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SPECTRAL STABILITY OF PERIODIC WAVES OF SECOND ORDER SYSTEMS

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Abstract: We study the spectral stability of space-periodic steady solutions of a class of multi-dimensional viscous conservation laws and a class of second-order systems in non-conservation form. In the framework of Floquet's theory and by means of the definition of the Evans function, we give an equivalent form of the Evans function at leading order around the origin in order to prove a necessary condition for spectral stability in both models. In addition, we describe the zero set of the Evans function by means of a formula, which involves macroscopic variables of first-order systems derived from the original models through homogenisation procedure.

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1. Introduction

The present paper is concerned with the study of the spectral stability of space-periodic stationary solutions to multi-dimensional models: we shall study a class of viscous conservation laws and a class of second order systems in non-conservation form. First we investigate the following models:

$$\partial_t u + \nabla_x f(u) = \nabla_x (B(u, \nabla_x u)), \tag{1}$$

where $x \in \mathbf{R}^d$, $t > 0$; the functions $f : \mathbf{R}^d \to \mathbf{R}$ and $B : \mathbf{R}^d \times \mathbf{R}^{d\times d} \to \mathbf{R}$, are given regular functions; the unknown function u is defined in $\mathbf{R}^d \times (0, \infty)$, with values in $\mathbf{R}^d$.

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Next, we shall study the problem of the spectral stability of stationary periodic waves of the systems:

$$\partial_t u = B(u, \partial_{x_j} u, \partial_{x_k} \partial_{x_j} u), \tag{2}$$

where $j, k = 1, \cdots, d$ and $B : \mathbf{R}^n \times \mathbf{R}^{n\times d} \times \mathbf{R}^{n\times d^2} \to \mathbf{R}^n$, is a given regular function. The unknown vector u is defined in $\mathbf{R}^d \times (0, \infty)$, with values in $\mathbf{R}^n$.

The spectral stability of periodic waves of one-dimensional viscous conservation laws has been studied by Serre in [5]. The spectral analysis has been carried out by means of Floquet's theory for linear differential operators with periodic coefficients and by means of the Evans function associated with the problem.

The same techniques have been applied in [2] to study the spectral stability of periodic stationary solutions of multi-dimensional models with a relaxation term.

As far as the systems (1) and (2) are concerned, we shall investigate below the spectral stability of space-periodic travelling wave solutions with velocity $s \in \mathbf{R}$ and direction $e_1 = (1, 0, \ldots, 0) \in \mathbf{R}^d$, i.e. solutions of the form $U(x,t) = U(x \cdot e_1 - st)$. Let us recall some basic notions. Consider a system of n equations in the form: $\partial_t u = L\,u$, where L is a one-dimensional differential operator. Assume that there exists a space-periodic travelling wave solution of period X_0 and denote it by U^0. Moreover, denote by L_0 the operator L linearized around the periodic wave U^0 and assume that the domain of L_0 is the space $L^2(\mathbf{R})$. Let us define the spectral stability of a periodic wave as follows.

Definition 1.1. The periodic wave U^0, is said to be spectrally unstable if the spectrum of L_0 contains an element of positive real part. It is weakly spectrally stable otherwise.

In accordance with Floquet's theory, consider the linearized problem associated with the operator L_0 and write it in equivalent form as a first-order system. Let $M(X_0, \lambda)$ be the associated monodromy matrix, where λ is an arbitrary complex number. From Floquet's theory, we know that λ belongs to the spectrum of L_0, if and only if $\det(M(X_0, \lambda) - \exp(i\theta) I) = 0$, for some real value θ. If $\lambda \in \sigma(L_0)$, the eigenvalue problem for L_0 admits a non-trivial bounded solution Y such that $Y(x + kX_0) = \exp(ik\theta) Y(x)$, $k \in \mathbf{N}$. The previous determinant can be written in equivalent form by choosing a basis in the kernel of $L_0 - \lambda$, as proved by Gardner [3]. Thanks to this property, one defines the Evans function $D(\lambda, \theta)$, of the problem; it holds true that $D(\lambda, \theta) = 0$, if and only if previous determinant is zero.

In the framework of Floquet's theory, the main theorem of [5] gives us a description of the zero set of the Evans function at leading order in a neighbourhood of the origin, by means of the flux of a first order system of conservation laws derived from the original system through rescaling and homogenization. In [5] it is proved a necessary condition for spectral stability of the periodic waves in terms of the hyperbolicity of the system in macroscopic variables, which describes the slow modulation of the periodic solutions.

By adapting the procedure described above, we shall study the spectral stability of a periodic wave of the conservation laws (1) and of system (2). We linearize the equations around the fixed periodic solution and by applying the Fourier transform we obtain a class of linear differential operators with periodic coefficients in the variable x_1. The union of the spectra of the transformed operators equals the spectrum of the original linearized operator. We shall perform the spectral analysis by means of Floquet's theory and the definition of the Evans function for a selected transformed operator.

As far as the first model (1) is concerned, we shall prove a necessary condition for spectral stability by means of the analysis up to the second order of the Evans function around the origin. Next, in accordance with the results of [5] for one-dimensional conservation laws, we establish the link between the zero set of the Evans function at leading order and the flux of a first order system of conservation laws obtained through homogenization procedure.

Section 3 is concerned with the study of the class of second order systems (2) in non-conservation form, by following the program described above. Because of the non-conservation form of the equations (2), the results turn out to be different with respect to the case studied in Section 2 and in previous papers (see references [5], [1] and [2]). Similarly as in the case of conservation laws, we shall prove a necessary condition for spectral stability thanks to the analysis of the Evans function up to the second order.

2. Spectral Stability: Viscous Conservation Laws

In the present section we point our attention on the class of viscous multidimensional conservation laws (1). A travelling wave solution of velocity $s \in \mathbf{R}$ and direction $e_1 = (1, 0, \dots, 0) \in \mathbf{R}^d$, is a function of the form $U(x,t) =$

$U(x \cdot e_1 - st)$ and satisfies the profile equations

$$\begin{cases} -s{U_1}' + \sum_{j=1}^{d} \frac{\partial f}{\partial u_j}(U){U_j}' = \sum_{j=1}^{d} \frac{\partial B}{\partial u_j}(U, U'){U_j}' \\ \qquad\qquad\qquad\qquad\qquad + \sum_{j=1}^{d} \frac{\partial B}{\partial v_{j1}}(U, U'){U_j}'', \\ -s{U_j}' = 0; \quad j = 2, \cdots, d. \end{cases} \tag{3}$$

Therefore, if $s \neq 0$, a space-periodic solution is given by a solution of

$$\begin{cases} -s{U_1}' + \frac{\partial f}{\partial u_1}(U){U_1}' = \frac{\partial B}{\partial u_1}(U, U'){U_1}' + \frac{\partial B}{\partial u_{11}}(U, U'){U_1}'', \\ U_j = \text{const.}; \quad j = 2, \cdots, d. \end{cases} \tag{4}$$

In the case where the periodic solution is stationary, the functions U_j, with $j = 1, \cdots, d$, are periodic solutions to the first equation of system (3).

Let us fix a periodic solution U^0 of system (3), with period X_0. Without loss of generality, we may suppose that U^0 is stationary. By following the procedure applied in previous papers (see [1] and [2]), let us linearize the equations of system (1) around the fixed periodic wave U^0

$$\partial_t u = \nabla_x \Bigg(- \sum_{j=1}^{d} \frac{\partial f}{\partial u_j}(U^0) u_j + \sum_{j=1}^{d} \frac{\partial B}{\partial u_j}(U^0, {U^0}') u_j + \sum_{j,k=1}^{d} \frac{\partial B}{\partial v_{jk}}(U^0, {U^0}') \frac{\partial u_j}{\partial x_k} \Bigg) . \tag{5}$$

In what follows, we assume that $\frac{\partial B}{\partial v_{11}}(U^0, {U^0}') \neq 0$. We apply now the Fourier transform with respect to the variables $(x_2, \cdots, x_d)$ in the equations of the linear system (5). Notice that the coefficients in (5) depend on x_1 only. Denote by $\hat{u}_j$, the Fourier transforms of the functions u_j, for every $j = 1, \cdots, d$ and set $\xi = (\xi_2, \cdots, \xi_d)$. We obtain the following system

$$\left\{\begin{array}{l}\partial_t\hat{u}_1=\dfrac{\partial}{\partial x_1}\left[-\displaystyle\sum_{j=1}^{d}\frac{\partial f}{\partial u_j}(U^0)\hat{u}_j+\sum_{j=1}^{d}\frac{\partial B}{\partial u_j}(U^0,U^{0\prime})\hat{u}_j\right.\\ \left.+\displaystyle\sum_{j=1}^{d}\left(\sum_{l=2}^{d}\frac{\partial B}{\partial v_{jl}}(U^0,U^{0\prime})i\xi_l\hat{u}_j+\frac{\partial B}{\partial v_{j1}}(U^0,U^{0\prime})\frac{\partial\hat{u}_j}{\partial x_1}\right)\right],\\ \partial_t\hat{u}_k=i\xi_k\left[-\sum_{j=1}^{d}\dfrac{\partial f}{\partial u_j}(U^0)\hat{u}_j+\displaystyle\sum_{j=1}^{d}\frac{\partial B}{\partial u_j}(U^0,U^{0\prime})\hat{u}_j\right.\\ \left.+\displaystyle\sum_{j=1}^{d}\left(\sum_{l=2}^{d}\frac{\partial B}{\partial v_{jl}}(U^0,U^{0\prime})i\xi_l\hat{u}_j+\frac{\partial B}{\partial v_{j1}}(U^0,U^{0\prime})\frac{\partial\hat{u}_j}{\partial x_1}\right)\right];\end{array}\right. \tag{6}$$

for every $k=2,\cdots,d$.

We are interested in studying the spectrum of the linearized operator L, associated with the equations (5) and defined on the space $L^2(\mathbf{R}^d)$. Denote by L_ξ, the operators (6) for every fixed $\xi\in\mathbf{R}^{d-1}$. The following identity holds true: $\sigma(L)=\cup_{\xi\in\mathbf{R}^{d-1}}\sigma(L_\xi)$. Thanks to previous result, we shall establish below a necessary condition for spectral stability by studying the spectrum of the operator L_ξ, in the case where $\xi=(0,\cdots,0)\in\mathbf{R}^{d-1}$. The corresponding system is

$$\left\{\begin{array}{l}\partial_t\hat{u}_1=\left(-\sum_{j=1}^{d}\dfrac{\partial f}{\partial u_j}(U^0)\hat{u}_j+\displaystyle\sum_{j=1}^{d}\frac{\partial B}{\partial u_j}(U^0,U^{0\prime})\hat{u}_j\right)'\\ +\left(\displaystyle\sum_{j=1}^{d}\frac{\partial B}{\partial v_{j1}}(U^0,U^{0\prime})\hat{u}_j'\right)',\\ \partial_t\hat{u}_k=0,\quad k=2,\cdots,d;\end{array}\right. \tag{7}$$

where prime denotes the derivative with respect to x_1. In order to carry out our program, we have to consider the following eigenvalue problem for the operator L_0,

$$\left\{\begin{array}{l}\lambda\hat{u}_1=\left(-\sum_{j=1}^{d}\dfrac{\partial f}{\partial u_j}(U^0)\hat{u}_j+\displaystyle\sum_{j=1}^{d}\frac{\partial B}{\partial u_j}(U^0,U^{0\prime})\hat{u}_j\right)'\\ +\left(\displaystyle\sum_{j=1}^{d}\frac{\partial B}{\partial v_{j1}}(U^0,U^{0\prime})\hat{u}_j'\right)',\\ \lambda\hat{u}_k=0,\quad k=2,\cdots,d;\end{array}\right. \tag{8}$$

where $\lambda\in\mathbf{C}$.

Due to the previous equations, if $\lambda \neq 0$, then $\hat{u}_k = 0$, for every $k = 2, \cdots, d$ and $\hat{u}_1$ is the solution of the following second order linear equation with periodic coefficients

$$\lambda \hat{u}_1 = \left(-\frac{\partial f}{\partial u_1}(U^0)\hat{u}_1 + \frac{\partial B}{\partial u_1}(U^0, U^{0\prime})\hat{u}_1 + \frac{\partial B}{\partial v_{11}}(U^0, U^{0\prime})\hat{u}_1' \right)'. \tag{9}$$

Let us recall that we have assumed $\frac{\partial B}{\partial v_{11}}(U^0, U^{0\prime}) \neq 0$, for every $x_1 \in \mathbf{R}$.

By means of the same techniques adopted in [5], [1] and [2], we study the eigenvalue problem (9) in the framework of Floquet's theory for linear differential operator with periodic coefficients and define the corresponding Evans function. Let us denote by $\tilde{L_0}$ the operator associated with the equation (9) and by Y_1, Y_2 two vectors in $\mathbf{R}^2$, holomorphic in terms of λ, which are a basis in

$$\ker(\tilde{L_0} - \lambda): \; Y_1 = (\hat{u}_{11}, \hat{u}_{11}')^t, \, Y_2 = (\hat{u}_{12}, \hat{u}_{12}')^t.$$

Set $\mu = \left(\frac{\partial f}{\partial u_1}(U^0(0)) - \frac{\partial B}{\partial u_1}(U^0(0), U^{0\prime}(0)) \right) \left(\frac{\partial B}{\partial v_{11}}(U^0(0), U^{0\prime}(0)) \right)^{-1}$. In accordance with the results due to Gardner [3] explained in the introduction, we define the Evans function associated with problem (9), as the determinant of the following matrix, where $\theta \in \mathbf{R}$

$$\begin{pmatrix} \hat{u}_{11}(X_0) - \exp(i\theta)\hat{u}_{11}(0) & \hat{u}_{12}(X_0) - \exp(i\theta)\hat{u}_{12}(0) \\ \hat{u}_{11}'(X_0) - \mu\hat{u}_{11}(X_0) - \exp(i\theta)(\hat{u}_{11}'(0) - \mu\hat{u}_{11}(0)) & \hat{u}_{12}'(X_0) - \mu\hat{u}_{12}(X_0) - \exp(i\theta)({\hat{u}_{12}}'(0) - \mu\hat{u}_{12}(0)) \end{pmatrix}. \tag{10}$$

Denote by $D(\lambda, \theta)$, the determinant of the matrix (10). Thanks to the result mentioned above, we have that $\lambda \in \sigma(\tilde{L_0})$ if and only if the Evans function $D(\lambda, \theta)$ associated with problem (9) vanishes for some real value θ. We shall prove below a necessary condition for the spectral stability of the fixed periodic wave U^0 of the viscous conservation laws (1), by studying the zero set of the Evans function $D(\lambda, \theta)$ in a neighbourhood of the origin $\lambda = 0$. Consider the Taylor expansion at $\lambda = 0$, of the linearly independent solutions $\hat{u}_{11}, \hat{u}_{12}$

$$\begin{aligned} \hat{u}_{11} &= \alpha_1(x_1) + \lambda\beta_1(x_1) + \lambda^2\gamma_1(x_1) + \cdots; \\ \hat{u}_{12} &= \alpha_2(x_1) + \lambda\beta_2(x_1) + \lambda^2\gamma_2(x_1) + \cdots; \end{aligned} \tag{11}$$

we may choose with no loss of generality, $\alpha_1(x_1) = U_1^{0\prime}(x_1)$. Set $[Y] = Y(X_0) - Y(0)$.

Let us state now our first theorem.

Theorem 2.1. *Consider the equations of system (1). Let U^0 be a non-trivial stationary space-periodic solution of (1). If $\hat{u}_{11}, \hat{u}_{12}$ are linearly independent solutions of the equation (9), and (11) is the Taylor expansion of $\hat{u}_{11}, \hat{u}_{12}$ in a neighbourhood of the origin $\lambda = 0$, then a necessary condition for the spectral stability of the periodic wave U^0 is given by the following inequality*

$$[\alpha_2][\gamma_1]([\gamma_2'] + [\gamma_2])([\gamma_1'] + [\gamma_1]) \geq 0. \tag{12}$$

Proof. Let us describe the Evans function $D(\lambda, \theta)$ around the origin $\lambda = 0$. To this aim, we have to determine the elements of the matrix (10) at leading order, by taking into account the Taylor formulas of $\hat{u}_{11}, \hat{u}_{12}$. By putting the asymptotic expansions (11) into the equation (9), we obtain the identities

$$\begin{aligned}
&\Big(-\frac{\partial f}{\partial u_1}(U^0)\alpha + \frac{\partial B}{\partial u_1}(U^0, U^{0\prime})\alpha + \frac{\partial B}{\partial v_{11}}(U^0, U^{0\prime})\alpha'\Big)' = 0;\\
&-\frac{\partial f}{\partial u_1}(U^0(0))[\beta_1] + \frac{\partial B}{\partial u_1}(U^0(0), U^{0\prime}(0))[\beta_1]\\
&\qquad\qquad + \frac{\partial B}{\partial v_{11}}(U^0(0), U^{0\prime}(0))[\beta_1{}'] = 0;\\
&-\frac{\partial f}{\partial u_1}(U^0(0))[\alpha_2] + \frac{\partial B}{\partial u_1}(U^0(0), U^{0\prime}(0))[\alpha_2]\\
&\qquad\qquad + \frac{\partial B}{\partial v_{11}}(U^0(0), U^{0\prime}(0))[\alpha_2{}'] = 0.
\end{aligned} \tag{13}$$

We find that the Evans function at leading order is given by the determinant of the following matrix

$$\begin{pmatrix} \lambda[\beta_1] - i\theta U^{0\prime}(0) & [\alpha_2] \\ 0 & \lambda[\beta_2{}'] - \lambda\mu[\beta_2] - i\theta\alpha_2{}'(0) + i\theta\mu\alpha_2(0) \end{pmatrix}. \tag{14}$$

Let us denote by A the matrix (14). Thus $D(\lambda, \theta) = \det(A) + O(|\lambda|^3 + |\theta|^3)$ and $\det(A)$, is a homogeneous polynomial of degree 2 in λ and θ. If we carry on the analysis of the Evans function around $\lambda = 0$ at next order we find that $D(\lambda, \theta) = \det(A + B) + O(|\lambda|^5 + |\theta|^5)$, where the matrix B is defined as follows

$$\begin{pmatrix} \lambda^2[\gamma_1] - i\theta\lambda\beta_1(0) + \frac{\theta^2}{2}U_1^{0\prime}(0) & \lambda[\beta_2] - i\theta\alpha_2(0) \\ \lambda^2[\gamma_1{}'] - \lambda^2\mu[\gamma_1] - i\theta\lambda(\beta_1{}'(0) - \mu\beta_1(0)) & \lambda^2[\gamma_2{}'] - \lambda^2\mu[\gamma_2] - i\theta\lambda\beta_2{}'(0) + \frac{\theta^2}{2}(\alpha_2{}'(0) - \mu\alpha_2(0)) \end{pmatrix}. \tag{15}$$

Set $\theta = 0$, and compute the elements of the matrix $A + B$. We get

$$A(\lambda, 0) + B(\lambda, 0) = \begin{pmatrix} \lambda[\beta_1] + \lambda^2[\gamma_1] & [\alpha_2] + \lambda[\beta_2] \\ \lambda^2[\gamma_1'] - \lambda^2\mu[\gamma_1] & \lambda[\beta_2'] - \lambda\mu[\beta_2] + \lambda^2[\gamma_2'] - \lambda^2\mu[\gamma_2] \end{pmatrix}. \quad (16)$$

Thus $D(\lambda, 0) = \det(A(\lambda, 0) + B(\lambda, 0)) + O(|\lambda|^5)$ and $\det(A(\lambda, 0) + B(\lambda, 0))$, is a fourth order polynomial with real coefficients. If $[\alpha_2][\gamma_1]([\gamma_2'] + [\gamma_2])([\gamma_1'] + [\gamma_1])$, is negative, then the equation $\det(A(\lambda, 0) + B(\lambda, 0)) = 0$, has a real positive solution. Since the Evans function $D(\lambda, \theta)$ is holomorphic, the result of Rouché Theorem about the persistence of zeros of holomorphic functions yields the necessary condition (12) for the weak spectral stability of the periodic wave U^0. □

Remark 2.1. The necessary condition for spectral stability proved in Theorem 2.1 may be regarded as a characterization of the stability index (see reference [5]), for multi-dimensional models. Let us consider the one-dimensional viscous conservation laws studied in [5]. If we carry on the analysis of the leading order term of the Evans function up to the second order, we obtain that

$$D(\lambda, \theta) = \det(A + B) + O(|\lambda|^{3n+2} + |\theta|^{3n+2}),$$

and

$$D(\lambda, 0) = \lambda^{n+1}(a_{2n}\lambda^{2n} + \cdots + a_0),$$

where $a_{2n} = \frac{\det(B)}{\lambda^{3n+1}}$, $a_0 = \frac{\det(A)}{\lambda^{n+1}}$.

A necessary condition for spectral stability turns out to be $\mathrm{sgn}(a_0 a_{2n}) \geq 0$.

In accordance with the result proved by Serre in [5] for one-dimensional conservation laws, we derive now a formula which describes the zero set of the Evans function at leading order by means of the flux of a first order system of conservation laws obtained through homogenization procedure from (1). Similarly as in [2], [1] and [5], we rescale the equations of system (1) and carry out the asymptotic analysis in order to find a new system of conservation laws in macroscopic variables. Let ϵ be a small positive parameter. After setting $(x, t) \to (\epsilon x, \epsilon t)$, in the equations of (1), we get

$$\partial_t u + \nabla_x f(u) = \nabla_x(B(u, \epsilon\nabla_x u)). \quad (17)$$

Consider solutions of (17) of the following form

$$u^\epsilon(x, t) = U^0\left(x, t, \frac{\varphi(x_1, t)}{\epsilon}\right) + \epsilon U^1\left(x, t, \frac{\varphi(x_1, t)}{\epsilon}\right) + \cdots. \quad (18)$$

Let us fix a periodic solution U^0, of (1) with period X_0, of the following form $U^0 = (U_1^0, c_2, \cdots, c_d)$, where c_j are real constants. We assume that the first term of the expansion (18) behaves like the fixed periodic solution U^0. Moreover, we require that the functions U^i, $i \geq 1$, are periodic with respect to the fast variable $y = \frac{\varphi(x_1,t)}{\epsilon}$, with period X_0. The phase function φ satisfies $\varphi_{x_1} \neq 0$. Let us put the asymptotic expansion (18) into the rescaled system (17) and derive the corresponding equations at order ϵ^0. We find the following system

$$\begin{aligned}
&\partial_t U_1^0 + \partial_y U_1^1 \varphi_t + \textstyle\sum_{j=1}^d \frac{\partial f}{\partial u_j}(U^0)\frac{\partial U_j^0}{\partial x_1} + \displaystyle\sum_{j=1}^d \frac{\partial f}{\partial u_j}(U^0)\frac{\partial U_j^1}{\partial y}\varphi_{x_1} \\
&= \sum_{j=1}^d \frac{\partial B}{\partial u_j}(U^0, U^{0\prime})\frac{\partial U_j^0}{\partial x_1} + \sum_{j=1}^d \frac{\partial B}{\partial u_j}(U^0, U^{0\prime})\frac{\partial U_j^1}{\partial y}\varphi_{x_1} \\
&+ \frac{\partial B}{\partial v_{11}}(U^0, U^{0\prime})\frac{\partial^2 U_1^0}{\partial x_1 \partial y}\varphi_{x_1} + \sum_{j=1}^d \frac{\partial B}{\partial v_{1j}}(U^0, U^{0\prime})\frac{\partial^2 U_j^1}{\partial y \partial y}(\varphi_{x_1})^2, \\
&\partial_t U_k^0 + \partial_y U_k^1 \varphi_t + \sum_{j=1}^d \frac{\partial f}{\partial u_j}(U^0)\frac{\partial U_j^0}{\partial x_k} = \sum_{j=1}^d \frac{\partial B}{\partial u_j}(U^0, U^{0\prime})\frac{\partial U_j^0}{\partial x_k};
\end{aligned} \tag{19}$$

where $k \geq 2$.

Notice that $U_j^0 = U_j^0(x,t)$, for every $j \geq 2$. Set $s = -\frac{\varphi_t}{\varphi_{x_1}}$, and take the average over the period $(0, X_0)$ of the functions in previous equations (19). If we denote the mean over the period by $< g >= \frac{1}{X_0}\int_0^{X_0} g(y)dy$, we obtain

$$\begin{aligned}
&\partial_t < U_1^0 > + \partial_{x_1} < f(U^0) >= \partial_{x_1} < B(U^0, U^{0\prime}) > - \frac{1}{X_0}\int_0^{X_0} \cdots dy; \\
&\partial_t < U_k^0 > + \partial_{x_k} < f(U^0) >= \partial_{x_k} < B(U^0, U^{0\prime}) >, \quad k \geq 2.
\end{aligned} \tag{20}$$

Set $s = S(U^0)$. Assume that $\partial_t(\varphi_{x_1}) = \partial_{x_1}(\varphi_t)$, by taking the mean over the period, we obtain the additional conservation law $\partial_t \Omega(U^0) = \partial_{x_1}(S(U^0)\Omega(U^0))$.

Consider now the first equation of system (19), denote by X the period of a generic periodic wave and define the following vector field

$$\begin{aligned}
&M(u_1, u_{1x_1}) = \frac{1}{X}\int_0^X u_1(x_1)dx_1; \\
&Q(u_1, u_{1x_1}) = \left(\frac{\partial f}{\partial u_1}(U^0(0)) - \frac{\partial B}{\partial u_1}(U^0(0), U^{0\prime}(0))\right) u_1 \\
&- \frac{\partial u_1}{\partial x_1}\frac{\partial B}{\partial v_{11}}(U^0(0), U^{0\prime}(0)).
\end{aligned} \tag{21}$$

In accordance with previous results proved for one-dimensional conservation laws in [5] and for van der Waals fluid models in [1], we shall describe the zero set of the Evans function at leading order, in terms of the vector field (M, Q).

Consider the matrix A defined in the proof of the previous Theorem. Replace the first element of the last row with the corresponding element of the matrix B defined in (15)

$$\begin{pmatrix} \lambda[\beta_1] - i\theta {U^0}'(0) & [\alpha_2] \\ \lambda^2[{\gamma_1}'] - \lambda^2\mu[\gamma_1] - i\theta\lambda({\beta_1}'(0) - \mu\beta_1(0)) & \lambda[{\beta_2}'] - \lambda\mu[\beta_2] - i\theta{\alpha_2}'(0) + i\theta\mu\alpha_2(0) \end{pmatrix}. \tag{22}$$

Denote by H the matrix (22), by ω_0 the constant $\omega_0 = \frac{1}{X_0}$.

Let us prove now the main result of the present section.

Theorem 2.2. *Let U^0 be a fixed periodic stationary solution of system (1). Let H be the matrix defined above. The definition of the vector field (M, Q) yields the following result*

$$\begin{aligned} \det(H) = &\left(\frac{\partial B}{\partial v_{11}}(U^0(0), {U^0}'(0))\right)^{-1} {\omega_0}^{-1} \\ &\times \det\left(\lambda\frac{\partial(M,\Omega)}{\partial u_1}(U^0) + i\omega_0\theta\frac{\partial(SM+Q, S\Omega)}{\partial u_1}(U^0)\right). \end{aligned} \tag{23}$$

Proof. The result will be proved by following the procedure of the proof of Theorem 1 in [5].

Without loss of generality, the functions α_2 and β_1, in the Taylor formula (11) may be chosen in such a way that either $[\beta_1]$ or $[\alpha_2]$ are not zero. Thus we may write $\mathbf{R} = \text{Span}\,\{[\beta_1], [\alpha_2]\}$.

Set $\delta = (\delta_1, \delta_2) \in \mathbf{R}^2$ and $\gamma \in \mathbf{R}$. Let w be the vector $w = \delta_1\beta_1 + \delta_2\alpha_2$, and Z the following functional

$$Z(\delta, \gamma) = \delta_1[\beta_1] + \delta_2[\alpha_2] - \gamma {U_1^0}'(0).$$

Consider the vector field (M, Q) and compute the differentials

$$
\begin{aligned}
&dX(\delta,\gamma) = \gamma, \quad dS(\delta,\gamma) = \delta_1,\\
&d(XM)(\delta,\gamma) = \delta_1 \int_0^{X_0} \beta_1(x_1)dx_1 + \delta_2 \int_0^{X_0} \alpha_2(x_1)dx_1\,,\\
&dQ(\delta,\gamma) = \delta_1 \left(\frac{\partial f}{\partial u_1}(U^0(0)) - \frac{\partial B}{\partial u_1}(U^0(0), {U^0}'(0)) \right) \beta_1\\
&\quad -\delta_1 \frac{\partial B}{\partial v_{11}}(U^0(0), {U^0}'(0)){\beta_1}'\\
&\quad +\delta_2 \left(\frac{\partial f}{\partial u_1}(U^0(0)) - \frac{\partial B}{\partial u_1}(U^0(0), {U^0}'(0)) \right) \alpha_2\\
&\quad -\delta_2 \frac{\partial B}{\partial v_{11}}(U^0(0), {U^0}'(0)){\alpha_2}'.
\end{aligned}
$$

According to standard rules, we have $\omega_0 d(XM) = dM + \omega_0 M_0 dX$. Consider the restriction to $\ker(Z)$ of $\lambda dM + i\omega_0\theta(M_0 dS + dQ)$, and compute the determinant of the matrix associated with the linear map

$$
(\delta,\gamma) \longrightarrow \begin{pmatrix} \lambda\gamma - i\theta\delta_1 \\ \delta_1[\beta_1] + \delta_2[\alpha_2] - \gamma {U_1^0}'(0) \\ \lambda dM + i\omega_0\theta(M_0 dS + dQ) \end{pmatrix}.
$$

Parametrize in the following way: $\gamma = i\theta\rho$, $\delta_1 = \lambda\rho$, $\delta_2 = \delta_2$, we obtain the map

$$
(\rho,\delta_2) \longrightarrow \begin{pmatrix} \lambda\rho[\beta_1] + \delta_2[\alpha_2] - i\theta\rho {U_1^0}'(0) \\ \lambda dM + i\omega_0\theta(M_0 dS + dQ) \end{pmatrix}.
$$

Compute the elements of the matrix defined by the previous linear map: the first column is given by

$$
\begin{pmatrix} \lambda[\beta_1] - i\theta {U_1^0}'(0) \\ \lambda^2\omega_0 \int_0^{X_0} \beta_1(x_1)dx_1 + i\omega_0\theta\lambda \left(\frac{\partial f}{\partial u_1}(U^0(0)) - \frac{\partial B}{\partial u_1}(U^0(0), {U^0}'(0)) \right) \\ \times\beta_1(0) - i\omega_0\theta\lambda \frac{\partial B}{\partial v_{11}}(U^0(0), {U^0}'(0)){\beta_1}'(0) \end{pmatrix}.
$$

As far as the second column is concerned, we get

$$
\begin{pmatrix} [\alpha_2] \\ \lambda\omega_0 \int_0^{X_0} \alpha_2(x_1)dx_1 + i\omega_0\theta \left(\frac{\partial f}{\partial u_1}(U^0(0)) - \frac{\partial B}{\partial u_1}(U^0(0), {U^0}'(0)) \right) \\ \times\alpha_2(0) - i\omega_0\theta \frac{\partial B}{\partial v_{11}}(U^0(0), {U^0}'(0)){\alpha_2}'(0) \end{pmatrix}.
$$

Thanks to the identities (13), we have

$$\lambda \int_0^{X_0} \alpha_2(x_1) dx_1 \left(\frac{\partial B}{\partial v_{11}} (U^0(0), {U^0}'(0)) \right)^{-1} = \lambda[{\beta_2}'] - \lambda\mu[\beta_2];$$

and

$$\int_0^{X_0} \beta_1(x_1) dx_1 \left(\frac{\partial B}{\partial v_{11}} (U^0(0), {U^0}'(0)) \right)^{-1} = [{\gamma_1}'] - \mu[\gamma_1].$$

Thus the result of the theorem is proved. □

3. Spectral Stability: Second Order Systems

In the present section we are interested in the spectral stability of periodic in space solutions of a class of second order systems, which are not conservation laws. We shall address our study to system (2) by applying the same techniques of previous section for systems (1) in conservation form.

We derive now the profile equations satisfied by a space-periodic travelling wave solution with velocity $s \in \mathbf{R}$ and direction $e_1 = (1, 0, \ldots, 0) \in \mathbf{R}^d$: $U(x,t) = U(x \cdot e_1 - st)$. We have

$$-sU' = B(U, U', U''), \tag{24}$$

where prime denotes the derivative with respect to the argument $z = x \cdot e_1 - st$.

Assume that system (24) admits a periodic in space solution U^0 with period X_0. With no loss of generality, suppose that the wave is stationary, i.e. $s = 0$.

As in the case of conservation laws, we linearize the equations of system (2) around the fixed periodic stationary wave U^0

$$\begin{aligned} \partial_t u = \sum_{j=1}^n \frac{\partial B}{\partial u_j}(U^0, {U^0}', {U^0}'')u_j + \sum_{j=1,\cdots,n} \sum_{k=1,\cdots,d} \frac{\partial B}{\partial v_{jk}}(U^0, {U^0}', {U^0}'')\partial_{x_k} u_j \\ + \sum_{j=1,\cdots,n} \sum_{k,l=1,\cdots,d} \frac{\partial B}{\partial w_{jkl}}(U^0, {U^0}', {U^0}'')\partial_{x_k}\partial_{x_l} u_j. \end{aligned} \tag{25}$$

For the sake of simplicity, let us set $b(U^0) = B(U^0, {U^0}', {U^0}'')$.

By applying the Fourier transform to (25), with respect to the variables $(x_2, \cdots, x_d)$, we get from the linearized system

$$\partial_t \hat{u} = \sum_{j=1}^n \frac{\partial b}{\partial u_j}(U^0)\hat{u}_j + \sum_{j=1,\cdots,n} \sum_{k=2,\cdots,d} \frac{\partial b}{\partial v_{jk}}(U^0) i\xi_k \hat{u}_j + \sum_{j=1}^n \frac{\partial b}{\partial v_{j1}}(U^0)\hat{u}_j'$$

$$+\sum_{j=1,\cdots,n}\sum_{k,l=2,\cdots,d}\frac{\partial b}{\partial w_{jkl}}(U^0)(-\xi_k\xi_l)\hat{u}_j+\sum_{j=1,\cdots,n}\sum_{k=2,\cdots,d}\frac{\partial b}{\partial w_{jk1}}(U^0)i\xi_k\hat{u}_j'$$
$$+\sum_{j=1,\cdots,n}\sum_{l=2,\cdots,d}\frac{\partial b}{\partial w_{j1l}}(U^0)i\xi_l\hat{u}_j'+\sum_{j=1,\cdots,n}\frac{\partial b}{\partial w_{j11}}(U^0)\hat{u}_j''. \quad (26)$$

The notations are the same as in Section 2. Consider now the operator (26) in the case where $\xi=(0,\ldots,0)\in\mathbf{R}^{d-1}$. We get

$$\partial_t\hat{u}=\sum_{j=1}^{n}\frac{\partial b}{\partial u_j}(U^0)\hat{u}_j+\sum_{j=1}^{n}\frac{\partial b}{\partial v_{j1}}(U^0)\hat{u}_j'+\sum_{j=1}^{n}\frac{\partial b}{\partial w_{j11}}(U^0)\hat{u}_j''. \quad (27)$$

By following an analogous procedure as in Section 2 for the conservation law (1), we shall study below the spectrum of the operator associated with system (27), in order to establish some condition for the spectral stability of the periodic wave U^0, of the original system (2). Let us denote by L_0 the operator (27) defined in the space $L^2(\mathbf{R}^d)$.

Concerning the related eigenvalue problem, we have the following equations, where $\lambda\in\mathbf{C}$,

$$\lambda\hat{u}=\sum_{j=1}^{n}\frac{\partial b}{\partial u_j}(U^0)\hat{u}_j+\sum_{j=1}^{n}\frac{\partial b}{\partial v_{j1}}(U^0)\hat{u}_j'+\sum_{j=1}^{n}\frac{\partial b}{\partial w_{j11}}(U^0)\hat{u}_j''. \quad (28)$$

We assume that $\sum_{j=1}^{n}\frac{\partial b}{\partial w_{j11}}(U^0)\neq 0$, for every $x_1\in\mathbf{R}$.

In order to apply Floquet's theory to the linear second order differential operator L_0, with periodic coefficients, we have to fix a basis in the kernel of $L_0-\lambda$. Denote by $W^{(k)}$, $k=1,\cdots,2n$, the vectors in $\mathbf{R}^{2n}$, defined in the following way: $W^{(k)}=(\hat{u}_j^k,\hat{u}_j^{k\prime})^t_{j=1,\cdots,n}$, where $\hat{u}^k=\hat{u}^k(x_1,\xi_2,\cdots,\xi_d)$ are linearly independent solutions to the second order linear homogeneous system (28). The vectors $W^{(k)}$, $k=1,\cdots,2n$, are a basis in $\ker(L_0-\lambda)$.

We intend to describe the spectrum of the operator L_0, in order to get information about the spectrum of the operator L defined by (25), since, as explained in section 2, $\sigma(L)=\cup_{\xi\in\mathbf{R}^{d-1}}\sigma(L_\xi)$.

Hence by following the procedure outlined in the Introduction, we define the Evans function for the eigenvalue problem (28). Let $\lambda\in\mathbf{C}$ and $\theta\in\mathbf{R}$; the Evans function $D(\lambda,\theta)$ is given by the determinant of the following $2n\times 2n$ matrix

$$\begin{pmatrix} W_j^{(k)}(X_0)-\exp(i\theta)W_j^{(k)}(0)\\ W_j^{(k)\prime}(X_0)-\exp(i\theta)W_j^{(k)\prime}(0)\end{pmatrix}_{k=1,\cdots,2n;j=1,\cdots,n}. \quad (29)$$

Let us now investigate the behaviour of $D(\cdot,\theta)$ in a neighbourhood of the origin, by writing the Taylor expansion of $W_j^{(k)}$, at $\lambda = 0$. Denote by $\alpha_j^{(k)}$, $\beta_j^{(k)}$, $\gamma_j^{(k)}$, the first three terms in Taylor formula of $W_j^{(k)}$

$$\hat{u}_j^k(x_1) = \alpha_j^{(k)}(x_1) + \lambda\beta_j^{(k)}(x_1) + \lambda^2\gamma_j^{(k)}(x_1) + \cdots ; \tag{30}$$

where $j = 1, \cdots, n$; $k = 1, \cdots, 2n$.

In the case where $\lambda = 0$, the functions $\alpha^{(k)}$ have to be a basis in the kernel of L_0. We may choose $\alpha^{(1)} = U^{0\prime}$.

In order to carry out the analysis of the Evans function at leading order around $\lambda = 0$, consider the elements of the matrix (29) and the Taylor formulas (30). We get that the leading order terms at first order are given by

$$\begin{pmatrix} \lambda[\beta_j^{(1)}] - i\theta {U_j^{(0)}}'(0) & [\alpha_j^{(k)}] \\ \lambda[{\beta_j^{(1)}}'] - i\theta {U_j^{(0)}}''(0) & [{\alpha_j^{(k)}}'] \end{pmatrix}_{k=2,\cdots,2n; j=1,\cdots,n} . \tag{31}$$

Denote by A the previous $2n \times 2n$ matrix. The determinant of A represents the leading order term of the Evans function in a neighbourhood of $\lambda = 0$. Therefore $D(\lambda,\theta) = \det(A) + O(|\lambda|^2 + |\theta|^2)$ and $\det(A)$ is a homogeneous polynomial of degree 1 in λ and θ.

Similarly as in the case studied in Section 2, we shall derive a new system describing the slow modulation of the periodic waves.

Denote by ϵ a small parameter; by means of the rescaling $(x,t) \to (\epsilon x, \epsilon t)$, in the equations of system (2), we obtain

$$\epsilon\partial_t u = B(u, \epsilon\partial_{x_j} u, \epsilon^2\partial_{x_k}\partial_{x_j} u). \tag{32}$$

Let us derive a homogenized system whose unknowns describe the mean behaviour of the solution to (2). We introduce the following asymptotic expansion of the solution in terms of powers of ϵ

$$u^\epsilon(x,t) = U^0\left(x, t, \frac{x_1 - st}{\epsilon}\right) + \epsilon U^1\left(x, t, \frac{x_1 - st}{\epsilon}\right) + \cdots . \tag{33}$$

The function U^0, above coincides with the fixed periodic wave. In addition, we assume that the functions U^i, $i \geq 1$,, are periodic of period X_0, with respect to the fast variable $y = \frac{x_1 - st}{\epsilon}$.

Let us put the asymptotic expansion (34) into the rescaled system (32) and derive the corresponding equations at order ϵ^{-1} and ϵ^0. At order ϵ^{-1}, we recover the profile equations satisfied by the periodic solution U^0. At next order, we get the following system

$$\epsilon\partial_t U^0 - s\epsilon\partial_y U^1 = B(\epsilon U^1, \epsilon\partial_{x_k} U^0 + \epsilon\partial_y U^1, \epsilon\partial_{x_k}\partial_y U^0 + \epsilon\partial_y^2 U^1). \tag{34}$$

If we linearize the previous equations around the periodic wave U^0 in order to drop the small parameter ϵ, we derive a system which is not in conservation form. After taking the average over the interval $(0, X_0)$ we obtain a system of the following form

$$\partial_t < U^0 >= \frac{1}{X_0}\int_0^{X_0} \sum_{j=1,\cdots,n}\sum_{k=1,\cdots,d} \frac{\partial B}{\partial v_{jk}} \partial_{x_k} U_j^0 dy + \frac{1}{X_0}\int_0^{X_0} \cdots dy. \tag{35}$$

In general, as a difference with the cases treated in [5], [1] and [2]), if system (2) is not in conservation form, the linearization of the rescaled equations does not yield a system of conservation laws. We shall investigate below a consequence of this fact. By adapting the techniques of section 2, we try to explain the link between previous system (35) and the determinant of the matrix A computed above, which gives us an equivalent of the Evans function at leading order.

Define the following vector field

$$M(\hat{u}) = \frac{1}{X}\int_0^X \hat{u}(x_1)dx_1. \tag{36}$$

We assume that the functions $\beta^{(1)}$, $\alpha^{(k)}$, $k = 2, \cdots, 2n$, in Taylor formula (30) may be chosen in such a way that

$$\mathbf{R}^n = \text{Span}\left\{[\beta^{(1)\prime}], [\alpha^{(2)\prime}], \cdots, [\alpha^{(2n)\prime}], -U^{0\prime\prime}(0)\right\}. \tag{37}$$

By taking into account the notations of previous section, we state now the main result of this section.

Theorem 3.1. *Let U^0 be a fixed periodic stationary solution to (2). Let A be the matrix defined in (31), whose determinant represents the Evans function associated with the eigenvalue problem (28), in a neighbourhood of $\lambda = 0$. If M is the vector field defined in (36) and condition (37) is fulfilled, then the following result holds true*

$$\det(A) = (-1)^{n^2}{\omega_0}^{-1}\lambda^{-1} \times \det\left(\lambda\frac{\partial(M,\Omega)}{\partial\dot{u}}(U^0) + i\omega_0\theta\frac{\partial(SM,S\Omega)}{\partial\dot{u}}(U^0)\right). \tag{38}$$

Proof. Denote by w the vector of $\mathbf{R}^n$, $w = \delta_1\beta^{(1)\prime} + \sum_{k=2}^{2n}\delta_k\alpha^{(k)\prime}$. Let $\delta = (\delta_1, \cdots, \delta_{2n}) \in \mathbf{R}^{2n}$ and $\gamma \in \mathbf{R}$.

Let Z be the following linear functional $Z : \mathbf{R}^{2n+1} \to \mathbf{R}^n$, $Z(\delta,\gamma) = \delta_1[\beta^{(1)\prime}] + \sum_{k=2}^{2n}\delta_k[\alpha^{(k)\prime}] - \gamma U^{0\prime\prime}(0)$.

We have the following differentials

$$dX(\delta,\gamma) = \gamma, \quad dS(\delta,\gamma) = \delta_1, \quad d\Omega(\delta,\gamma) = -\omega_0^2\gamma,$$

$$d(XM)(\delta,\gamma) = -\gamma U^{0\prime}(0) + \delta_1[\beta_1] + \sum_{k=2}^{2n}[\alpha^{(k)}].$$

We consider the restriction to $\ker Z$ of the linear map

$$\begin{pmatrix} \lambda d\Omega + i\omega_0^2\theta dS \\ \lambda dM + i\omega_0\theta M_0 dS \end{pmatrix};$$

and the determinant of the matrix associated with the linear map

$$(\delta,\gamma) \longrightarrow \begin{pmatrix} \lambda\gamma - i\theta\delta_1 \\ \delta_1[\beta^{(1)\prime}] + \sum_{k=2}^{2n}\delta_k[\alpha^{(k)\prime}] - \gamma U^{0\prime\prime}(0) \\ \lambda dM + i\omega_0\theta M_0 dS \end{pmatrix}.$$

Parametrize as follows: $\gamma = i\theta\rho$, $\delta_1 = \lambda\rho$, $\delta_k = \delta_k$, $k = 2, \cdots, 2n$. We obtain the map

$$(\rho, \delta_2, \cdots, \delta_{2n}) \longrightarrow \begin{pmatrix} \lambda\rho[\beta^{(1)\prime}] + \sum_{k=2}^{2n}\delta_k[\alpha^{(k)\prime}] - i\theta\rho U^{0\prime\prime}(0) \\ \lambda^2\omega_0\rho[\beta^{(1)}] + \lambda\omega_0\sum_{k=2}^{2n}\delta_k[\alpha^{(k)}] - \lambda\omega_0 i\theta\rho U^{0\prime}(0) \end{pmatrix}.$$

The matrix defined by previous linear map is given by

$$\begin{pmatrix} \lambda[\beta^{(1)\prime}] - i\theta U^{0\prime\prime}(0) & [\alpha^{(k)\prime}] \\ \lambda^2\omega_0[\beta^{(1)}] - \lambda\omega_0 i\theta U^{0\prime}(0) & \lambda\omega_0[\alpha^{(k)}] \end{pmatrix}. \tag{39}$$

By computing the determinant of (39), we obtain the formula (3). □

In accordance with the results proved by Serre in [5], we obtain the following proposition.

Proposition 3.1. *Consider system (2) under the assumptions of Theorem 3.1. Suppose moreover that* $\det\left(\frac{\partial(M,\Omega)}{\partial\dot{u}}(U^0)\right) \neq 0$, *and denote by* $K(u)$ *the following matrix* $K(u) = \left(\frac{\partial(M,\Omega)}{\partial\dot{u}}(U^0)\right)^{-1}\left(\frac{\partial(SM,S\Omega)}{\partial\dot{u}}(U^0)\right)$. *If the periodic wave* U^0 *of system (2) is weakly spectrally stable, then the spectrum of the matrix* $K(u)$ *is real.*

The result can be achieved in the same way as in Theorem 2 of [5].

Finally, let us remark that, as a difference with the case of conservation laws, in formula (3) does not appear a flux vector, since system (35) obtained by means of homogenization, is not in conservation form.

Remark 3.1. If we carry on the analysis of the Evans function in a neighbourhood of the origin up to the second order, we obtain $D(\lambda,\theta) = \det(A+B) + O(|\lambda|^{2n+2} + |\theta|^{2n+2})$, where B is the matrix defined as follows

$$\begin{pmatrix} \lambda^2[\gamma_j^{(1)}] - i\theta\lambda\beta_j^{(1)}(0) + \frac{\theta^2}{2}U_j^{0\prime}(0) & \lambda[\beta_j^{(k)}] - i\theta\alpha_j^{(k)}(0) \\ \lambda^2[\gamma_j^{(1)\prime}] - i\theta\lambda\beta_j^{(1)\prime}(0) + \frac{\theta^2}{2}U_j^{0\prime\prime}(0) & \lambda[\beta_j^{(k)\prime}] - i\theta\alpha_j^{(k)\prime}(0) \end{pmatrix},$$

and $\det(A+B)$ is a polynomial of degree $2n+1$ in λ and θ. Moreover, we have that $\det(A+B)(\lambda,0) = \lambda P(\lambda)$, where $P(\lambda)$ is a polynomial of degree $2n$ with real coefficients. Thus the necessary condition for spectral stability proved in Remark 2.1 holds true in this case too.

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