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HYPOTHESIS TESTING FOR THE POPULATION MEAN USING UNBIASED RANKED SET SAMPLING DESIGNS

Yaprak Arzu Özdemır[1 §], Fikri Gökpınar[2]
[1,2]Department of Statistics
Faculty of Science and Arts
Gazi University
Besevler, Ankara, 06500, TURKEY
[1]e-mail: yaprak@gazi.edu.tr
[2]e-mail: fikri@gazi.edu.tr

Abstract: Ranked set sampling is a cost efficient sampling technique when actually measuring sampling units is difficult but ranking them is relatively easy. In 1952 McIntyre [3] observed that the sample mean obtained by ranked set sample is an unbiased estimator of the population mean with a smaller variance than the sample mean of a simple random sample with the same sample size. Özdemır et al [5] considered all unbiased ranked set sampling designs for the population mean in normal distribution with the sample size of $n = 3, 4, 5$. In this study, we compared these ranked set sampling designs with simple random sampling in terms of power of all type of hypothesis tests about normal mean with known variance.

AMS Subject Classification: 26A33
Key Words: ranked set sampling, simple random sampling, power of test, extreme ranked set sampling, median ranked sampling

1. Introduction

Ranked set sampling (RSS) is a common sampling technique, which is often used recently in some areas such as, environment, ecology and agriculture. In these areas, measurements of the units according to variables of interest can be quite difficult in some cases, in terms of cost, time and other factors. In such

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§Correspondence author

conditions, by using RSS, the sample selection process is done by less cost and in less time than the simple random sampling (SRS).

It can be said that the sample which is selected with RSS has a better expansion on the distribution of the interested variable. This situation is sourced by the sample selection which is based on two stages. The random sample of size m^2 that is chosen to make ranked set sample consists of m sets of m size each. Those m sets are random samples which are elements of j-th set; $X_{j,1}, X_{j,2}, \ldots, X_{j,m}$ $(j = 1, 2, \ldots, m)$, and each set has the same $F(x;\theta)$ distribution function. The order statistics for the j-th set are defined as $Y_{(j,1)} \leq Y_{(j,2)} \leq \ldots \leq Y_{(j,m)}$. If we assume that the j-th order statistic of the j-th set is determined by visual or costless methods, we form a ranked set sample by $Y_{(1:1)}, Y_{(2:2)}, \ldots, Y_{(m:m)}$ which $Y_{(j:j)}$ is j.th order statistic from j-th set. While ranking units by visual or costless methods, ranking error may be occurred. But in this study, we only considered perfect ranking case. This kind of RSS is called Balanced RSS (BRSS) (McIntyre [3]).

Generally, for identifying all RSS designs, following definitions are required. $Y_{(j:i)}$ denotes the observation of i-th order statistic in the j-th set $\{X_{j,1}, X_{j,2} \ldots, X_{j,m}\}$. $Y_{(j:i)}$, has the same distribution as $Y_{i:m}$ which is i-th order statistic from the random sample with size m. As i_j would be the rank score, the sample of $Y_{(1:i_1)}, Y_{(2:i_2)}, \ldots, Y_{(m:i_m)}$ is corresponding to $(i_1, i_2, \ldots, i_m)$ RSS design. So there are m^m RSS designs (Al-Saleh [1], Gökpınar et al [2]). Most popular of within these designs are extreme and median RSS designs. Median RSS design is explained as follows: when the sample size n is odd, select for measurement the $(n+1)/2$-th smallest unit from each sample. When the sample size is even, select for measurement from each of the first $n/2$ sets the $(n/2)$-th smallest unit and from the second $n/2$ sets the $(n+2)/2$-th smallest unit. These n units constitute median ranked set sample. Extreme RSS design is explained as follows: when the sample size n is odd, select from $(n-1)/2$ sets the smallest unit, from the other $(n-1)/2$, the largest unit and from one set the median of the sample for actual measurement. When the sample size is even, select from $(n/2)$ sets the smallest unit and from the other $(n/2)$ sets the largest unit for actual measurement. These n units constitute extreme ranked set sample.

RSS is first used by McIntyre [3] in 1952, in order to estimate of mean pasture yields. RSS is also used for estimating parameters of distribution in recent years. But the efficiency of RSS changes according to the distribution type. Because of this reason, instead of the BRSS design, different RSS designs are suggested according to distribution type. Sinha et al [9] defined the best possible RSS design for the estimations of the normal and exponential distribution parameters. Al-Saleh [1] compared all different RSS designs by using normal, uniform and exponential distributions according to mean square errors (MSE) without separating unbiased or biased estimators for the population mean and the variance when the sample size is 3.

BRSS design can be also used in hypothesis testing for the population mean μ and the population variance σ^2. Muttlak and Abu Dayyeh [4] made comparison between BRSS design and SRS by the power of hypothesis testing for the population mean and variance. Also they compared RSS and SRS under the two sided alternative hypothesis.

Pan and Sien [6] made comparison between BRSS design and SRS by power of hypothesis testing for population mean when the population variance unknown. Shen [8] derived alternative tests for a normal mean μ when the variance known.

In this study, we used all unbiased estimators for the population mean when the sample size are $n = 3, 4, 5$ in normal distribution (Özdemır et al [5]). For the all possible alternative hypothesis ($H_1 : \mu \neq \mu_0$, $H_1 : \mu < \mu_0$, $H_1 : \mu > \mu_0$) we compare power of tests under the all unbiased RSS designs. Also these RSS designs compared with SRS due to power of tests.

In Section 2, we consider calculation of critical values for all possible alternative hypothesis and all unbiased RSS estimators for the normal mean. In RSS, since identification of distribution for the test statistics is difficult, we calculate the critical values for these test statistics by simulation study.

In Section 3, the SRS and all unbiased RSS designs are compared according to power of test under all possible hypothesis for the population mean.

2. Critical Values for All Possible Alternative Hypothesis

In this section, we consider the critical values for the test of $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$, $H_1 : \mu < \mu_0$, $H_1 : \mu > \mu_0$, based on SRS and different RSS designs.

Let $X_1, X_2, ..., X_n$ have a normal distribution with the mean μ and the variance σ^2. So the probability density function of X_i is known as,

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}(x_i-\mu)^2}, \quad -\infty < x_i < \infty .$$

Hypothesis testing for μ when the variance σ^2 is known under SRS, well known test statistic is,

$$Z = \frac{\overline{X} - \mu}{\sigma},$$

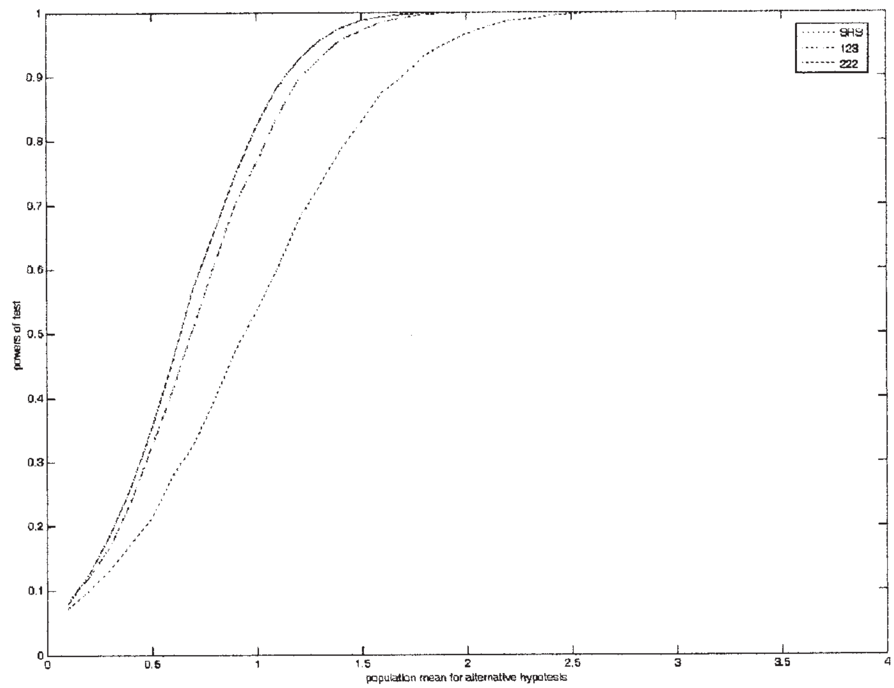

Figure 2.1: The powers for $H_1 : \mu > \mu_0$, where $n = 3$

where $\overline{X}$ is the sample mean under SRS. When $H_0 : \mu = 0$ and $\sigma^2 = 1$, the test statistic can be written as

$$Z = \overline{X}\sqrt{n}\,, \tag{2.1}$$

where Z has a distribution of $Z \sim N(0,1)$.

The power of test for the alternative hypothesis $H_1 : \mu \neq \mu_0$, $H_1 : \mu < \mu_0$, $H_1 : \mu > \mu_0$, respectively;

$$\text{Power } \{\mu = \mu_1/SRS\} = \Pr\left\{Z \geq Z_{1-\frac{\alpha}{2}}, Z \leq -Z_{1-\frac{\alpha}{2}}/\mu = \mu_1\right\}, \quad \text{where } \mu_1 \neq \mu_0\,,$$

$$\text{Power } \{\mu = \mu_1/SRS\} = \Pr\{Z \geq Z_{1-\alpha}/\mu = \mu_1\}\,, \text{ where } \mu_1 > \mu_0\,,$$

$$\text{Power } \{\mu = \mu_1/SRS\} = \Pr\{Z \leq -Z_{1-\alpha}/\mu = \mu_1\}\,, \text{where } \mu_1 < \mu_0$$

(Rohatgi and Saleh [7]). When the sample selected by RSS then the distribution of the test statistic is not known. Therefore, by simulation study the critical values are generated for all possible hypothesis tests. For the population mean,

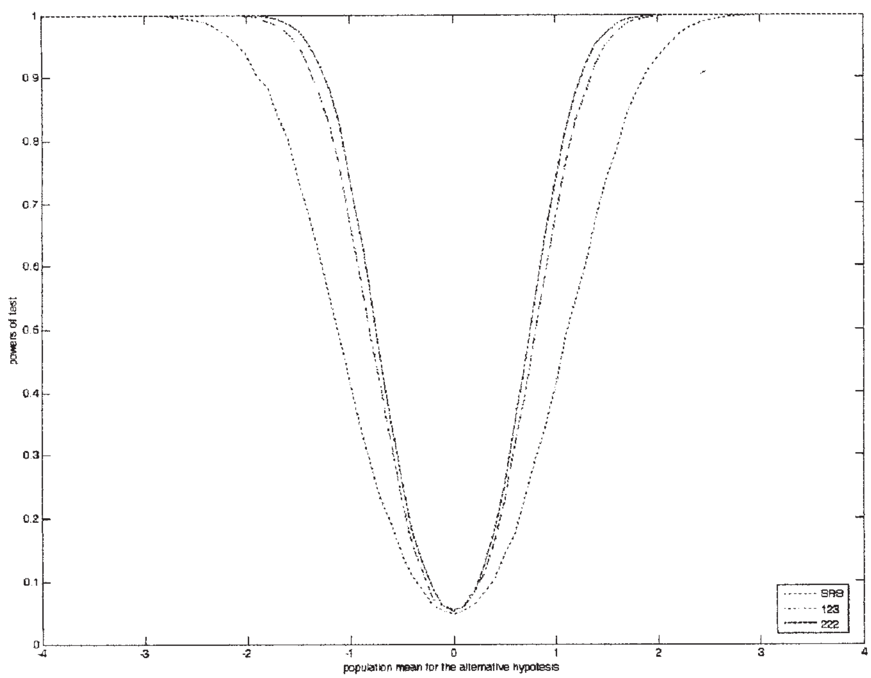

Figure 2.2: The powers for $H_1 : \mu \neq \mu_0$, where $n = 3$

Muttlak and Abu-Dayyeh [4] proposed a test statistic under RSS that is given below

$$T_{RSS} = \overline{Y}_{RSS}\sqrt{n}\,, \tag{2.2}$$

where the alternative hypothesis is $H_1 : \mu \neq \mu_0$, the critical values are calculated by simulation study. The critical values are calculated by the following steps below;

1) nxn=n^2 random sample are generated from standard normal distribution.

2) The ranked set sample with size n is obtained from the RSS procedure which is described in Section 1.

3) The test statistic in equation (2.2) is computed.

4) First, second and third steps are repeated 50000 times and calculated T_{RSS} values are sorted.

5) i) For the alternative hypothesis of $H_1 : \mu \neq \mu_0$, the critical values are respectively $C_1 = T_{k_1}$, $C_2 = T_{k_2}$, where $k_2 = \left(1 - \frac{\alpha_0}{2}\right) n$ and $k_1 = \left(\frac{\alpha_0}{2}\right) n$ for given $\alpha = \alpha_0$ values.

ii) For the alternative hypothesis of $H_1 : \mu > \mu_0$; the critical value is $C_3 = T_{k_3}$, where $k_3 = (1 - \alpha_0)\, n$ for given $\alpha = \alpha_0$ values.

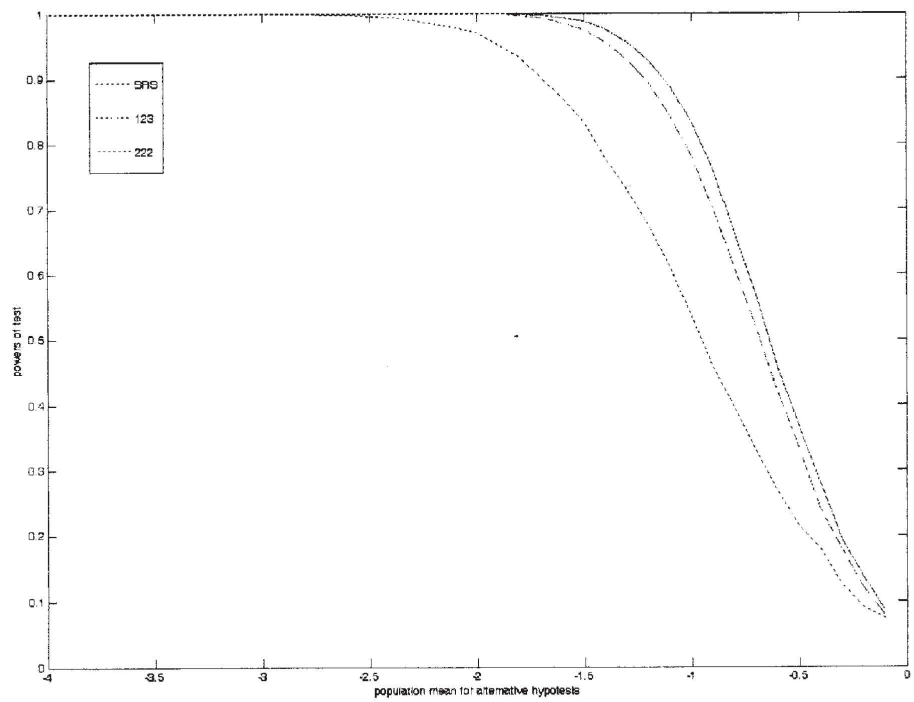

Figure 2.3: The powers for $H_1 : \mu < \mu_0$, where $n = 3$

iii) For the alternative hypothesis of $H_1 : \mu < \mu_0$; the critical value is $C_4 = T_{k_4}$, where $k_4 = \alpha_0 n$ for given $\alpha = \alpha_0$ values.

Using the critical values C_1, C_2, C_3 and C_4, the power of tests for the alternative hypothesis $H_1 : \mu \neq \mu_0$, $H_1 : \mu > \mu_0$, $H_1 : \mu < \mu_0$, can be written as follows respectively,

Power $\{\mu = \mu_1/RSS\} = \Pr\{Z \geq C_1, Z \leq C_2/\mu = \mu_1\}$, where $\mu_1 \neq \mu_0$,

Power $\{\mu = \mu_1/RSS\} = \Pr\{Z \geq C_3/\mu = \mu_1\}$, where $\mu_1 > \mu_0$,

Power $\{\mu = \mu_1/RSS\} = \Pr\{Z \leq C_4/\mu = \mu_1\}$, where $\mu_1 < \mu_0$.

The critical values that calculated for $n = 3, 4, 5$ and $\alpha_0 = 0.05$ are given in Table 2.1. The all unbiased RSS estimators for the population mean with the sample size $3, 4, 5$ are determined by Özdemır et al [5].

Table 2.1 shows that when the sample size n is increases for all types of hypothesis, absolute values of critical values are decreases. This is a natural situation like occurred in SRS. Also in all RSS designs, extreme RSS (for $n = 3$ $(1, 2, 3)$, for $n = 4$ $(1, 1, 4, 4)$, for $n = 5$ $(1, 1, 3, 5, 5)$) gives largest absolute critical values but median RSS (for $n = 3$ $(2, 2, 2)$, for $n = 4$ $(2, 2, 3, 3)$, for $n = 5$

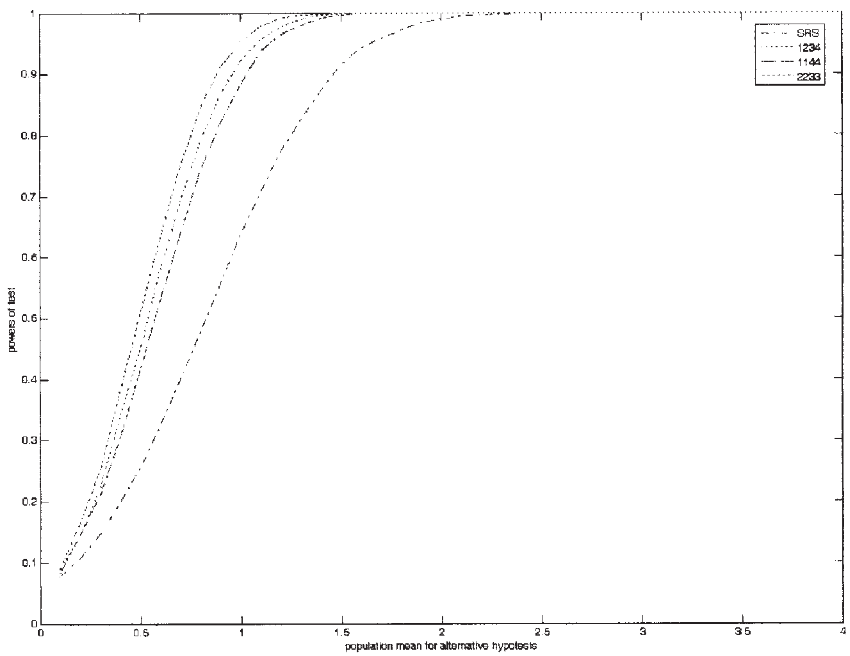

Figure 2.4: The powers for $H_1 : \mu > \mu_0$, where $n = 4$

n	RSS Design	$H_1 : \mu \neq \mu_0$	$H_1 : \mu > \mu_0$	$H_1 : \mu < \mu_0$
3	(1,2,3)	(-1.4101, 1.4045)	1.1922	-1.1864
3	(2,2,2)	(-1.2971, 1.3023)	1.1076	-1.1000
4	(1,2,3,4)	(-1.2815, 1.2805)	1.0779	-1.0679
4	(1,1,4,4)	(-1.3768, 1.3667)	1.1496	-1.1648
4	(2,2,3,3)	(-1.1858, 1.1853)	0.9881	-0.9883
5	(1,2,3,4,5)	(-1.1883, 1.1796)	0.9858	-0.9814
5	(1,3,3,3,5)	(-1.1659, 1.1769)	0.9777	-0.9690
5	(1,1,3,5,5)	(-1.2592, 1.2680)	1.0545	-1.0585
5	(3,3,3,3,3)	(-1.0514, 1.0590)	0.8905	-0.8707
5	(2,2,3,4,4)	(-1.0928, 1.0826)	0.9064	-0.9147
5	(2,3,3,3,4)	(-1.0642, 1.0618)	0.8934	-0.8961

Table 2.1: The critical values for hypothesis tests under different RSS designs for $\alpha_0 = 0.05$.

$(3, 3, 3, 3, 3)$) gives smallest absolute critical values under the same sample size n.

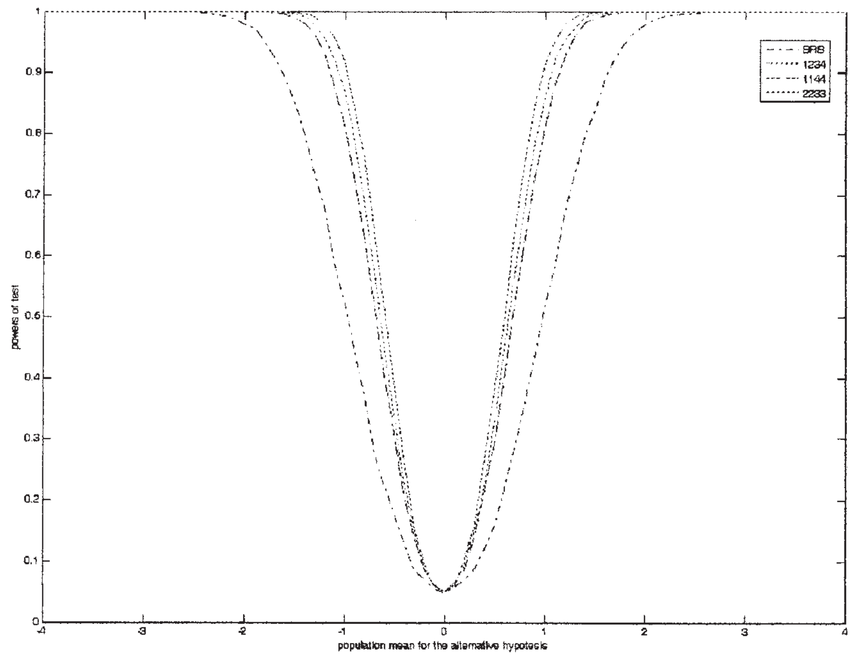

Figure 2.5: The powers for $H_1 : \mu \neq \mu_0$, where $n = 4$

3. Comparison of the Power of Tests Using Different RSS Designs and SRS

In this section, the power of tests under different RSS designs and SRS are computed with the sample size of 3, 4 and 5 for all type of hypothesis. To calculate the power of these tests under different RSS designs the critical values which are given in Table 2.1 are used.

A simulation program which is written in *Matlab 7.0* is used for the obtaining of the power of tests. For testing the null hypothesis H_0, the sample was generated from normal distribution with mean $\mu_1 = -4(0.1)4$ for $H_1 : \mu \neq \mu_0$, $\mu_1 = 0.1(0.1)4$ for $H_1 : \mu > \mu_0$, $\mu_1 = -4(0.1) - 0.1$ for $H_1 : \mu < \mu_0$ and $\sigma^2 = 1$. The steps of this simulation program are as follows:

1) $nxn = n^2$ random sample are generated from normal distribution with mean μ_1 and variance $\sigma^2 = 1$.

2) The ranked set sample with size n are obtained from the RSS procedure which was described in Section 1.

3) The test statistic in equation (2.2) is computed.

4) First, second and third steps are repeated 10000 times.

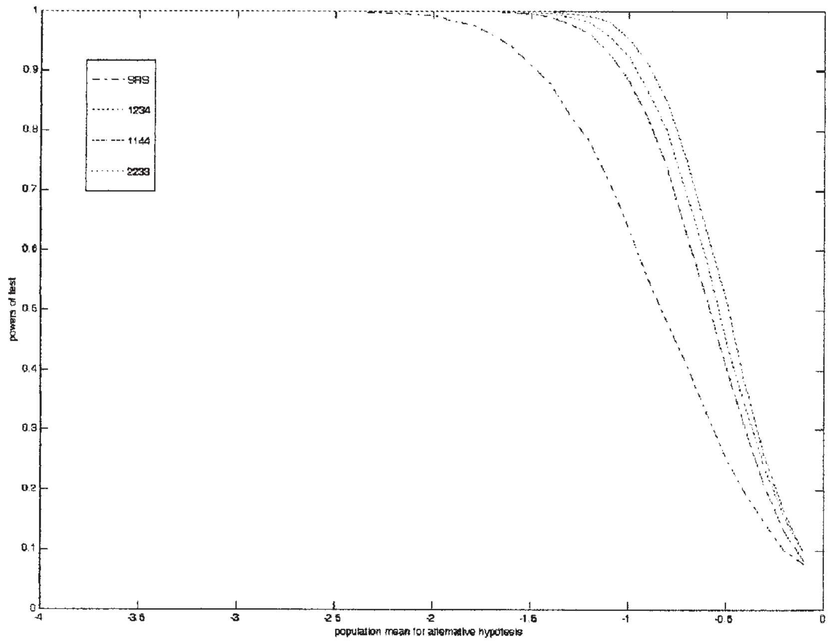

Figure 2.6: The powers for $H_1 : \mu < \mu_0$, where $n = 4$

5) The power of tests, P_1, P_2,and P_3 can be calculated approximately below

$$d_1 = |\{T_{RSS}|T_{RSS} > C_1 or\ T_{RSS} < C_2\}|\ ,\ \text{for}\ H_1 : \mu_1 \neq \mu_0\ , \quad P_1 \cong \frac{d_1}{10000}\ ,$$

$$d_2 = |\{T_{RSS}|T_{RSS} > C_3\}|\ ,\ \text{for}\ H_1 : \mu_1 > \mu_0\ , \quad P_2 \cong \frac{d_2}{10000}\ ,$$

$$d_3 = |\{T_{RSS}|T_{RSS} < C_4\}|\ ,\ \text{for}\ H_1 : \mu_1 < \mu_0\ , \quad P_3 \cong \frac{d_3}{10000}\ ,$$

The power values calculated from this simulation program are expressed by Figure 2.1-2.9. For all sample size and the hypothesis tests, RSS designs are more powerful than SRS as seen at Figures 2.1-2.9.

For the all type of alternative hypothesis and for all sample size, the median RSS designs (for $n = 3$ $(2, 2, 2)$, for $n = 4$ $(2, 2, 3, 3)$, for $n = 5$ $(3, 3, 3, 3, 3)$) have greatest power than the other RSS designs but while the sample size increases from $n = 3$ to 5 this difference between median RSS and the other RSS designs are decreases. For the all type of alternative hypothesis and for the all sample size, although the extreme RSS designs (for $n = 3$ $(1, 2, 3)$, for $n = 4$ $(1, 1, 4, 4)$, for $n = 5$ $(1, 1, 3, 5, 5)$) have lowest power between all the other RSS designs,

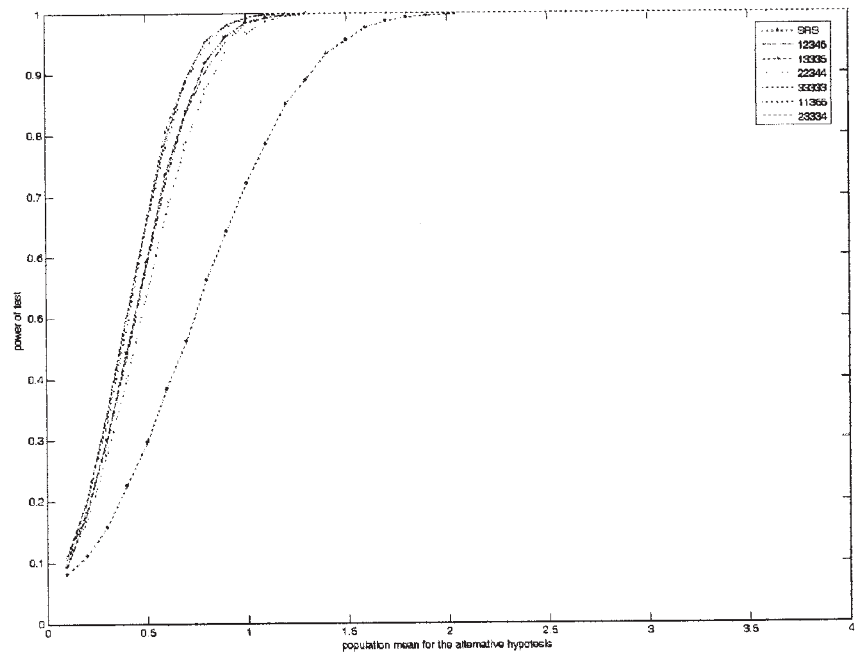

Figure 2.7: The powers for $H_1 : \mu > \mu_0$, where $n = 5$

it is more powerful than SRS. When the sample size n is increases, for the all type of hypothesis and sampling designs (RSS and SRS) the powers are also increases. Moreover, with increasing of the sample size, the difference between power of the RSS designs becomes closer.

4. Conclusions

In this study, we investigate the different RSS designs which are unbiased for the normal mean, by respect to power of test with the sample size of $n = 3, 4, 5$. It is found that, the powers of test values for all RSS designs are higher than SRS. But between the all RSS designs the median RSS designs indicate highest power values than the other RSS designs. For this reason, under the normal distribution, the hypothesis test about the population mean when the variance is known, the median RSS design has more powerful test than the other unbiased RSS designs and SRS for the sample size of $n = 3, 4, 5$. Especially when the sample size n is small, the median RSS is more powerful than the other unbiased RSS designs. When the sample size increases, the difference between powers of

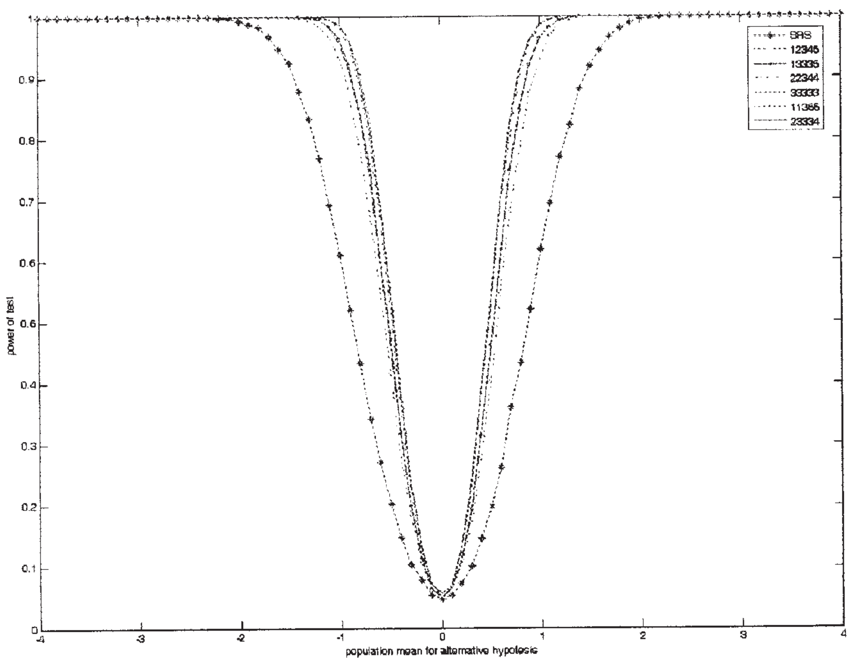

Figure 2.8: The powers for $H_1 : \mu \neq \mu_0$, where $n = 5$

all unbiased RSS designs are decreases. Further study could be made for the population mean when the population variance is unknown.

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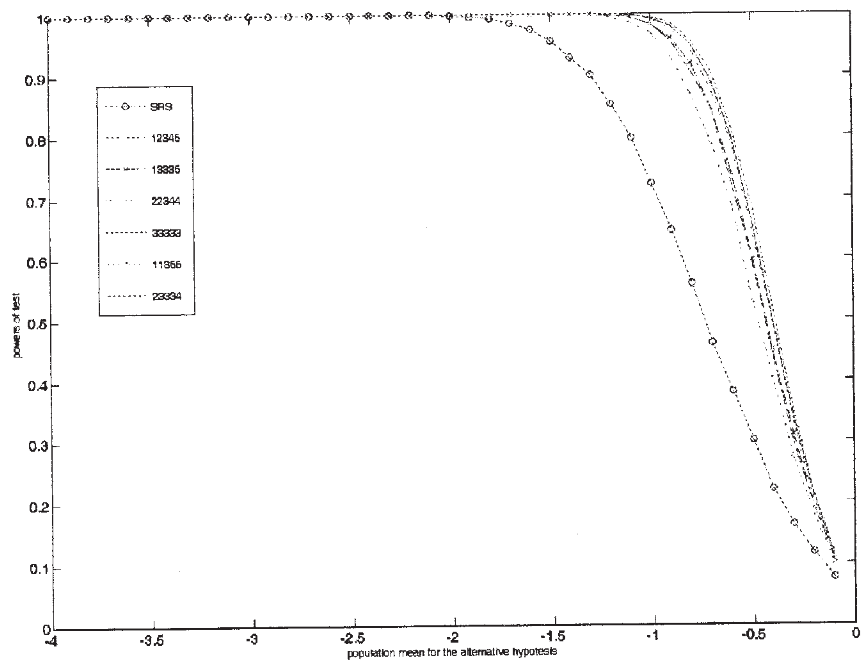

Figure 2.9: The powers for $H_1 : \mu < \mu_0$, where $n = 5$

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