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VARIATIONAL-LIKE INEQUALITIES WITH GENERALIZED MONOTONE MAPPINGS IN BANACH SPACES

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Abstract: In this paper, we introduce two classes of variational-like inequalities with generalized monotone mappings in Banach spaces. Using the KKM technique, we obtain the existence of solutions for variational-like inequalities with M-η-monotone hemicontinuous mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results of [5], [7]-[10].

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1. Introduction

Variational inequality theory plays an important role in many fields such as mechanics, physics, optimization, control, nonlinear programming, economics and transportation equilibrium, engineering sciences, etc. Because of their wide applicability, variational inequality problems have been generalized in various directions for the past several years. For details, we refer to [1], [6], [9], [11] and references therein.

It is known that monotonicity plays an important role in the study of variational inequalities. In recent years, a number of authors have obtained many important generalizations of monotonicity such as quasimonotonicity, pseudomonotonicity, etc; see for example [1], [2].

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In this paper, we introduce two classes of variational-like inequalities with generalized monotone mappings in Banach spaces. Using the KKM technique, we obtain the existence of solutions for variational-like inequalities with M-η-monotone hemicontinuous mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results of [5], [7]-[10].

2. Preliminaries

Let E be a real Banach space with dual space E^*, K a nonempty convex subset of E and a line segment $\text{seg}[x,y] = \{\alpha x + (1-\alpha)y : \alpha \in [0,1]\}$ for $x, y \in K$. Given three mappings $T : K \to E^*$, $\eta : K \times K \to E$ and $f : K \to R \cup \{+\infty\}$, we consider the following problem:

(GMVLIP) find $x \in K$ such that for each $y(\neq x) \in K$, there exists a $u \in \text{seg}(y,x]$ such that

$$< T(u), \eta(y,u) > + f(y) - f(u) \geq 0. \tag{2.1}$$

The inequalities (2.1) is known as a generalized mixed variational-like inequality problem.

In particular, if $u = x$ for all $y \in K$, then we obtain the following mixed variational-like inequality problem (MVLIP): find $x \in K$ such that

$$< T(x), \eta(y,x) > + f(y) - f(x) \geq 0 \tag{2.2}$$

for all $y \in K$.

If $f \equiv 0$, then (MVLIP) (2.2) reduces to the following variational-like inequality problem (VLIP): find $x \in K$ such that

$$< T(x), \eta(y,x) > \geq 0 \tag{2.3}$$

for all $y \in K$. If $E = H$ is a Hilbert space, then (VLIP) (2.3) has been studied in [7,8,10].

If $\eta(y,x) = y - x$, then (VLIP) (2.3) reduces to the variational inequality problem (VIP) considered by Hadjisavvas and Schaible (see [5]): find $x \in K$ such that

$$< T(x), y - x > \geq 0 \tag{2.4}$$

for all $y \in K$.

It is clear from these special cases that our problem (2.1) is a more general unifying one, which is one of the main motivations for this paper.

Let us recall the following definitions.

Definition 2.1. Let K be a nonempty convex subset of a real Banach space E, $\eta : K \times K \to E$ and $T : K \to E^*$ be two mappings. Then T is said to be M-η-monotone on K if for every pair of distinct points $x, y \in K$ and $\varepsilon \in [0,1]$, there exists a $u \in \mathrm{seg}[\varepsilon x + (1-\varepsilon)y, y]$ such that

$$< T(x) - T(u), \eta(x,u) > \geq 0.$$

Definition 2.2. A mapping $T : K \to E^*$ is said to be hemicontinuous if T is continuous from the line segment of K to the weak* topology of E^*.

Definition 2.3. Let K be a nonempty subset of a Banach space E and $\eta : K \times K \to E$ a mapping. A point $x_0 \in K$ is an inner point of K if and only if for any $\xi \in E^* - \{0\}$ and $y \in K$, the following implication is valid:

if $< \xi, \eta(x,y) > \leq < \xi, \eta(x_0,y) >$ for all $x \in K$, then $< \xi, \eta(x,y) > = < \xi, \eta(x_0,y) >$ for all $x \in K$.

The set of all inner points of K is denoted by $\mathrm{inn}K$.

Lemma 2.1. *Let K be a convex subset of a Banach space E and $\eta : K \times K \to E$ be a mapping satisfying the following condition:*

$$\eta(\sum_{i=1}^{n} \alpha_i x_i, y) = \sum_{i=1}^{n} \alpha_i \eta(x_i, y)$$

for $x_i, y \in K$, where $\alpha_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \alpha_i = 1$. If $y_n = (1 - \frac{1}{n})y + \frac{1}{n}z$ $(n = 1, 2, \cdots)$, where $y \in K$ and $z \in \mathrm{inn}K$, then $y_n \in \mathrm{inn}K$.

Proof. Let $z \in K$ and suppose that there is some $\xi \in E^* - \{0\}$ such that

$$< \xi, \eta(x,w) > \leq < \xi, \eta(y_n, w) >, \quad \forall x \in K.$$

Then

$$< \xi, \eta(x,w) > \leq (1 - \frac{1}{n}) < \xi, \eta(y,w) > + \frac{1}{n} < \xi, \eta(z,w) >, \ \forall x \in K. \tag{2.5}$$

Substituting z for x in (2.5), we have

$$< \xi, \eta(z,w) > \leq < \xi, \eta(y,w) > . \tag{2.6}$$

Also, setting $x = y$ in (2.5), we obtain

$$< \xi, \eta(y,w) > \leq < \xi, \eta(z,w) > . \tag{2.7}$$

From (2.6) and (2.7) we get

$$< \xi, \eta(z,w) >=< \xi, \eta(y,w) > . \tag{2.8}$$

By (2.5) and (2.8), we have

$$< \xi, \eta(x,w) >\leq< \xi, \eta(z,w) >, \quad \forall x \in K.$$

By the assumption that $z \in \text{inn}K$,

$$< \xi, \eta(x,w) >=< \xi, \eta(z,w) >, \quad \forall x \in K.$$

Hence

$$\begin{aligned} < \xi, \eta(x,w) > &=< \xi, \eta(z,w) >=< \xi, (1-\frac{1}{n})\eta(z,w) + \frac{1}{n}\eta(z,w) > \\ &=< \xi, (1-\frac{1}{n})\eta(y,w) + \frac{1}{n}\eta(z,w) > \\ &=< \xi, \eta((1-\frac{1}{n})y + \frac{1}{n}z, w) >=< \xi, \eta(y_n,w) >, \quad x \in K. \end{aligned}$$

Thus, $y_n \in \text{inn}K$. □

Lemma 2.2. *Let $T : K \to E^*$ be a M-η-monotone mapping and $f : K \to R \cup \{+\infty\}$ be a proper convex function. Assume that $\eta : K \times K \to E$ is a mapping satisfying the following conditions:*

(i) $\eta(\sum_{i=1}^n \alpha_i x_i, y) = \sum_{i=1}^n \alpha_i \eta(x_i, y)$ for $x_i, y \in K$, where $\alpha_i \geq 0$ ($i = 1, 2, \cdots, n$) and $\sum_{i=1}^n \alpha_i = 1$.

(ii) $\eta(x,y) + \eta(y,x) = 0$ for all x, y in K.

Then, the following problems (2) and (3) are equivalent:

$$x \in K, \quad < T(x), \eta(y,x) > +f(y) - f(x) \geq 0, \quad \forall y \in K; \tag{2.9}$$

$$x \in K, \quad < T(y), \eta(y,x) > +f(y) - f(x) \geq 0, \quad \forall y \in K. \tag{2.10}$$

Proof. Let $x \in K$ be a solution of problem (2.9). Since T is M-η-monotone, we have

$$\begin{aligned} &< T(y), \eta(y,x) > +f(y) - f(x) \\ &=< T(y) - T(x), \eta(y,x) > + < T(x), \eta(y,x) > +f(y) - f(x) \geq 0. \end{aligned}$$

for all y in K. Thus, $x \in K$ is a solution of problem (2.10).

Conversely, let $x \in K$ be a solution of problem (2.10) and let $y \in K$ be any point with $f(y) < \infty$. From (2.10), we know that $f(x) < \infty$. Letting $y_n = (1 - \frac{1}{n})x + \frac{1}{n}y$, $n \in N$, then $y_n \in K$. Since $x \in K$ is a soultion of problem (2.10), it follows that

$$< T(y_n), \eta(y_n, x) > + f(y_n) - f(x) \geq 0. \tag{2.11}$$

The convexity of f and condition (i), (ii) imply that

$$\begin{aligned} f(y_n) - f(x) &= f((1 - \frac{1}{n})x + \frac{1}{n}y) - f(x) \\ &\leq (1 - \frac{1}{n})f(x) + \frac{1}{n}f(y) - f(x) = \frac{1}{n}(f(y) - f(x)) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} < T(y_n), \eta(y_n, x) > &= < T(y_n), \eta((1 - \frac{1}{n})x + \frac{1}{n}y, x) > \\ &= (1 - \frac{1}{n}) < T(y_n), \eta(x, x) > + \frac{1}{n} < T(y_n), \eta(y, x) > \\ &= \frac{1}{n} < T(x + \frac{1}{n}(y - x)), \eta(y, x) > . \end{aligned} \tag{2.13}$$

It follows from (2.11)-(2.13) that

$$\begin{aligned} < T(x + \frac{1}{n}(y - x)), \eta(y, x) > &= n < T(y_n), \eta(y_n, x) > \\ &\geq n(f(x) - f(y_n)) \geq f(x) - f(y). \end{aligned}$$

Hence, we get

$$< T(x + \frac{1}{n}(y - x)), \eta(y, x) > + f(y) - f(x) \geq 0$$

for all y in K. Since $y_n = x + \frac{1}{n}(y - x) \to x$ on seg$[x, y]$ as $n \to 0$, by the hemicontinuity of T, $T(x + \frac{1}{n}(y - x))$ converges weakly* to $T(x)$. Thus,

$$< T(x + \frac{1}{n}(y - x)), \eta(y, x) > \to < T(x), \eta(y, x) > \quad \text{as} \quad n \to \infty.$$

Therefore, we obtain

$$< T(x), \eta(y, x) > + f(y) - f(x) \geq 0$$

for all y in K with $f(y) < \infty$. □

Lemma 2.3. (see [4]) *Let K be a nonempty subset of a Hausdorff topological vector space E and let $F : K \to 2^E$ be a KKM mapping. If $F(x)$ is closed in E for every x in K and compact for some $x \in K$, then $\cap_{x \in K} F(x) \neq \phi$.*

3. Main Results

Now we are ready to prove the main result of this paper.

Theorem 3.1. *Let E be a real Banach space and K be a nonempty convex, weakly compact subset of E with $\mathrm{inn}K \neq \phi$. Let $T : K \to E^*$ be an M-η-monotone and hemicontinuous mapping, $f : K \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\eta : K \times K \to E$ be a mapping satisfying the following conditions:*

(i) $\eta(\sum_{i=1}^{n} \alpha_i x_i, y) = \sum_{i=1}^{n} \alpha_i \eta(x_i, y)$ for all $x_i, y \in K$, where $\alpha_i \geq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \alpha_i = 1$;

(ii) $\eta(x, y) + \eta(y, x) = 0$ for all x, y in K;

(iii) $x \mapsto \eta(x, \cdot)$ and $x \mapsto \eta(\cdot, x)$ are weakly continuous.

Then (GMVLIP) is solvable.

Proof. Define $F : K \to 2^E$ be a set-valued mapping as follows:

$$F(y) = \{x \in K : \text{for} \quad \varepsilon \in [0, 1], \quad \text{there exists a} \quad u \in \mathrm{seg}[\varepsilon y + (1-\varepsilon)x, x] \\ \text{such that} \quad < T(u), \eta(y, u) > + f(y) - f(u) \geq 0\}.$$

Then, for $y \in K$, $y \in F(y)$. Hence $F(y)$ is nonempty. We claim that F is a KKM mapping. If F is not a KKM mapping, then there exist $\{y_1, y_2, \cdots, y_n\} \subset K$ and $t_i \geq 0$, $i = 1, 2, \cdots, n$ such that

$$\sum_{i=1}^{n} t_i = 1, \quad w = \sum_{i=1}^{n} t_i y_i \notin \cup_{i=1}^{n} F(y_i).$$

Since $w \notin F(y_i)$ for $i = 1, 2, \cdots, n$, there exists an $\varepsilon_i > 0$ such that for all

$$u_i \in \mathrm{seg}[\varepsilon_i y_i + (1-\varepsilon_i)w, w] \subset \mathrm{seg}[\varepsilon y_i + (1-\varepsilon)w, w],$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$,

$$< T(u_i), \eta(y_i, u_i) > + f(y_i) - f(u_i) < 0.$$

In particular,

$$< T(w), \eta(y_i, w) > + f(y_i) - f(w) < 0$$

for each $i = 1, 2, \cdots, n$. Since $\sum_{i=1}^n t_i = 1$ for $t_i \geq 0$ $(i = 1, 2, \cdots, n)$, we have

$$\sum_{i=1}^n t_i[< T(w), \eta(y_i, w) > + f(y_i) - f(w)] < 0.$$

It follows from the convexity of f and condition (i) that

$$0 =< T(w), \eta(\sum_{i=1}^n t_i y_i, w) >= \sum_{i=1}^n t_i < T(w), \eta(y_i, w) >$$
$$< \sum_{i=1}^n t_i(f(w) - f(y_i)) \leq f(w) - f(\sum_{i=1}^n t_i y_i) = 0,$$

which is a contradiction. Thus, F is a KKM mapping.

Define $\bar{F} : K \to 2^E$ be a set-valued mapping defined by $\bar{F}(y) = \overline{F(y)}$ for each $y \in K$, where $\overline{F(y)}$ is the closure of $F(y)$ with respect to the weak topology of E. Since $F(y) \subset \overline{F(y)}$ for $y \in K$, $\bar{F}$ is also a KKM mapping. Since each $\bar{F}(y)$ is a weakly closed subset of weakly compact set K, it is weakly compact. It follows from Lemma 2.3, $\cap_{y\in K}\bar{F}(y) \neq \phi$.

Moreover, $\cap_{y\in K}F(y)$ is nonempty. In fact, let $\bar{y} \in \cap_{y\in K}\bar{F}(y)$, $z \in \text{inn}K$ and y be an arbitrary point of K. For each $n \in N$, set $y_n = \frac{1}{n}z + (1 - \frac{1}{n})y$, then $y_n \in \text{inn}K$ by Lemma 2.1. For each fixed $n \in N$, since $\bar{y} \in \bar{F}(y_n)$, there exists a net $\{y^n_\alpha\}_{\alpha\in I_n}$ in $F(y_n)$ such that $y^n_\alpha \to \bar{y}$ weakly in K. The fact that y^n_α belongs to $F(y_n)$ for each $\alpha \in I_n$ guarantees the existence of $u^n_{\alpha,m}$ in $\text{seg}[\frac{1}{m}y_n + (1 - \frac{1}{m})y^n_\alpha, y^n_\alpha]$ $(m \in N)$ satisfying the following inequality:

$$< T(u^n_{\alpha,m}), \eta(y_n, u^n_{\alpha,m}) > + f(y_n) - f(u^n_{\alpha,m}) \geq 0.$$

By Lemma 2.2,

$$< T(y_n), \eta(y_n, u^n_{\alpha,m}) > + f(y_n) - f(u^n_{\alpha,m}) \geq 0.$$

From the hemicontinuity of T and the weak continuity of $x \mapsto \eta(x, \cdot)$, we have

$$< T(y), \eta(y, u^n_{\alpha,m}) > + f(y) - f(u^n_{\alpha,m}) \geq 0$$

for all $\alpha \in I_n$. Since $\{u^n_{\alpha,m}\}_{m\in N}$ converges to y^n_α as $m \to \infty$ and $\{y^n_\alpha\}_{\alpha\in I_n}$ converges to $\bar{y}$, by the weak continuity of $x \mapsto \eta(\cdot, x)$ and the proper convex, lower semicontinuity of f, we have

$$< T(y), \eta(y, \bar{y}) > + f(y) - f(\bar{y}) \geq 0.$$

By Lemma 2.2, we obtain

$$< T(\bar{y}), \eta(y, \bar{y}) > + f(y) - f(\bar{y}) \geq 0.$$

Since y is arbitrary, $\bar{y} \in \cap_{y\in K} F(y)$, which says that $\bar{y}$ is a solution of (GMVLIP). Consequently, (GMVLIP) is solvable. □

Theorem 3.2. *Let E be a real reflexive Banach space and K be a nonempty closed convex subset of E with inn$K \neq \phi$. Suppose there exists $r > 0$ such that for each $y \in K$ with $\|y\| \geq r$, there exists $x \in K$ satisfying $\|x\| < \rho$ and*

$$< T(y), \eta(y,x) > + f(y) - f(x) \geq 0.$$

Let $T : K \to E^*$ be an M-η-monotone and hemicontinuous mapping, $f : K \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\eta : K \times K \to E$ be a mapping satisfying the following conditions:

(i) $\eta(\sum_{i=1}^n \alpha_i y_i, x) = \sum_{i=1}^n \alpha_i \eta(y_i, x)$ for all $y_i, x \in K$;

(ii) $\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in K$;

(iii) $x \mapsto \eta(x,\cdot)$ and $x \mapsto \eta(\cdot,x)$ are weakly continuous.

Then, (GMVLIP) is solvable.

Proof. Let $K_1 = \{y \in K : \|y\| \leq r\}$. First, we claim that inn$K_1 \neq \phi$. From the assumption, K_1 is nonempty. So, we can find $x_0 \in K$ with $\|x_0\| < r$. Let $z \in$ inn$K$ and choose $n \in N$ sufficiently large so that $z_0 = (1 - \frac{1}{n})x_0 + \frac{1}{n}z \in K_1$. By Lemma 2.1, we know that $z_0 \in$ inn$K$. Suppose that there is some $\xi \in E^* - \{0\}$ and let $x \in K_1$ such that

$$< \xi, \eta(w,x) > \leq < \xi, \eta(z_0,x) >, \quad \forall w \in K_1.$$

For each $y \in K$, we can find $n \in N$ such that $(1 - \frac{1}{n})z_0 + \frac{1}{n}y \in K_1$, so

$$< \xi, \eta((1 - \frac{1}{n})z_0 + \frac{1}{n}y, x) > \leq < \xi, \eta(z_0,x) > .$$

By the condition (i), we deduce that

$$< \xi, \eta(y,x) > \leq < \xi, \eta(z_0,x) >$$

for all $y \in K$. From the fact that $z_0 \in$ inn$K$, we have

$$< \xi, \eta(y,x) > = < \xi, \eta(z_0,x) >$$

for all $y \in K$, which implies that $z_0 \in$ innK_1.

Since every bounded closed and convex subset of a reflexive Banach space is weakly compact, K_1 is weakly compact (see [3]). It follows from Theorem 3.1 that there exists $\bar{y} \in K_1$ such that

$$< T(\bar{y}), \eta(y,\bar{y}) > + f(y) - f(\bar{y}) \geq 0. \tag{3.1}$$

for all $y \in K_1$.

Now if $\|\bar{y}\| = r$, by the hypothesis, there exists $x \in K$ such that $\|x\| < r$ and

$$< T(\bar{y}), \eta(\bar{y}, x) > + f(\bar{y}) - f(x) \geq 0. \tag{3.2}$$

On the other hand, by condition (ii) and (3.1), we obtain

$$< T(\bar{y}), \eta(\bar{y}, x) > + f(\bar{y}) - f(x) \leq 0.$$

Thus,

$$\begin{aligned} &< T(\bar{y}), \eta(x, \bar{y}) > + f(x) - f(\bar{y}) \\ &\qquad = -[< T(\bar{y}), \eta(\bar{y}, x) > + f(\bar{y}) - f(x)] = 0. \end{aligned} \tag{3.3}$$

For each $y \in K$, we may choose $n \in N$ large enough such that $\frac{1}{n}y + (1 - \frac{1}{n})x \in K_1$. The convexity of f, (3.3) and condition (i) imply that

$$\begin{aligned} 0 &\leq < T(\bar{y}), \eta(\frac{1}{n}y + (1 - \frac{1}{n})x, \bar{y}) > + f(\frac{1}{n}y + (1 - \frac{1}{n})x) - f(\bar{y}) \\ &\leq \frac{1}{n}\{< T(\bar{y}), \eta(y, \bar{y}) > + f(y) - f(\bar{y})\} \\ &\quad + (1 - \frac{1}{n})\{< T(\bar{y}), \eta(x, \bar{y}) > + f(x) - f(\bar{y})\} \\ &= \frac{1}{n}\{< T(\bar{y}), \eta(y, \bar{y}) > + f(y) - f(\bar{y})\}. \end{aligned}$$

This implies that

$$< T(\bar{y}), \eta(y, \bar{y}) > + f(y) - f(\bar{y}) \geq 0.$$

If $\|\bar{y}\| < r$, then we can find $n \in N$ such that for each $y \in K$, $\frac{1}{n}y + (1 - \frac{1}{n})\bar{y} \in K_1$. It follows from (3.1) that

$$0 \leq \frac{1}{n}\{< T(\bar{y}), \eta(y, \bar{y}) > + f(y) - f(\bar{y})\}.$$

This implies that

$$< T(\bar{y}), \eta(y, \bar{y}) > + f(y) - f(\bar{y}) \geq 0$$

for all $y \in K$ and so $\bar{y} \in K$ is a solution of (MVLIP). Therefore, $\bar{y} \in K$ is a solution of (GMVLIP).

Remark 3.1. Theorem 3.1 and Theorem 3.2 improve and generalize the corresponding results of [5], [7]-[10] in several aspects.

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