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MINIMAL 1-CYCLES GENERATING A CANONICAL BASIS OF 2-MANIFOLD'S HOMOLOGY GROUP

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Abstract: We studied closed triangulated 2-dimensional manifolds and used the homology groups with coefficients from $\mathbb{Z}_2$. The main result of our study is to create and validate algorithm 4 which finds in an orientable 2-manifold P 1-cycles $z_1^1, \dots, z_r^1$ with the following properties: 1) each cycle z_k^1, $k = 1, \dots, r$, has weight less than weight of any other cycle homologous to z_k^1, 2) homological classes $[z_1^1], \dots, [z_r^1]$ generate a canonical basis of the group $H_1(P)$. A number of auxiliary results will be presented as well. Specifically, we will define an algorithm that computes the basis of the group $H_1(P)$ in an arbitrary closed 2-manifold P without using of incidence matrices. Another proposed algorithm indexes the edges of polyhedron P relative to any 1-cycle z. The latter can be used to calculate the intersection index for any 1-cycle with z and to construct a regular covering $p : \hat{P} \to P$ with the group of covering transformations $G \cong H_1(P)$. We will also provide several solutions to the problem of topological noise removal in computer models.

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1. Computing a Basis of the Group $H_1(P)$ without Using of Matrices

We first consider an auxiliary problem.

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Let Q be a polyhedron, $\dim Q = n$, and let $K(Q)$ be its simplicial complex. Suppose σ^n is the unique n-dimensional simplex of polyhedron Q that has the face $\sigma^{n-1} \in K(Q)$. Let Q' be a union of all simplices from $K(Q)$ that are different from σ^n and from σ^{n-1}. Then there exists a deformation retraction $r_{\sigma^{n-1}} : Q \to Q'$. Such deformation retraction is called an elementary collapse of the polyhedron Q to the polyhedron Q' through the boundary simplex σ^{n-1}. If there is a sequence of the elementary collapses $r_j : Q_j \to Q_{j+1}$, $j = 0, 1, \ldots, k-1$, $Q_0 = Q$ is the initial polyhedron and $Q_1, \ldots, Q_k$ are its subpolyhedra, then the composition $R = r_{k-1} \circ \cdots \circ r_1 \circ r_0$ is called a collapse. By definition, if it is possible to collapse the polyhedron Q to the sybpolyhedron $\bar{Q}$, then the inclusion $\imath : \bar{Q} \to Q$ is a homotopic equivalence.

Algorithm 1. (n-Dimensional Collapse)

Input:

1) lists $K^n(Q)$ and $K^{n-1}(Q)$ of simplices for the polyhedron Q;

2) for each $\sigma^{n-1} \in K^{n-1}(Q)$ the list $\partial^{-1}(\sigma^{n-1}, Q)$ and the number $[\sigma^{n-1} : K^n(Q)]$ of n-simplices from the polyhedron Q that are incident to σ^{n-1}.

Output:

1) lists $K^n(\bar{Q})$ and $K^{n-1}(\bar{Q})$ of simplices for the subpolyhedron $\bar{Q} \subset Q$;

2) for each $\sigma^{n-1} \in K^{n-1}(\bar{Q})$ the list $\partial^{-1}(\sigma^{n-1}, Q)$ and the number $[\sigma^{n-1} : K^n(\bar{Q})]$ of n-simplices from the polyhedron $\bar{Q}$ that are incident to σ^{n-1}.

Algorithm Description.

Step 1. Create copies $K^n(\bar{Q}) := K^n(Q)$ and $K^{n-1}(\bar{Q}) := K^{n-1}(Q)$. For each simplex $\sigma^{n-1} \in K^{n-1}(\bar{Q})$ assume that $\partial^{-1}(\sigma^{n-1}, \bar{Q}) = \partial^{-1}(\sigma^{n-1}, Q)$ and that $[\sigma^{n-1} : K^n(\bar{Q})] = [\sigma^{n-1} : K^n(Q)]$.

Step 2. Choose a simplex $\sigma_0^{n-1} \in K^{n-1}(Q)$ and make a queue $R = \{\sigma_0^{m-1}\}$ from it.

Step 3. Take the first element σ^{n-1} from the queue R and remove it from the lists $K^{n-1}(Q)$ and R.

Step 4. Check whether the number $[\sigma^{n-1} : K^n(\bar{Q})]$ is equal to 1. If this is not the case go to Step 7.

Step 5. Delete the simplex σ^{m-1} from the variable list $K^{n-1}(\bar{Q})$ and the unique n-simplex $\sigma^n \in \partial^{-1}(\sigma^{n-1}, \bar{Q})$ from the list $K^n(\bar{Q})$.

Step 6. For each face σ_*^{n-1} of the simplex σ^n different from σ^{n-1} execute the following operations: enqueue σ_*^{n-1} into R, remove σ^n from $\partial^{-1}(\sigma_*^{n-1}, \bar{Q})$ and set $[\sigma_*^{n-1} : K^n(\bar{Q})] = [\sigma_*^{n-1} : K^n(\bar{Q})] - 1$.

Step 7. If $R \neq \emptyset$ then go to Step 3.

Step 8. Check if the list $K^{n-1}(Q)$ is empty. If it is not then return to Step 2.

End of Algorithm.

Proposition 1. *Let Q be a polyhedron, let $K^n(\bar{Q})$ and $K^{n-1}(\bar{Q})$ be the lists provided by algorithm 1 for this polyhedron, and let $\bar{Q}$ be a union of simplices from $K^n(\bar{Q})$, $K^{n-1}(\bar{Q})$ and $K^p(\bar{Q}) = K^p(Q)$ for $p < n-1$. Then:*

— Q collapses to the subpolyhedron $\bar{Q}$ and $\bar{Q}$ does not allow elementary collapses through the $(n-1)$-dimensional simplices;

— if Q is a strongly connected polyhedron, each $(n-1)$-dimensional simplex of Q is incident at most to two n-simplices, and there exists a $(n-1)$-simplex that is incident to only one n-simplex, then $\dim \bar{Q} = n-1$.

Proof. We assume that $Q_0 = Q$ and introduce the symbol Q_k to denote the subpolyhedron of the polyhedron Q resulting from the k-th iteration on Step 5 of algorithm 1. According to the criterion verified in Step 4, the considered iteration is only applied if σ^n is the unique n-simplex of the polyhedron Q_{k-1} that has the face σ^{n-1}. There exists then an elementary collapse such that $r_i : Q_{k-1} \to Q_k$. If $\bar{Q} = Q_l$ then the composition $r = r_l \circ \cdots \circ r_1$ represents a collapse of the polyhedron $Q = Q_0$ to the resulting subpolyhedron $\bar{Q}$.

Assume now that after the algorithm completes there is still a possibility to perform the collapse $\bar{Q}$ through some simplex $\sigma^{n-1} \in K^{n-1}(\bar{Q})$. Then $[\sigma^{n-1} : K^n(\bar{Q})] = 1$.

Note that simplices are removed from the list $K^{n-1}(Q)$ in Step 3 only. But before removing the simplex is necessarily enqueued into R and picked from this queue for further processing. According to Step 8, this implies that any $(n-1)$-dimensional simplex of the polyhedron Q is processed by Step 4.

If $[\sigma^{n-1} : K^n(Q)] = 1$, then in accordance with our assumption, $[\sigma^{n-1} : K^n(\bar{Q})] = 1$ from the beginning to the end of the course of the algorithm. But after Step 4 the simplex σ^{n-1} will necessarily go to Step 5 where it will be removed from the list $K^{n-1}(\bar{Q})$. This fact contradicts our assumption.

If $[\sigma^{n-1} : K^n(Q)] = k > 1$, then the algorithm removes $(k-1)$ simplices from the list $\partial^{-1}(\sigma^{n-1}, \bar{Q})$ that is initially coincident with $\partial^{-1}(\sigma^{n-1}, Q)$. Consider the moment when the last of these $k-1$ simplices is being removed and $\partial^{-1}(\sigma^{n-1}, \bar{Q})$ is containing the only one n-dimensional simplex σ^n. This can only happen on Step 5. But it will necessarily be followed by Step 6 where σ^{n-1} is enqueued to R. Since it is impossible to be removed from R anywhere but in Step 3, then according to Step 7, the simplex σ^{n-1} will sooner or later be taked from the queue and verified in Step 4. According to the assumption, the simplex σ^n could not be removed from $K^n(\bar{Q})$. Then the number $\partial^{-1}(\sigma^{n-1}, \bar{Q})$ will turn out to be equal to 1 at this verification. But then σ^{n-1} will be removed from the list $K^{n-1}(\bar{Q})$ on Step 5 . Thus we face with a contradiction again.

The above implies that our assuming is wrong, and the polyhedron $\bar{Q}$ cannot collapse through the simplices of dimension $n-1$.

Assume now that the conditions of the second statement are met. Then the polyhedron Q allows at least one elementary collapse and hence $K^n(Q) \setminus K^n(\bar{Q}) \neq \emptyset$. At the same time if $\dim \bar{Q} = n$ then $K^n(\bar{Q}) \neq \emptyset$ as well. Now choose simplices $\tau^n \in K^n(\bar{Q})$ and $\tau^n_* \in K^n(Q) \setminus K^n(\bar{Q})$. Due to the strong connection of Q, there can be found a sequence of its n-dimensional simplices $\tau^n_0, \tau^n_1, \ldots, \tau^n_q$ such that $\tau^n_0 = \tau^n$, $\tau^n_q = \tau^n_*$ and the simplices τ^n_{i-1} and τ^n_i have a common $(n-1)$-dimensional face for all $i = 1, \ldots, q$. Let τ^n_{k-1} be the last element of this sequence that belongs to the list $K^n(\bar{Q})$. Then $1 \leq k \leq q$ and $\tau^n_k \in K^n(Q) \setminus K^n(\bar{Q})$. Denote by τ^{n-1} the common $(n-1)$-dimensional face of the simplices τ^n_{k-1} and τ^n_k. By condition, the simplex τ^{n-1} cannot be incident to the n-simplices of the polyhedron Q that are different from τ^n_{k-1} and from τ^n_k. Since τ^{n-1} does not belong to the subpolyhedron $\bar{Q}$, $[\tau^{n-1} : K^n(\bar{Q})] = 1$. But in this case $\bar{Q}$ allows an elementary collapse through the $(n-1)$-dimensional simplex τ^{n-1} and this contradicts the above statements. Hence the equality $\dim \bar{Q} = n$ is impossible and $\dim \bar{Q} = n - 1$. □

Corollary 1. *If Q is a connected n-dimensional manifold with a nonempty boundary, then algorithm 1 will provide us with a subpolyhedron $\bar{Q} \subset Q$ of dimension $n-1$ that is homotopically equivalent to Q.*

Let P be a triangulated 2-dimensional compact manifold. We introduce the symbols $C_k(P)$, $Z_k(P)$ and $H_k(P)$ to denote its groups of simplicial chains, cycles and homologies respectively with coefficients from $\mathbb{Z}_2$, $k = 0, 1, 2$.

Algorithm 2. (Computing Basic 1-Cycles of a 2-Manifold)

Input:

1) lists $V(P)$, $E(P)$ and $T(P)$ consisting of vertices, edges and triangles of the polyhedron P respectively;

2) for each edge $a \in E(P)$ the list $\partial^{-1}(a, P)$ and the number $[a : T(P)]$ of triangles from the polyhedron P that are incident to a.

Output:

the set of cycles $z_i \in Z_1(P)$, $i = 1, \ldots, r$.

Algorithm Description.

Step 1. If $\partial P = \emptyset$, then take a triangle $t_0 \in T(P)$ and remove it from $T(P)$. As a result we will obtain the lists of vertices $V(P') = V(P)$, edges $E(P') = E(P)$ and triangles $T(P') = T(P) \setminus \{t_0\}$ of the subpolyhedron $P' \subset P$. Having $\partial P \neq \emptyset$, assume that $T(P') = T(P)$ and $P' = P$.

Step 2. Using algorithm 1 for $n = 2$, we create the lists of vertices $V(Q) = K^0(Q)$ and edges $E(Q) = K^1(Q)$ for the 1-dimensional subpolyhedron $Q \subset P'$. And we will also get lists $\partial^{-1}(a, Q)$ and numbers $[v : E(Q)]$ for all vertices $v \in V(Q)$.

Step 3. Using algorithm 1 for $n = 1$, we construct an uncollapsable subpolyhedron $Q' \subset Q$. As the output we will get the lists of its vertices $V(Q')$, its edges $E(Q')$, and also lists $\partial^{-1}(a, Q')$ and numbers $[v : E(Q')]$ for all vertices $v \in V(Q')$.

Step 4. Find the basic cycles $z_1, \ldots, z_r$ of the polyhedron Q' with the help of the algorithm for searching fundamental cycles on a graph (see [4], Chapter 2).

End of Algorithm.

Theorem 1. *Let P be a connected 2-manifold, and $r > 0$. Then the homological classes $[z_1], \ldots, [z_r] \in H_1(P)$ of the cycles $z_1, \ldots, z_r$ computed by algorithm 2 generate a basis of the homology group $H_1(P)$.*

Proof. If $\partial P \neq \emptyset$, then according to Step 1 of algorithm 2, the following equalities take place: $P' = P$ and $H_1(P') = H_1(P)$.

Let then $\partial P = \emptyset$. For the triad $(P; P', t_0)$ consider the Mayer-Vietoris-sequence (see [1], Chapter III)

$$\ldots \to H_2(P') \oplus H_2(t_0) \xrightarrow{\jmath_*} H_2(P) \xrightarrow{\partial_*} \\ \xrightarrow{\partial_*} H_1(\partial t_0) \xrightarrow{\imath_*} H_1(P') \oplus H_1(t_0) \xrightarrow{\jmath_*} H_1(P) \xrightarrow{\partial_*} \\ \xrightarrow{\partial_*} H_0(\partial t_0) \xrightarrow{\imath_*} H_0(P') \oplus H_0(t_0) \xrightarrow{\jmath_*} H_0(P) \to 0. \quad (1)$$

Since P and ∂t_0 are closed manifolds of dimensions 2 and 1 respectively, then $H_2(P) \cong H_1(\partial t_0) \cong \mathbb{Z}_2$. Then under the strong connectivity of the polyhedra P, P', t_0 and ∂t_0 the following isomorphisms take place: $H_0(P) \cong H_0(P') \cong H_0(t_0) \cong H_0(\partial t_0) \cong \mathbb{Z}_2$. Finally, $H_2(P') = H_2(t_0) = H_1(t_0) = 0$. So in the considered case our sequence (1) is represented in the form:

$$0 \xrightarrow{\jmath_*} \mathbb{Z}_2 \xrightarrow{\partial_*} \mathbb{Z}_2 \xrightarrow{\imath_*} H_1(P') \xrightarrow{\jmath_*} H_1(P) \xrightarrow{\partial_*} \\ \xrightarrow{\partial_*} \mathbb{Z}_2 \xrightarrow{\imath_*} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\jmath_*} \mathbb{Z}_2 \to 0,$$

where the homomorphism $\jmath_* : H_1(P') \to H_1(P)$ is induced by the inclusion $j : P' \to P$. The resulting sequence can be exact only when the homomorphisms

$\imath_* : \mathbb{Z}_2 \to H_1(P')$ and $\partial_* : H_1(P) \to \mathbb{Z}_2$ are equal to zero. Hence (1) contains an exact subsequence

$$0 \to H_1(P') \xrightarrow{\jmath_*} H_1(P) \to 0.$$

Thus the inclusion $j : P' \to P$ induces the isomorphism $\jmath_* : H_1(P') \to H_1(P)$ in any case.

Since P' is a connected manifold, then according to the corollary 1, $\dim Q = 1$ and Step 2 is well defined.

The polyhedron Q' is generated by collapsing P'. So the inclusion $\imath' : Q' \to P'$ is a homotopic equivalence. Then the composition $\jmath' = \jmath \circ \imath'$, which is the inclusion $\jmath' : Q' \to P$, induces the isomorphism $\jmath'_* : H_1(Q') \to H_1(P)$.

According to Step 4, $z_1, \ldots, z_r$ is a basis of the group $H_1(Q')$. By construction, $\jmath'_*(z_i) = [z_i]$ for all $i = 1, \ldots, r$, where the square brackets denote the homological classes of cycles z_i in the manifold P. Hence $[z_1], \ldots, [z_r]$ is a basis of the group $H_1(P)$. □

Remark 1. Another method for computing a basis of the group $H_1(P)$ without using of matrices was proposed in [7].

2. Indexation of Edges Relative to the Given Cycle

Here we will think that P is a triangulated closed 2-manifold, and $\mathrm{Ind} : H_1(P) \times H_1(P) \to \mathbb{Z}_2$ is the intersection index (see [6], Chapter X).

Algorithm 3. (Indexation of Edges Relative to the Given 1-Cycle)
Input:

1) a 1-cycle z;

2) the list $E(P)$ of edges for the polyhedron P;

3) the list $T(P, z)$ consisting of triangles from neighbourhood of the cycle z.

Output:

1) the element $J_z(a) \in \mathbb{Z}_2 = \{0, 1\}$ for all edges $a \in E(P)$;

2) the chain M of edges indexed relative to the cycle z;

3) lists $M(u), u \in z$ of edges that we add to M when the vertex u is considered;

4) sets $\Sigma(u), u \in z$ of triangles incident to edges from $M(u)$.

Algorithm Description.

Step 1. Assume $M = \emptyset$, $J_z(a) := 0$ for all $a \in E(P)$. We create then lists of vertices and edges of cycle z, $V(z)$ $E(z)$ respectively.

Step 2. For each vertex $u \in V(z)$ execute Step 2.1 – Step 2.4.

Step 2.1. Create a list $T(P,u) \subset T(P,z)$ of triangles of the polyhedron P that contain u, and a list $E(P,u)$ of all edges of triangles from $T(P,u)$. At the same time for each edge $a \in E(P,u)$ we get a list $\partial^{-1}(a,u)$ of triangles from $T(P,u)$ those are incident to a, and assume $\mu(a) := 0$. Then we create empty lists $M(u) := \emptyset$ and $\Sigma(u) := \emptyset$.

Step 2.2. We chose a triangle $t_0 \in T(P,u)$, create a queue $R := \{t_0\}$ and remove t_0 from $T(P,u)$. Set $\Sigma(u)\{t_0\}$.

Step 2.3. While the queue R is not empty we will do the following actions. Take the first triangle $t \in R$ and remove it from the queue R. For each edge a of the triangle t we check the following: whether it belongs to the cycle z, whether $\mu(a)$ is equal to zero, whether the list $\partial^{-1}(a,u)$ contains any triangle different from t. If all above conditions are satisfied we will execute Step 2.3.1 – Step 2.3.2.

Step 2.3.1. Take the simplex $t_* \in \partial^{-1}(a,u) \setminus \{t\}$, remove it from $T(P,u)$ and enqueue to R; set $\mu(a) := 1$ and $\Sigma(u)\Sigma(u) \cup \{t_*\}$.

Step 2.3.2. Set $J_z(a) := J_z(a) + 1 \mod 2$, $M(u) = M(u) \cup \{a\}$, $M := M + a \mod 2$.

Step 2.4. If the list $M(u)$ is empty then go back to Step 2.2.

Step 3. For each edge $a = [w_1 w_2] \in E(z)$ we search any edges $b \in M(w_1)$ $c \in M(w_2)$ such that $b \cap c \neq \emptyset$ and a, b c are sides of some triangle of the polyhedron P. If the edges b and c do not exist then we set $J_z(a) := 1$ $M = M + a \mod 2$.

End of Algorithm.

Theorem 2. *If P is a closed 2-dimensional manifold, z is a simple cycle, $x = a_1 + \cdots + a_l \in Z_1(P)$ and $J_z(x) = \sum_{i=1}^{l} J_z(a_i)$, then $J_z(x) = \mathrm{Ind}([x],[z])$.*

Proof. Consider a vertex $u_p \in V(z)$ and its barycentric star $\mathrm{bst}(u_p, P)$.

Let $\Sigma^*(u_p)$ be the set of all triangles from the barycentric subdivision of $\Sigma(u_p)$. Construct the chain $c(u_p)$ of triangles $t_b \in \mathrm{bst}(u_p,P) \cap \Sigma^*(u_p)$.

Then we write the boundary of the chain $c(u_p)$ as a sum $Y_1 + Y_2$, where Y_1 is the sum of edges $a \in \partial c(u_p)$ that belong to cycle z and Y_2 is the sum of all remaining edges from the chain $\partial c(u_p)$. Set $z_p^* = z_{p-1}^* + Y_1 + Y_2 \mod 2$.

By construction $z_p^* \sim z_{p-1}^*$ for all $p = 1, \ldots, N$. Hence the cycle $z^* = z_N^*$ is homologous to the cycle $z = z_0^*$.

Let us now prove that for any edge $a = [uv] \in E(P)$ and $c_b \in \mathrm{bst}(a)$ the edge c_b belong to z^* if and only if $a \in M$.

Let us view all possible positions of the edge a. At the same time we also agree to think that $M(u) = \emptyset$ and that $\Sigma(u) = \emptyset$ for all $u \notin V(z)$.

0. If $a \notin M(u) \cup M(v)$ and $a \notin E(z)$, then according to the algorithm, $a \notin M$. On the other hand, the edge a cannot be incident to triangles from the lists $\Sigma(u)$ and $\Sigma(v)$ and hence $c_b \notin z^*$.

1. Let $u \in V(z)$, $a \in M(u)$, and $v \notin V(z)$. Then the edge a will be still in the chain M when algorithm 3 is completed. At the same time the barycentric star $\mathrm{bst}(a)$ belongs to the boundary of the chain $c(u)$ and does not belong to the cycle z. Thus in this case $a \in M$ and the chain $\mathrm{bst}(a)$ belongs to the cycle z^*.

2. Further, assume that $u, v \in V(z)$ and $a \in M(u)$. At that, $a \notin E(z)$.

2.1. If $a \in M(v)$, then $a \notin M$, and simplices of its barycentric star will be added twice to the initial cycle z and will not be in the resulting cycle z^*.

2.2. If $a \notin M(v)$, then $a \in M$ and any edge $c_b \in \mathrm{bst}(a)$ is added to the cycle z^* exactly once. So $c_b \in z^*$.

3. Finally, let $a \in E(z)$.

3.1. Let assume that the condition from Step 3 of algorithm 3 is satisfied, i.e.:

($*$) there exist edges $b \in M(u)$ and $c \in M(v)$ such that $b \cap c \neq \emptyset$ and a, b and c are sides of some triangle $t' \in T(P)$.

In this case, according to the algorithm, $a \notin M$.

The triangle t' from ($*$) belong both to $\Sigma(u)$ and $\Sigma(v)$. Since P is a closed 2-dimensional manifold, there exists exactly one triangle $t'' \neq t'$ incident to the edge a; at that t'' does not belong to the both sets $\Sigma(u)$ and $\Sigma(v)$. Consider any edge $c_b \in \mathrm{bst}(a)$ and a triangle t such that $c_b \in t$. Then t either belong to the both sets $\Sigma(u)$ and $\Sigma(v)$ or does not belong to them. Hence the edge c_b either is not added to the cycle z^* or is added twice. Therefore, $c_b \notin z^*$.

3.2. Assume now that condition ($*$) is not satisfied. Then according to Step 3 of algorithm 3, $a \in M$.

By construction, the set $\Sigma(u)$ cannot be empty. Moreover, if the edge $\tilde{a} \in z$ is incident to the vertex u, then $\tilde{a}$ is a face of some triangle from $\Sigma(u)$. So there exists a triangle $t_1 \in \Sigma(u)$ that contains the edge a.

This implies, in accordance with our assumption, that the triangle t_1 cannot belong to the set $\Sigma(v)$.

The set $\Sigma(v)$ cannot be empty also. Since each edge of z incident to the vertex v is a face of some triangle from $\Sigma(v)$, it follows that there exists a triangle $t_2 \in \Sigma(v)$ that contains the edge a.

Consider $c_b \in \mathrm{bst}(a)$. Since P is a closed 2-dimensional manifold, $c_b \in t_1$ or $c_b \in t_2$. It follows that c_b is involved in the cycle z^* exactly once, so $c_b \in z^*$.

Thus we have proved that the cycle $z^* \sim z$ consists of barycentric stars of the edges from the chain M. That means that this cycle intersects transversally only the edges of the cycle x that are in the list M. According to algorithm 3, $J_z(a) = 1$ for all $a \in M$ and $J_z(b) = 0$ for all edges $b \notin M$. So

$$\mathrm{Ind}([x],[z]) = \mathrm{Ind}([x],[z^*]) = \sum_{a\in x} J_z(a) \bmod 2 = J_z(x). \qquad \square$$

Definition 1. Let $[z_1^1], \ldots, [z_r^1]$ be a basis of the homology group $H_1(P)$. A homomorphism $J : C_1(P) \to \mathbb{Z}_2^r$, $J = (J^1, \ldots, J^r)$ is called *a index vector-function* if $J^k(y) = \mathrm{Ind}([y],[z_k^1])$ for any cycle $y \in Z_1(P)$ and each $k \in \{1, \ldots, r\}$. For any chain $x \in C_1(P)$ the value $J(x)$ is called its index relative to the basis $[z_1^1], \ldots, [z_r^1]$.

Remark 2. We can apply algorithm 3 to an arbitrary set of simple 1-cycles of the manifold P. In particular, this set may consist of cycles $z_1^1, \ldots, z_r^1$ such that $[z_1^1], \ldots, [z_r^1]$ is some basis of the homology group $H_1(P)$. So we can use algorithm 3 to construct the index vector-function $J : C_1(P) \to \mathbb{Z}_2^r$.

Remark 3. If $J : C_1(P) \to \mathbb{Z}_2^r$ is the index vector-function relative to the basis $\{[z_1^1], \ldots, [z_r^1]\}$ of the group $H_1(P)$, $x, y \in C_1(P)$ and $\partial x = \partial y$, then $J(x) = J(y)$ if and only if $x \sim y$.

3. Computing Minimal 1-Cycles that Generate a Canonical Basis of the Group $H_1(P)$

Let P be a closed 2-dimensional manifold, $S = (V, K)$ its simplicial scheme, $r = \operatorname{rank} H_1(P)$, and $J : C_1(P) \to \mathbb{Z}_2^r$ the index vector-function relative to some basis of the homology group $H_1(P)$. Then we can construct an abstract simplicial scheme $\hat{S} = (\hat{V}, \hat{K})$ as follows.

Set $\hat{V} = V \times G$, where $G = \mathbb{Z}_2^r$. Let $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_m \in \hat{V}$, where $\hat{v}_i = (v_i, g_i)$ for all $i = 0, 1, \ldots, m$. We will think that $\{\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_m\} \in \hat{K}$ if the below conditions are satisfied:

(U1) $\{v_0, v_1, \ldots, v_m\} \in K$;

(U2) $g_0 + g_i = J([v_0 v_i])$ for any $i = 1, \ldots, m$; here $J([v_0 v_i])$ is the index of the edge $[v_0 v_i]$.

Let define now a mapping $p^0 : \hat{V} \to V$ and a left action $\lambda^0 : G \times \hat{V} \to \hat{V}$ of the group G on $\hat{V}$: $p^0((v, g)) = v$ and $\lambda^0(g', (v, g)) = g' \cdot (v, g) = (v, g' + g)$ for all $(v, g) \in \hat{V}$ and $g' \in G$.

Let $\hat{P}_J$ define some realization of the scheme $\hat{S} = (\hat{V}, \hat{K})$. At that we identify the set of vertices of the polyhedron $\hat{P}_J$ with $\hat{V}$. In the paper [3] we proved the following proposition.

Proposition 2. *For the mapping $p^0 : \hat{V} \to V$ there exists the unique continuation $p : \hat{P}_J \to P$ that is a simplicial regular covering with the group of covering transformations $G \cong H_1(P)$.*

Let $L : C_1(R) \to \mathbb{R}$ be some non-negative weight function. We can define a weight function $\hat{L} : C_1(\hat{P}_J) \to \mathbb{R}$ assuming that $\hat{L}(\hat{x}) = L(p(\hat{x}))$ for an arbitrary chain $\hat{x} \in C_1(\hat{P}_J)$

Assume that the manifold P is orientable.

Definition 2. Any basis $[x_1^1], \dots, [x_r^1]$ of the homology group $H_1(P)$ is called canonical if $\mathrm{Ind}([x_{2k-1}^1], [x_{2k}^1]) = 1$, $k = 1, \dots, m$, and $\mathrm{Ind}([x_i^1], [x_j^1]) = 0$ for all the remaining pairs of indices i and j. If only the first of above conditions is met we say that $[x_1^1], \dots, [x_r^1]$ is a semicanonical basis.

Algorithm 4. (Computing a Canonical Basis Consisting of Minimal Cycles)

Input:

1) the list $V(P)$ of vertices for polyhedron P;
2) basis cycles $x_1^1, \dots, x_r^1$;
3) a weight function $L : C_1(P) \to \mathbb{R}$.

Output:

cycles $z_1^1, \dots, z_r^1$.

Algorithm Description.

Step 1. Construction of the index function. Use algorithm 3 to compute the index vector-function $J : C_1(P) \to \mathbb{Z}_2^r$ relative to the basis $[x_1^1], \dots, [x_r^1]$.

Step 2. Computing a semicanonical basis $[z_1^1], \dots, [z_r^1]$.

Step 2.1. Assume that $q := 1$.

Step 2.2. If $q = r - 1$ then assume that $z_q^1 := x_q^1$, $z_{q+1}^1 := x_{q+1}^1$ and go to Step 3.

Step 2.3. Assume that $z_q^1 := x_q^1$ and $p := q + 1$.

Step 2.4. If $\mathrm{Ind}(x_p^1, x_q^1) = J^q(x_p^1) = 1$ then go to Step 2.6.

Step 2.5. Assign the variable p to the value $p + 1$ and repeat Step 2.4.

Step 2.6. Compute the determinant of the matrix B_q built from the intersection indices $\mathrm{Ind}(x_i^1, x_j^1) = J^j(x_i^1)$, $i, j \in \{q + 1, \dots, r\} \setminus \{p\}$. If $\det B_q = 0$ then return to Step 2.5., assume that $z_{q+1}^1 := x_p^1$ otherwise.

Step 2.7. Having $p > q + 1$, assume that $x_p^1 := x_{q+1}^1$, $q := q + 2$ and return to Step 2.2.

Step 2.8. Rearrange the elements J^k, $k = 1, \ldots, r$ in the index vector-function $J = \left(J^1, \ldots, J^r\right)$ in accordance with the new order of cycles in the basis.

Step 3. Computing a canonical basis. For all $k = 1, \ldots, r$ run Step 3.1 – Step 3.6.

Step 3.1. If k is an even number assume $k_* = k - 1$, otherwise assume $k_* = k + 1$. Also assume that $Z := \emptyset$.

Step 3.2. Compute the vector I following the rule: $I^{k_*} := 1$, $I^j := 0$ for $j \in \{1, \ldots, r\} \setminus \{k_*\}$.

Step 3.3. For each vertex $v \in V(z^1_{k_*})$ run the loop 3.3.1 – 3.3.2.

Step 3.3.1. In the 1-dimensional skeleton of the polyhedron $\hat{P}_J$ compute the path $\hat{z}_v$ from the vertex $(v, 0)$ to the vertex (v, I) with minimal weight $\hat{L}(\hat{z}_v)$. Assume that $z_v = p(\hat{z}_v)$.

Step 3.3.2. Add the cycle z_v into the list Z.

Step 3.4. From the list Z select a cycle z such that $L(z) = \min\limits_{z' \in Z} L(z')$.

Step 3.5. Assume $z^1_k := z$.

Step 3.6. Use algorithm 3 to re-compute the k-th element of the vector-function J.

End of Algorithm.

Theorem 3. *Algorithm 4 is well defined. The homological classes of cycles $z^1_1, \ldots, z^1_r$ provided by algorithm 4 form a canonical basis of the homology group $H_1(P)$, and weight of each cycle z^1_k, $k = 1, \ldots, r$, is minimal among all cycles homologous to it.*

Proof. Assume that either $q = 1$ or $1 < q = 2n + 1 < r$ and that Step 2 of algorithm 4 has picked from the set $\{x_1, \ldots, x_r\}$ the cycles $z^1_1, \ldots, z^1_{2n}$ such that $\mathrm{Ind}(z^1_{2l-1}, z^1_{2l}) = 1$ for all $l = 1, \ldots, n$. Consider now the matrix B_q built from the intersection indices $b_{ij} = \mathrm{Ind}(x^1_{2n+i}, x^1_{2n+j})$, $i, j \in \{1, \ldots, r - 2n\}$, of the cycles $x^1_q, \ldots, x^1_r$, where $x^1_l \notin \{z^1_1, \ldots, z^1_{2n}\}$, $l = q, \ldots, r$. At the same time due to the nonsingularity of the form Ind on $H_1(P)$ and according to Step 2.6, $\det B_q = 1$.

Note that the matrix B_q is skew-symmetric. So

$$\det B_q = \sum_{j=2}^{r-q+1} b_{1j}\sqrt{U_{1j}}, \tag{2}$$

where U_{1j} is an algebraic complement of the minor of order 2 lying on the intersection of rows and columns numbered 1 and j within the matrix B_q ([5]). It follows from the condition $\det B_q \neq 0$ that at least one item of the decomposition (2) is non-zero. Let p' be the number of such item and $p = q + p'$. Then $\operatorname{Ind}(x_q^1, x_p^1) = 1$ and $\det B_{q+2} \neq 0$, where the matrix B_{q+2} derives from the matrix B_q through deletion of columns and rows numbered 1 and $p' = p-q$.

Thus the computing of the semicanonical basis is well defined. Let us make one important remark. For all $q = 1, 3, \ldots, r-1$ the matrix A_q created after Step 2 derives from B_q through rows and columns permutation. So, $\det A_q = \det B_q = 1$.

Let Step 3 of algorithm 4 has transformed first k cycles $z_1^1, \ldots, z_k^1$, $k \in \{1, \ldots, r\}$. We prove now that after these transformations the homological classes of cycles $z_1^1, \ldots, z_r^1$ are still linearly independent.

According to 3.1 – 3.6, when the cycles $z_1^1, \ldots, z_k^1$ have been transformed, the matrix A_0 of intersection indices for all cycles can be put into form

$$A_0^k = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & 0 & a_{lq} & \ldots & a_{lr} \\ 0 & 0 & \ldots & 0 & a_{ql} & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & a_{rl} & * & \ldots & * \end{pmatrix}, \tag{3}$$

where $l = \max\{k, k_*\}$, $q = l + 1$, and the asterisk denotes the elements of the matrix A_q. If k is an even number, the following equalities take place: $k = l$ and $a_{lj} = a_{il} = 0$ for all $i, j = q, \ldots, r$.

It is evident that $\det A_0^k = \det A_{q-2}^k$. Using decomposition (2) and form (3), we get the equality $\det A_0^k = \det A_q^k = \det A_q$. Since $\det A_q = 1$, it proves the linear independence of homological classes $[z_1^1], \ldots, [z_r^1]$.

So, if $k < r$ then after Step 3.6 we get a well-defined index vector-function J and may now repeat the loop 3.1 – 3.6 for the next value of parameter k.

If $k = r$, the matrix A_0^k is in a canonical form. Hence, when algorithm 4 finishes its course, the homological classes of cycles $z_1^1, \dots, z_r^1$ form a canonical basis of the group $H_1(P)$.

Let then $k \in \{1, \dots, r\}$, $y \in Z_1(P)$ and $[y] = [z_k^1]$, where z_k^1 is a cycle output by the algorithm. Consider the index vector-function J relative to the computed before Step 3.3 basis of the group $H_1(P)$. Then $J(y) = J(z_k^1)$.

By construction, $z_k^1 = z_v$ for a certain vertex $v \in V(z_{k_*}^1)$. At the same time $z_v = p(\hat{z}_v)$, where $\hat{z}_v$ is a path in $\hat{P}_J$ running from the vertex $(v, 0)$ to the vertex (v, I). Assume that $\hat{z}_v = \sum_{i=1}^{q} [\hat{v}_{i-1}\hat{v}_i]$, where $\hat{v}_0 = (v_0, g_0) = (v, 0)$ and $\hat{v}_i = (v_i, g_i)$ for all $i = 1, \dots, q$. Then according to conditions (U1) and (U2), $z_v = \sum_{i=1}^{q} [v_{i-1}v_i]$ and $g_i = g_{i-1} + J([v_{i-1}v_i])$ for all $i = 1, \dots, q$. As a result we have the equalities $J(y) = J(z_v) = g_q = I$, and especially $\mathrm{Ind}([y], [z_{k_*}^1]) = I^{k_*} = 1$. The latter means that the cycles y and $z_{k_*}^1$ have the common vertex w.

Let $y = \sum_{j=1}^{s} [w_{j-1}w_j]$, where $w_0 = w = w_s$. Assume that $\hat{w}_0 = (w_0, h_0) = (w, 0)$, $h_j = h_{j-1} + J([w_{j-1}w_j])$, $\hat{w}_j = (w_j, h_j)$ for all $j = 1, \dots, s$ and $\hat{y} = \sum_{j=1}^{s} [\hat{w}_{j-1}\hat{w}_j]$. Then $\hat{y}$ is a edge path in the $\hat{P}_J$ connecting $\hat{w}_0 = (w, 0)$ and $\hat{w}_s = (w, h_s)$ and covering the cycle y. By construction, $h_s = J(y) = I = J(z_w)$, where $z_w = p(\hat{z}_w)$ is a cycle computed in Step 2.3.1 for the vertex $w \in V(z_{k_*}^1)$. Since $\hat{z}_w$ is a path in the 1-dimensional skeleton $\hat{P}_J^1$ running from $(w, 0)$ to (w, I) and having minimal weight among all such paths, then $L(y) = \hat{L}(\hat{y}) \geq \hat{L}(\hat{z}_w) = L(z_w) \geq L(z_k^1)$. □

Remark 4. When we search the path z_v in Step 3.3.1 there is no need to precompute a 1-dimensional skeleton for the polyhedron $\hat{P}_J$. For instance, if one wants to apply Dijkstra's algorithm, it will suffice to know for each vertex $\hat{v} = (v, g)$, $v \in \hat{V}$, the rules of finding adjacent vertices and lengths of the edges that connect these vertices with $\hat{v}$. According to conditions (U1) and (U2), if the vertices $w_1, \dots, w_t \in P$ are all adjacent to v, then the vertices $\hat{w}_i = (w_i, g + J([vw_i])$, $i = 1, \dots, t$, will form the neighborhood for $\hat{v} = (v, g)$ in $\hat{P}_J$. At the same time $\hat{L}([\hat{v}\hat{w}_i]) = L([vw_i])$. So, in Step 3.6 we will need to update the index vector-function J only. The detailed description for this procedure is given in [3].

4. Some Applications

Let P be a triangulated and orientable 2-manifold of genus m and let $T(P)$ be the list of its triangles. We use the term of handle allocation to denote the creation of a list $T(R) \subset T(P)$ such that the union R of all its elements is homeomorphic to a bounded cylinder, and the union of simplices from the complement $T(P) \setminus T(R)$ is a surface of genus $m-1$ with two boundary components.

The algorithms proposed above allow to allocate all handles on the surface P. The general search scheme is as follows:

1. Using algorithm 2, we find cycles $y_1, \dots, y_r$, $r = 2m = \operatorname{rank} H_1(P)$ such that homological classes $[y_1], \dots, [y_r]$ form a basis of the group $H_1(P)$.

2. Using algorithm 3, we compute the index vector-function $J_0 : C_1(P) \to \mathbb{Z}_2^r$ relative to the basis $[y_1], \dots, [y_r]$ of the group $H_1(P)$.

3. Using algorithm 4, we compute the cycles $z_1, \dots, z_r$ such that each of them has minimal weight in its homology class, and these classes form a canonical basis of the group $H_1(P)$. At the same time according to Step 3.6, we also get the index vector-function $J : C_1(P) \to \mathbb{Z}_2^r$ relative to the basis $[z_1], \dots, [z_r]$.

3. For each pair of cycles z_{2k-1} and z_{2k}, $k = 1, \dots, m$, run the following steps:

3.1. Choose a main cycle $x \in \{z_{2k-1}, z_{2k}\}$ and a complementary cycle $y \in \{z_{2k-1}, z_{2k}\}$, $y \neq x$ (by length for instance).

3.2. For each vertex $v_i \in x$, $i = 1, \dots, n_k$, we search through the 1-dimensional skeleton of the covering polyhedron $\hat{P}_J$ for the shortest path $\hat{y}_i$ running from the vertex $(v_i, 0)$ to the vertex $(v_i, J(y))$, and assume that $y_i = p(\hat{y}_i)$.

3.3. Starting from the triangle incident to the edge $[v_i v_{i+1}]$ ($i+1$ is computed modulo n_k), we build a list T_i of triangles lying between the cycles y_i and y_{i+1}. Then we compute the group $H_1(U_i)$ for the union U_i of simplices from T_i.

3.4. Finally, we compute a maximal by inclusion strongly connected union R_k of the areas U_i, where $\operatorname{rank} H_1(U_i) \leq 1$.

We make use of handle allocation to locate and remove topological noises on the computer models of real objects.

In a number of cases computer models may contain topological artifacts (boundary components, handles, lines of branching, singular vertices) that are not present on the initial objects (see, for example, [2], [8]). The totality of these redundant elements, which represent modeling defects, is called the topological noise.

The problem of location of redundant handles and their removal is one of the most difficult aspects of topological denoising. If an identified handle R_k is a defect of computer modeling, then the model is corrected: the list of handle's triangles $T_k(R_k)$ is removed from the general list $T(P)$, and two resulting holes are sealed.

But a handle of the surface P is not determined in a unique way and can be modified. It is useful to grow the region R_k in order to get the resulting surface smoother.

Let $y_l, y_{l+1}, \ldots, y_{l'}$ be the cycles computed above that belong to the handle R_k, $1 \leq l < l' \leq n_k$, and let n be the integer part of the number $(l'-l)/2$. We introduce the symbol $S(R_k)$ to denote the list $\{n+1, n-1, n+2, n-2, \ldots, l', l\}$ if $l'-l$ is an odd number and the list $\{n+1, n-1, n+2, n-2, \ldots, l, l'\}$ if $l'-l$ is an even one. For each j from $S(R_k)$ Step 3.2 finds a cycle y'_j running through the vertex v_j and sharing edges neither with y_n nor with the cycles $y'_{j'}$ founded before y'_j. The symbol R'_k will denote the union of triangles lying between the cycles y'_l and $y'_{l'}$ and containing the handle R_k.

R'_k removal provides us a better result than the method of cutting the surface on one almost shortest noncontractible cycle on the handle R_k or on two close cycles (see [2], [8]).

Having the list $T_k(R_k)$, we can compute area and diameter of the handle R_k. This gives us a way to evaluate the ratio of handle sizes to surface P sizes, so we can correct classify this handle as the topological noise.

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