint union of σ_1 and σ_2, where $\sigma_1 \subseteq K_{3,3}^{(1)}$ and $\sigma_2 \subseteq K_{3,3}^{(2)}$.

We want to show that exactly for 81 choices of (σ_1, σ_2), no $RG-$sequence exists such that its maximum edges occur among the edges of (σ_1, σ_2).

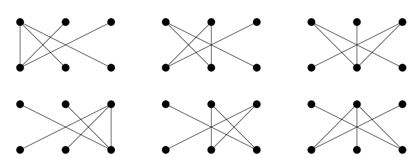
Since $|\sigma| = 14$, all possible choices for (σ_1, σ_2) can be grouped into the following five cases:

(1) $\sigma_1 = K_{3,3}^{(1)}$, $|\sigma_2| = 5$.
(2) $|\sigma_1| = 8$, $|\sigma_2| = 6$.
(3) $|\sigma_1| = 7$, $|\sigma_2| = 7$.
(4) $|\sigma_1| = 6$, $|\sigma_2| = 8$.
(5) $|\sigma_1| = 5$, $\sigma_2 = K_{3,3}^{(2)}$.

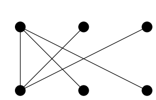
It turns out that case (1) yields 6 simplices, and so does case (5) which is dual to (1). Case (2) yields 21 simplices, and so does case (4) which is dual to case (2). Case (3) yields 27 simplices. Hence 81 simplices altogether, as claimed.

In order to be short, we discuss only case (1). The other cases are left to the reader.

Assume that $\sigma_1 = K_{3,3}^{(1)}$ and $|\sigma_2| = 5$. Recalling the description of all degree 4 and degree 6 $RG-$sequences given in Section 3, if no $RG-$sequence has maximum edges occurring among the edges of (σ_1, σ_2), then σ_2 cannot have "non-intersecting" edges. That is, σ_2 must be one of the following subgraphs of $K_{3,3}^{(2)}$:



This proves that there are 6 simplices belonging to case (1). For instance, the pair formed by $\sigma_1 = K_{3,3}^{(1)}$ and $\sigma_2 =$



corresponds to:

$$\sigma = \{(111), (112), (113), (121), (122), (123),$$
$$(131), (132), (133), (211), (212), (213), (221), (231)\}.$$

It is hard to use in general, even with the help of a computer, the procedure for getting the degree of the variety $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ illustrated by Example 10.

It is better to adopt the following point of view.

Remark 11. Notation as in Remark 9. The statement that σ is a $(5r+3)-$simplex is further equivalent to saying that $|\sigma| = 5r+4$ and

$$NF(\prod_{(ijk)\in\sigma} x_{ijk},\quad in_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})) \neq 0,$$

where the left hand side of the previous inequality denotes the normal form of the monomial $\prod_{(ijk)\in\sigma} x_{ijk}$ modulo the monomial ideal $in_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$ (i.e., the uniquely determined remainder on division of $\prod_{(ijk)\in\sigma} x_{ijk}$ by the (unique) minimal basis of $in_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$).

This suggests a simple algorithm for computing the maximal simplices of the regular triangulation $\Delta_{<_{Lex}}(I_{\mathcal{A}_{\underline{2}\times\underline{3}\times\underline{3}}})$. Presumably a very effective algorithm, if one uses the reduced lexicographic Gröbner basis of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ as computed by the algorithm indicated in [2, Remark 7.3] in order to find the minimal basis of $in_{<_{Lex}}(I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$. The only drawback is that we need producing all $(5r+4)-$subsets of a set of cardinality $9r$.

For the rest of this section, we assume that the field K is algebraically closed and of characteristic zero. We get another consequence of Proposition 6

An irreducible projective variety embedded in projective space is called projectively normal if its coordinate ring is integrally closed (in its field of fractions).

Corollary 12. *(a) For every $r \geq 2$, the variety $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ is projectively normal.*

(b) Let $H_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(n)$ (resp., $P_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(t)$) be the Hilbert function (resp., polynomial) of the variety $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$. For every $r \geq 2$, one has:

$$H_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(n) = P_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(n)$$

for every $n \in \mathbb{N}$, i.e., for every $r \geq 2$, the index of regularity of $I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ is zero.

Proof. (a) It follows from [6, Proposition 13.5] that for $K[\underline{x}]/I_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ to be integrally closed, it is enough to show that the semigroup $\mathbb{N}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$ is equal to

$$\mathbb{Z}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}} \cap \operatorname{pos}(\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}),$$

where $\operatorname{pos}(\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}})$ is given by all $\mathbb{R}_+$−linear combinations of the columns $\underline{a}^{(ijk)}_{\underline{r}\times\underline{3}\times\underline{3}}$, and $\mathbb{R}_+$ denotes the set of nonnegative reals.

The stated equality follows from Proposition 6 in the way illustrated in the proof of part (i) of [6, Proposition 13.15].

(b) It follows from [6, Chapter 4] that

$$H_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(n) = \left| \ \mathbb{N}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}} \cap n \cdot Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}} \ \right|$$

and that

$$E_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(n) = \left| \ \mathbb{Z}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}} \cap n \cdot Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}} \ \right|,$$

where $E_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(t)$ is the normalized Ehrhart polynomial of $Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ (a polynomial of degree $\dim(Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}})$). Hence it suffices to show that:

$$\mathbb{Z}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}} \cap n \cdot Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}} \subseteq \mathbb{N}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}} \cap n \cdot Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}.$$

Let $\underline{b} \in \mathbb{Z}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}} \cap n \cdot Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$, say $\underline{b} = n \cdot \underline{b}'$ with $\underline{b}' \in Q_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$. Then

$$\underline{b}' = \sum_{(ijk)} \lambda_{ijk} \ \underline{a}^{(ijk)}_{\underline{r}\times\underline{3}\times\underline{3}},$$

where $\lambda_{ijk} \in \mathbb{R}_+$ and $\sum_{(ijk)} \lambda_{ijk} = 1$. It follows that

$$\underline{b} \in \operatorname{pos}(\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}),$$

indeed

$$\underline{b} \in \mathbb{Z}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}} \cap \operatorname{pos}(\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}).$$

As in the proof of part (a), the latter intersection equals $\mathbb{N}\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}$, and we are through. □

One should notice that part (b) above implies $P_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(t) = E_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(t)$, a known characterization of the normality of $Y_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}$ (cf. [6, Theorem 13.11]).

We think a complete description of $H_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(n)$ and $P_{\mathcal{A}_{\underline{r}\times\underline{3}\times\underline{3}}}(t)$ can be obtained thanks to the description of all RG−sequences given in Section 3.

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