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KOEBE TYPE THEOREMS OF QUASICONFORMAL MAPPINGS OF THE EXTREIOR OF THE UNIT BALL

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Abstract: Let $D \subset R^n$ be a domain, $B = \{x \in R^n : |x| < 1\}$ be the unit ball, and D^* and B^* be the exterior of D and B respectively. In this paper, we obtain two Koebe type distortion theorems of quasiconformal mappings which map B^* onto D^* or D^* onto B^*.

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1. Introduction

Let R^n be the n-dimensional Euclidean space, $\bar{R}^n = R^n \bigcup \{\infty\}$. Throughout this paper, we assume that $D \subseteq R^n$ is a domain, $\bar{D}$ is the closure of D, $D^* = \bar{R}^n \backslash \bar{D}$ is the exterior of D and $B(0,1) = \{x \in R^n : |x| < 1\}$ is the unit ball. For $z \in R^n$, $d(z, \partial D)$ denotes the Euclidean distance from z to the boundary ∂D of D.

The following well-known Koebe Theorem due to Koebe [7].

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Theorem A. *Suppose that D and D' are domains in R^2. If $f : D \to D'$ is conformal, then*

$$\frac{1}{4}\frac{d(f(x), \partial D')}{d(x, \partial D)} \le |f'(x)| \le 4\frac{d(f(x), \partial D')}{d(x, \partial D)}$$

for $x \in D$.

Theorem A has been used extensively in the research fields of the variational theory [3], the inverse problem theory [8], the capacity theory [5], the extremality theory [4] and the Clifford-algebra theory [6].

Suppose next that D and D' are domains in R^n and that $f : D \to D'$ is a K-quasiconformal mapping with Jacobian J_f. Then $\log J_f$ is integrable over each ball $B \subset D$, and we set

$$(\log J_f)_B = \frac{1}{m(B)} \int_B \log J_f dm. \tag{1.1}$$

In particular, for each $x \in D$ we let

$$a_f(x) = \exp\left(\frac{1}{n}(\log J_f)_{B(x)}\right), \tag{1.2}$$

where $B(x) = B(x, d(x, \partial D))$, the open ball with center x and radius equal to the $d(x, \partial D)$. If $n = 2$ and f is conformal in D, then $\log J_f$ is harmonic,

$$(\log J_f)_{B(x)} = \log J_f(x) = 2\log|f'(x)|$$

and hence $a_f(x) = |f'(x)|$.

K. Astala and F.W. Gehring [2] found that for certain distortion properties of quasiconformal mappings the function a_f plays a role exactly analogous to that played by $|f'|$ when $n = 2$ and f is conformal. They investigated this analogy further and established the following two results in [1].

Theorem B. *Suppose that D and D' are domains in R^n. If $f : D \to D'$ is a K-quasiconformal mapping, then*

$$\frac{1}{c_1}\frac{d(f(x), \partial D')}{d(x, \partial D)} \le a_f(x) \le c_1\frac{d(f(x), \partial D')}{d(x, \partial D)} \tag{1.3}$$

for $x \in D$, where c_1 is a constant which depends only on K and n.

Theorem C. *Suppose that D, D', D'' are domains in R^n. If $f : D \to D'$ and $g : D' \to D''$ are K_1- and K_2-quasiconformal mappings, then*

$$\frac{1}{c_2}a_g(f(x))a_f(x) \le a_{g\circ f}(x) \le c_2 a_g(f(x))a_f(x) \tag{1.4}$$

for $x \in D$, where c_2 is a constant which depends only on K_1, K_2 and n.

The main purpose of this paper is to prove the following two results.

Theorem 1. *Suppose that D is a domain in R^n. If $f : B^*(0,1) \to D^*$ is a K-quasiconformal mapping and $f(\infty) = \infty$, then*

$$\frac{1}{c}\frac{d(f(x), \partial D^*)}{d(x, \partial B^*(0,1))} \leq a_f(x) \leq c\frac{d(f(x), \partial D^*)}{d(x, \partial B^*(0,1))} \tag{1.5}$$

for $x \in B^*(0,1)\backslash\{\infty\}$, where c is a constant which depends only on K and n.

Theorem 2. *Suppose that D is a domain in R^n. If $f : D^* \to B^*(0,1)$ is a K-quasiconformal mapping and $f(\infty) = \infty$, then*

$$\frac{1}{c}\frac{d(f(x), \partial B^*(0,1))}{d(x, \partial D^*)} \leq a_f(x) \leq c\frac{d(f(x), \partial B^*(0,1))}{d(x, \partial D^*)} \tag{1.6}$$

for $x \in D^*\backslash\{\infty\}$, where c is a constant which depends only on K and n.

2. Proof of Theorems

Proof of Theorem 1. For any $x_0 \in B^*(0,1)\backslash\{\infty\}$, let

$$G' = \{z \in D^* : |z - f(x_0)| < d(f(x_0), \partial D^*)\}$$

and

$$G = f^{-1}(G'),$$

then G, $G' \subset R^n$. For $x \in B(0,1)$, let

$$h(x) = d(f(x_0), \partial D^*)x + f(x_0),$$

then $h : B(0,1) \to G'$ is a conformal mapping and $h(0) = f(x_0)$, and hence $f^{-1} \circ h : B(0,1) \to G$ is a K-quasiconformal mapping. Applying Theorem B to $f^{-1} \circ h$ we have

$$a_{f^{-1}\circ h}(0) \leq c_1\frac{d(f^{-1} \circ h(0), \partial G)}{d(0, \partial B(0,1))} = c_1 d(x_0, \partial G). \tag{2.1}$$

Making use of Theorem C to $f^{-1} \circ h$ and $f^{-1} \circ f$ we get

$$a_{f^{-1}\circ h}(0) \geq \frac{1}{c_2}a_{f^{-1}}(f(x_0))a_h(0) \tag{2.2}$$

and

$$a_{f^{-1}\circ f}(x_0) \leq c_3 a_{f^{-1}}(f(x_0))a_f(x_0). \tag{2.3}$$

The above constants c_1, c_2 and c_3 depend only on K and n.

Since

$$a_{f^{-1}\circ f}(x_0) = 1 \tag{2.4}$$

and

$$a_h(0) = d(f(x_0), \partial D^*), \tag{2.5}$$

hence (2.1), (2.2), (2.3), (2.4) and (2.5) imply

$$a_f(x_0) \geq \frac{1}{c_1 c_2 c_3} \frac{d(f(x_0), \partial D^*)}{d(x_0, \partial G)}. \tag{2.6}$$

(2.6) and $d(x_0, \partial G) \leq d(x_0, \partial B^*(0,1))$ yield

$$a_f(x_0) \geq \frac{1}{c_1 c_2 c_3} \frac{d(f(x_0), \partial D^*)}{d(x_0, \partial B^*(0,1))}. \tag{2.7}$$

Next, for any $x_0 \in B^*(0,1) \backslash \{\infty\}$, let

$$G_1 = \{z \in B^*(0,1) : |z - x_0| < d(x_0, \partial B^*(0,1))\}$$

and

$$G_1^{'} = f(G_1),$$

then G_1, $G_1^{'} \subset R^n$. For $x \in B(0,1)$, let

$$h(x) = d(x_0, \partial B^*(0,1))x + x_0,$$

then $h : B(0,1) \to G_1$ is a conformal mapping and $h(0) = x_0$, and hence $f \circ h : B(0,1) \to G_1^{'}$ is K-quasiconformal mapping. Applying Theorem B to $f \circ h$, we have

$$a_{f\circ h}(0) \leq c_1 \frac{d(f \circ h(0), \partial G_1^{'})}{d(0, \partial B(0,1))} = c_1 d(f(x_0), \partial G_1^{'}). \tag{2.8}$$

Making use of Theorem C to $f \circ h$ we get

$$a_{f\circ h}(0) \geq \frac{1}{c_2} a_f(h(0)) a_h(0) = \frac{1}{c_2} a_f(x_0) a_h(0). \tag{2.9}$$

(2.8), (2.9) and $a_h(0) = d(x_0, \partial B^*(0,1))$ imply

$$a_f(x_0) \leq c_1 c_2 \frac{d(f(x_0), \partial G_1^{'})}{d(x_0, \partial B^*(0,1))}. \tag{2.10}$$

(2.10) and $d(f(x_0), \partial G_1^{'}) \leq d(f(x_0), \partial D^*)$ yield

$$a_f(x_0) \leq c_1 c_2 \frac{d(f(x_0), \partial D^*)}{d(x_0, \partial B^*(0,1))}. \tag{2.11}$$

(2.7) and (2.11) complete the proof of Theorem 1 with $c = c_1 c_2 c_3$. □

Proof of Theorem 2. For any $x_0 \in D^* \backslash \{\infty\}$, let

$$G' = \{z \in B^* : |z - f(x_0)| < d(f(x_0), \partial B^*(0,1))\}$$

and

$$G = f^{-1}(G'),$$

then G, $G' \subset R^n$. For $x \in B(0,1)$, let

$$h(x) = d(f(x_0), \partial B^*(0,1))x + f(x_0),$$

then $h : B(0,1) \to G'$ is a conformal mapping and $h(0) = f(x_0)$, and hence $f^{-1} \circ h : B(0,1) \to G$ is a K-quasiconformal mapping. Applying Theorem B to $f^{-1} \circ h$ we have

$$a_{f^{-1} \circ h}(0) \leq c_1 \frac{d(f^{-1} \circ h(0), \partial G)}{d(0, \partial B)} = c_1 d(x_0, \partial G). \tag{2.12}$$

Making use of Theorem C to $f^{-1} \circ h$ and $f^{-1} \circ f$ we get

$$a_{f^{-1} \circ h}(0) \geq \frac{1}{c_2} a_{f^{-1}}(h(0)) a_h(0) = \frac{1}{c_2} a_{f^{-1}}(f(x_0)) d(f(x_0), \partial B^*(0,1)) \tag{2.13}$$

and

$$1 = a_{f^{-1} \circ f}(x_0) \leq c_3 a_{f^{-1}}(f(x_0)) a_f(x_0). \tag{2.14}$$

The above constant c_1, c_2 and c_3 depend only on K and n. Combining (2.12), (2.13), (2.14) and $d(x_0, \partial G) \leq d(x_0, \partial D^*)$ we obtain

$$a_f(x_0) \geq \frac{1}{c_1 c_2 c_3} \frac{d(f(x_0), \partial B^*(0,1))}{d(x_0, \partial D^*)}. \tag{2.15}$$

Next for any $x_0 \in D^* \backslash \{\infty\}$, let

$$G_1 = \{z \in D^* : |z - x_0| < d(x_0, \partial D^*)\}, \quad G_1' = f(G_1),$$

then G_1, $G_1' \subset R^n$. For $x \in B(0,1)$, let

$$h(x) = d(x_0, \partial D^*)x + x_0,$$

then $h : B(0,1) \to G_1$ is a conformal mapping and $h_1(0) = x_0$. Hence $f \circ h : B(0,1) \to G_1'$ is a K-quasiconformal mapping. Applying Theorem B to $f \circ h$, we have

$$a_{f \circ h}(0) \leq c_1 \frac{d(f \circ h(0), \partial G_1')}{d(0, \partial B(0,1))} = c_1 d(f(x_0), \partial G_1'). \tag{2.16}$$

Making use of Theorem C to $f \circ h$ and $a_h(0) = d(x_0, \partial D^*)$ we get

$$a_{f \circ h}(0) \geq \frac{1}{c_2} a_f(h(0)) a_h(0) = \frac{1}{c_2} a_f(x_0) d(x_0, \partial D^*). \tag{2.17}$$

(2.16), (2.17) and $d(f(x_0), \partial G_1') \leq d(f(x_0), \partial B^*(0,1))$ imply

$$a_f(x_0) \leq c_1 c_2 \frac{d(f(x_0), \partial B^*(0,1))}{d(x_0, \partial D^*)}. \tag{2.18}$$

(2.15) and (2.18) complete the proof of Theorem 2 with $c = c_1 c_2 c_3$. □

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